National Curriculum (Vocational) Mathematics Level 3

National Curriculum (Vocational) Mathematics Level 3

DYLAN BUSA, NATASHIA BEARAM-EDMUNDS, AND GILL SCOTT

MICHAEL ATKINSON

National Curriculum (Vocational) Mathematics Level 3 by Department of Higher Education and Training is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

Contents

Subject Outcome I. Complex numbers: Represent complex numbers in a form appropriate to the context

Unit 1: Introduction to imaginary and complex numbers 3
Dylan Busa
Unit 1: Assessment 14
Unit 1: Solutions 15
Unit 2: Represent complex numbers using the Argand diagram 19
Dylan Busa
Unit 2: Assessment 33
Unit 2: Solutions 33
Unit 3: Represent complex numbers in polar form 43
Dylan Busa
Unit 3: Assessment 51
Unit 3: Solutions 51

Subject Outcome II. Complex numbers: Perform operations on complex numbers

Unit 1: Basic complex number arithmetic 57
Dylan Busa
Unit 1: Assessment 64
Unit 1: Solutions 65
Unit 2: Multiply and divide complex numbers in polar form 71
Dylan Busa
Unit 2: Assessment 76
Unit 2: Solutions 76

Subject Outcome III. Functions and algebra: Manipulate and simplify algebraic expressions

Unit 1: Multiplying algebraic fractions 83
Dylan Busa
Unit 1: Assessment 91
Unit 1: Solutions 92

Unit 2: Simplifying algebraic fractions 97
Dylan Busa
Unit 2: Assessment 103
Unit 2: Solutions 104
Unit 3: Completing the square 111
Dylan Busa
Unit 3: Assessment 118
Unit 3: Solutions 119

Subject Outcome IV. Functions and algebra: Solve algebraic equations and inequalities

Unit 1: Solve quadratic equations by factorisation 125
Dylan Busa
Unit 1: Assessment 134
Unit 1: Solutions 134
Unit 2: Solve quadratic equations by completing the square 139
Dylan Busa
Unit 2: Assessment 143
Unit 2: Solutions 144
Unit 3: Solve quadratic equations using the quadratic formula 147
Dylan Busa
Unit 3: Assessment 157
Unit 3: Solutions 157
Unit 4: Solve simultaneous equations 165
Dylan Busa
Unit 4: Assessment 172
Unit 4: Solutions 172
Unit 5: Solve quadratic inequalities 183
Dylan Busa
Unit 5: Assessment 195
Unit 5: Solutions 196

Subject Outcome V. Functions and algebra: Use a variety of techniques to sketch and interpret information from graphs of functions

Unit 1: Sketch quadratic functions 209
Dylan Busa
Unit 1: Assessment 221
Unit 1: Solutions 221

Unit 2: Find the equation of quadratic functions 231
Dylan Busa
Unit 2: Assessment 241
Unit 2: Solutions 242
Unit 3: Sketch hyperbolic functions 247
Dylan Busa
Unit 3: Assessment 260
Unit 3: Solutions 261
Unit 4: Find the equation of hyperbolic functions 267
Dylan Busa
Unit 4: Assessment 272
Unit 4: Solutions 273
Unit 5: Sketch exponential functions 275
Dylan Busa
Unit 5: Assessment 289
Unit 5: Solutions 289
Unit 6: Find the equation of an exponential function 295
Dylan Busa
Unit 6: Assessment 300
Unit 6: Solutions 301
Unit 7: The sine function 305
Dylan Busa
Unit 7: Assessment 325
Unit 7: Solutions 326
Unit 8: Horizontal transformation of the sine graph 333
Dylan Busa
Unit 8: Assessment 345
Unit 8: Solutions 346
Unit 9: The cosine function 355
Dylan Busa
Unit 9: Assessment 369
Unit 9: Solutions 370
Unit 10: Horizontal transformation of the cosine graph 377
Dylan Busa
Unit 10: Assessment 391
Unit 10: Solutions 392
Unit 11: The tangent function 401
Dylan Busa
Unit 11: Assessment 416
Unit 11: Solutions 417

Unit 12: Horizontal transformation of the tangent graph 425
Dylan Busa
Unit 12: Assessment 434
Unit 12: Solutions 435

Subject Outcome VI. Functions and algebra: Use Mathematical models to investigate linear programming problems

Unit 1: Linear programming 447
Dylan Busa
Unit 1: Assessment 459
Unit 1: Solutions 460

Subject Outcome VII. Functions and algebra: Investigate and use instantaneous rate of change

Unit 1: Limits of a function 471
Natashia Bearam-Edmunds
Unit 1: Assessment 478
Unit 1: Solutions 479
Unit 2: Find the derivative by first principles 481
Natashia Bearam-Edmunds
Unit 2: Assessment 493
Unit 2: Solutions 493
Unit 3: Find the derivative using rules for differentiation 499
Natashia Bearam-Edmunds
Unit 3: Assessment 505
Unit 3: Solutions 505
Unit 4: Use derivatives to solve maxima and minima problems 509
Natashia Bearam-Edmunds
Unit 4: Assessment 516
Unit 4: Solutions 517

Subject Outcome VIII. Space, shape and measurement: Calculate the surface area and volume of two- and three-dimensional shapes

Unit 1: Finding the volume of 3-D shapes 523
Natashia Bearam-Edmunds
Unit 1: Assessment 530
Unit 1: Solutions 531

Unit 2: Finding the surface area of 3-D shapes 535
Natashia Bearam-Edmunds
Unit 2: Assessment 542
Unit 2: Solutions 544

Subject Outcome IX. Space, shape and measurement: Use the Cartesian co-ordinate system to derive and apply equations

Unit 1: Finding the equation of a line 549
Dylan Busa
Unit 1: Assessment 560
Unit 1: Solutions 563
Unit 2: Equations of parallel and perpendicular lines 569
Dylan Busa
Unit 2: Assessment 581
Unit 2: Solutions 582
Unit 3: Determine the angle of inclination of a line 589
Dylan Busa
Unit 3: Assessment 595
Unit 3: Solutions 599

Subject Outcome X. Space, shape and measurement: Solve problems by constructing and interpreting trigonometric models

Unit 1: Special angles 609
Dylan Busa
Unit 1: Assessment 620
Unit 1: Solutions 622
Unit 2: Use reduction formulae to simplify trig ratios 627
Dylan Busa
Unit 2: Assessment 645
Unit 2: Solutions 645
Unit 3: Prove trig identities 651
Dylan Busa
Unit 3: Assessment 657
Unit 3: Solutions 657
Unit 4: Find the general solution to trig equations 663
Dylan Busa
Unit 4: Assessment 672
Unit 4: Solutions 673

Unit 5: Apply the area rule 677
Dylan Busa
Unit 5: Assessment 685
Unit 5: Solutions 687
Unit 6: Apply the sine rule 691
Dylan Busa
Unit 6: Assessment 702
Unit 6: Solutions 704
Unit 7: Apply the cosine rule 711
Dylan Busa
Unit 7: Assessment 719
Unit 7: Solutions 721
Unit 8: Solve 2-D trig problems 727
Dylan Busa
Unit 8: Assessment 735
Unit 8: Solutions 738

Subject Outcome XI. Data handling: Calculate, represent and interpret measures of central tendency and dispersion in univariate numerical ungrouped data

Unit 1: Revise measures of central tendency and dispersion for ungrouped data 747
Natashia Bearam-Edmunds
Unit 1: Assessment 756
Unit 1: Solutions 757
Unit 2: The five-number summary and box-and-whisker diagram 761
Natashia Bearam-Edmunds
Unit 2: Assessment 770
Unit 2: Solutions 771

Subject Outcome XII. Data handling: Calculate, represent and interpret measures of central tendency and dispersion in univariate numerical grouped data

Unit 1: Grouping data using frequency distributions 779
Natashia Bearam-Edmunds
Unit 1: Assessment 784
Unit 1: Solutions 784
Unit 2: Construct the ogive 787
Natashia Bearam-Edmunds
Unit 2: Assessment 794
Unit 2: Solutions 796

Unit 3: Constructing histograms 801
Natashia Bearam-Edmunds
Unit 3: Assessment 811
Unit 3: Solutions 812
Unit 4: Central tendency and dispersion for grouped data 817
Natashia Bearam-Edmunds
Unit 4: Assessment 826
Unit 4: Solutions 827

Subject Outcome XIII. Financial mathematics: Plan and describe how to manage finances of social clubs

Unit 1: Financial control for social clubs 835
Natashia Bearam-Edmunds
Unit 1: Assessment 843
Unit 1: Solutions 845

Subject Outcome XIV. Financial mathematics: Use simple and compound interest to explain and define a variety of situations

Unit 1: Calculations involving simple interest 851
Gill Scott
Unit 1: Assessment 856
Unit 1: Solutions 856
Unit 2: Calculations with compound interest 861
Unit 2: Assessment 872
Unit 2: Solutions 873
Unit 3: Using timelines for calculations of simple and compound interest 879
Unit 3: Assessment 885
Unit 3: Solutions 885

SUBJECT OUTCOME I

COMPLEX NUMBERS: REPRESENT COMPLEX NUMBERS IN A FORM APPROPRIATE TO THE CONTEXT

Subject outcome

Subject outcome 1.1: Represent complex numbers in a form appropriate to the context

Learning outcomes

- Write imaginary numbers in their simplest form where $i^2 = -1$.
- Simplify negative roots into imaginary numbers.
- Simplify and perform addition, subtraction, multiplication and division on imaginary numbers.
- Construct Argand diagrams to find and represent the modulus and positive argument.
- Represent complex numbers in polar form with positive argument.

Unit 1 outcomes

By the end of this unit you will be able to:

- Define imaginary numbers.
- Perform basic operations on imaginary numbers.
- Define complex numbers.
- Represent complex numbers in standard rectangular coordinate form.

Unit 2 outcomes

By the end of this unit you will be able to:

- Represent complex numbers using an Argand diagram.
- Find the modulus and argument of a complex number $i^2 = -1$.

Unit 3 outcomes

By the end of this unit you will be able to:

- Express complex numbers in polar form.
- Convert complex numbers from standard/rectangular form to polar form.
- Convert complex numbers from polar form to standard/rectangular form.

Unit 1: Introduction to imaginary and complex numbers

DYLAN BUSA

Unit 1 outcomes

By the end of this unit you will be able to:

- Define imaginary numbers.
- Perform basic operations on imaginary numbers.
- Define complex numbers.
- Represent complex numbers in standard rectangular coordinate form.

What you should know

Before you start this unit, make sure you can:

- Describe the different parts of the real number system. Refer to Level 2 subject outcome 1.2 if you need help with this.
- Simplify expressions containing irrational numbers, exponents and surds. Refer to Level 2 subject outcome 1.2 if you need help with this.

Introduction

The history of numbers is long and interesting. At several points in Mathematics history, we have encountered problems that forced us to discover or invent new types of types of numbers to fill in the gaps.

First there were the counting numbers such as 1, 20, $1\ 140$. These are the numbers we first learn about as young children. We call these **natural numbers** ($\mathbb{N}$) because they exist in the natural world around us – three chairs, five cows, etc. If we add the number zero to this set, we get **whole numbers** ($\mathbb{N}_0$).

But there were some gaps. Some of these gaps became evident when people tried to work out what $4 - 7$ was equalled. To solve this problem, negative natural numbers such as -1, -12 and $-2\ 321$ were added to the set. We called this new expanded set of numbers the **integers** ($\mathbb{Z}$).

But there were still gaps. For example, dividing 6 by 3 is not a problem. The answer is 2. Even $\frac{6}{-3} = -2$ is fine. But what about $\frac{3}{6}$? To solve these kinds of problems, we needed to add the fractions to the set. We called this new expanded set of numbers **rational numbers** ($\mathbb{Q}$). Every number that can be written as one integer divided by another integer is a rational number.

But there were still more gaps which we discovered when we tried to figure out the answer to $\sqrt{2}$. This cannot be written as one integer divided by another, so we needed to expand the set of numbers yet again to include **non-rational numbers** ($\mathbb{Q}'$).

All these numbers together were called **real numbers** ($\mathbb{R}$) (see Figure 1).

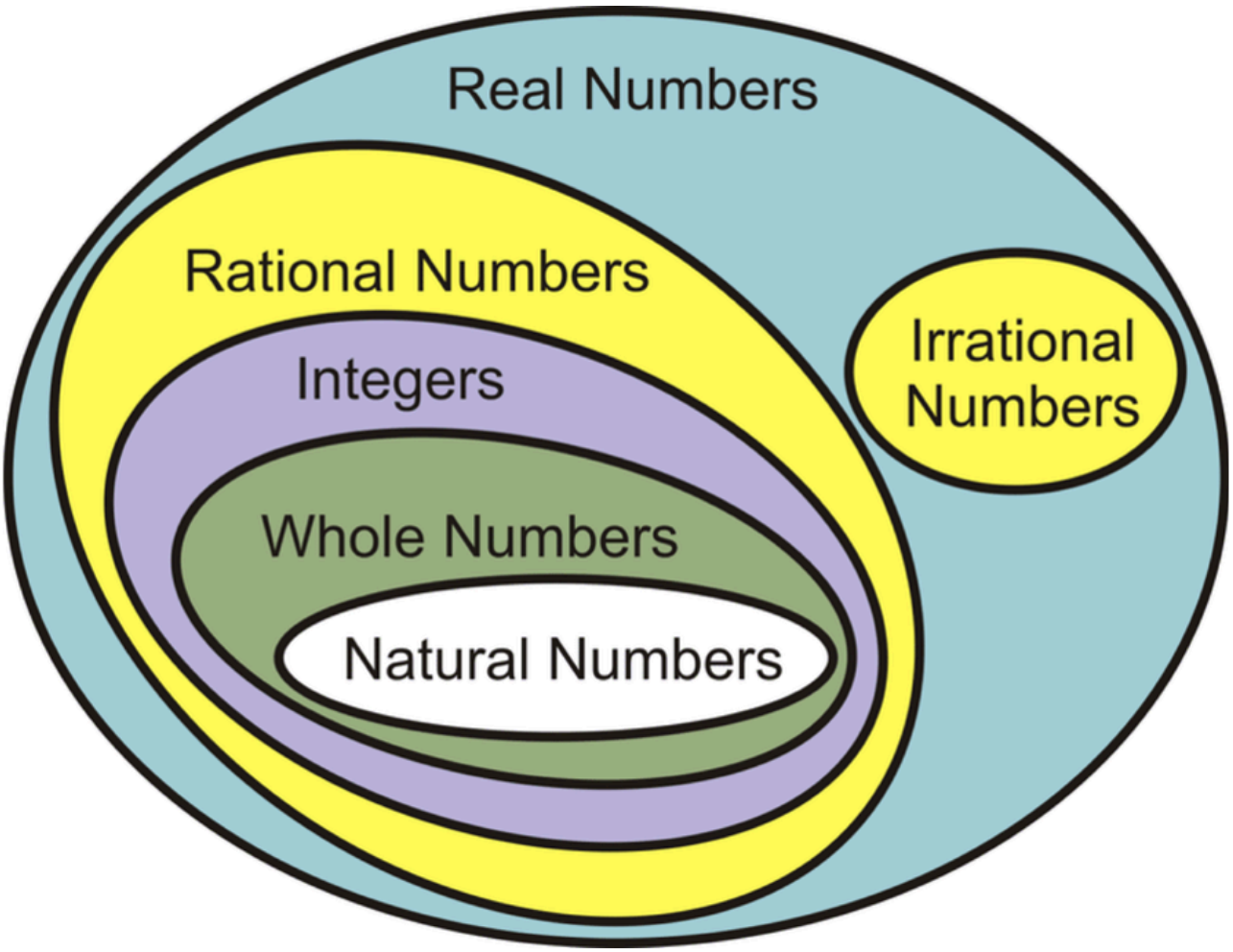


Figure 1: The Real Number System

At this point we seemed to be done. Phew! But...

Then someone tried to figure out what the answer to $\sqrt{-1}$ is. At first, this seemed to be impossible. $1 \times 1 = 1$ and $-1 \times -1 = 1$. There is no real number that when we multiply it by itself gives us a negative answer. What to do?

The solution turned out to be to invent or discover (depending on your view) a new kind of number. This number is called i, the **imaginary number** and is defined as $i^2 = -1$.

The imaginary number:

$i^2 = -1$ or $\sqrt{-1} = i$

But what is this imaginary number good for? Well, now we can solve equations like this:

$$x^2 + 9 = 0$$
$$\therefore x^2 = -9$$
$$\therefore x = \pm\sqrt{-9}$$
$$\therefore x = \pm\sqrt{(9)(-1)}$$
$$\therefore x = \pm\sqrt{9} \times \sqrt{-1}$$
$$\therefore x = \pm 3 \times i$$
$$\therefore x = \pm 3i$$

The name chosen for this number is a bit unfortunate because it turns up in all sorts of very non-imaginary places like three-phase electricity, sound analysis, all sorts of engineering problems and quantum physics.

If you have an internet connection, you should watch both these excellent videos that help explain why the imaginary numbers are as 'real' as any of the real numbers.

- Imaginary Numbers Are Just Regular Numbers (Duration: 9.01).
- What are imaginary numbers? (Duration: 3.54).

Imaginary numbers

We now know how to find the square root of negative numbers. We simply write the square root of the corresponding positive number as a multiple of i.

$\sqrt{-49} = 7i$
$\sqrt{-25} = 5i$
$\sqrt{-2} = \sqrt{2}i$

Multiplying the imaginary number i by every possible real number would yield the set of all imaginary numbers and, in general, i follows the rules of real number arithmetic.

Just like if we add (or subtract) two real numbers, the answer is always another real number, when we add (or subtract) two imaginary numbers, the answer is always an imaginary number. For example, $3i + 5i = 8i$ and $\sqrt{2}i - 3i = \left(\sqrt{2} - 3\right)i$.

Example 1.1

Simplify the following, expressing your answers as imaginary numbers (where necessary):

1. $\sqrt{-16}$
2. $\sqrt{-24}$
3. $\sqrt{-32} - \sqrt{-2}$
4. $\sqrt{80}$

Solutions

1. $$\begin{aligned}\sqrt{-16} &= \sqrt{(16)(-1)} \\ &= 4\sqrt{-1} \\ &= 4i\end{aligned}$$
2. $$\begin{aligned}\sqrt{-24} &= \sqrt{(24)(-1)} \\ &= \sqrt{(4)(6)(-1)} \\ &= 2\sqrt{6}i\end{aligned}$$
3. $$\begin{aligned}\sqrt{-32} - \sqrt{-2} &= \sqrt{16(-2)} - \sqrt{-2} \\ &= 4\sqrt{-2} - \sqrt{-2} \\ &= \sqrt{-2}\,(4-1) \\ &= 3\sqrt{-2} \\ &= 3\sqrt{2}i\end{aligned}$$
4. $$\begin{aligned}\sqrt{80} &= \sqrt{(16)(5)} \\ &= 4\sqrt{5}\end{aligned}$$

Exercise 1.1

Simplify the following, expressing your answers as imaginary numbers (where necessary):

1. $\sqrt{-400}$
2. $\sqrt{-49} - \sqrt{-100}$
3. $\sqrt{-4} + \sqrt{-25} - \sqrt{3}$
4. $5i - \sqrt{-25} + 2\sqrt{2}i$

The full solutions are at the end of the unit.

Multiply and divide imaginary numbers

Just as imaginary numbers obey the normal (real number) rules of addition and subtraction, they also obey the normal (real number) rules of multiplication and division. However, we always need to remain aware of the fact that $i^2 = -1$. Whenever we see i^2, we can replace this with -1.



Example 1.2

Simplify the following:

1. $3i \times 5i$
2. $(-6i)(2i)$
3. $7i^3 \times 4i^2$

Solutions

1. We need to multiply the numbers and the i's.
$$\begin{aligned} 3i \times 5i &= 15i^2 && \text{But } i^2 = -1 \\ &= 15(-1) \\ &= -15 \end{aligned}$$

2.
$$\begin{aligned} &(-6i)(2i) && \text{Multiply the numbers and the } i\text{'s} \\ &= -12i^2 && \text{But } i^2 = -1 \\ &= -12(-1) \\ &= 12 \end{aligned}$$

3.
$$\begin{aligned} 7i^3 \times 4i^2 &= 28i^5 && \text{The normal exponent rules still apply} \\ &= 28 \times i^2 \times i^2 \times i && \text{But } i^2 = -1 \\ &= 28 \times (-1) \times (-1) \times i \\ &= 28i \end{aligned}$$



Example 1.3

Simplify the following:

1. $\dfrac{6i}{2i}$
2. $\dfrac{36i^2}{6i}$
3. $\dfrac{20i^2}{4}$

4. $\dfrac{18i^6 \times 2i^4}{6i^3}$

Solutions

1. We need to divide the numbers and the i's. If $i = \sqrt{-1}$ then $\dfrac{i}{i} = \dfrac{\sqrt{-1}}{\sqrt{-1}} = 1$.

   $\dfrac{6i}{2i} = 3$

2. $\dfrac{36i^2}{6i} = 6i$

3. $\dfrac{20i^2}{4} = 5i^2 \qquad \text{But } i^2$
   $= -1$
   $= -5$

4. $\dfrac{18i^6 \times 2i^4}{6i^3} = \dfrac{36i^{10}}{6i^3}$
   $= 6i^7$
   $= 6 \times i^2 \times i^2 \times i^2 \times i$
   $= 6(-1)(-1)(-1)i$
   $= -6i$

The powers of i are cyclic. Look at what happens when we raise i to increasing powers.

$i^1 = i$
$i^2 = -1 \quad \text{By definition}$
$i^3 = i^2.i = -1.i = -i$
$i^4 = i^3.i = -i.i = -i^2 = -(-1) = 1$
$i^5 = i^4.i = 1.i = i \quad \text{This is the same as } i^1$
$i^6 = i^5.i = i.i = i^2 = -1 \quad \text{This is the same as } i^2$
$i^7 = i^6.i = -1.i = -i \quad \text{This is the same as } i^3$
$i^8 = i^7.i = -i.i = -i^2 = -(-1) = 1 \quad \text{This is the same as } i^4$

We can see that the cycle repeats itself every **fourth** power of i. This means that we can simplify large powers of i by factoring out as many factors of i^4 as possible. For example:

i^{35}
$= i^{(4\times 8)+3}$
$= \left(i^4\right)^8.i^3$
$= 1^8.i^3 \quad (i^4 = 1)$
$= i^3$
$= i^2.i$
$= -i$

Note

Watch the video called “Ex: Raising the imaginary unit *i* to powers” if you would like to see more examples.

Ex: Raising the imaginary unit *i* to powers (Duration: 4.42)



i_to_powers" data-url="https://youtu.be/ZMLFfTX615w">i_to_powers">

Exercise 1.2

Simplify the following:

1. $\dfrac{7i^5}{49i^7}$
2. $\dfrac{\sqrt{6}i}{\sqrt{-24}}$
3. $\dfrac{7i^3.3i^8.2i^4}{21i^5.3i^5}$
4. $\dfrac{\sqrt{12}i.(-4i).3i^5}{\sqrt{-24}.6i^4}$

The full solutions are at the end of the unit.

Complex numbers

We have seen that when we add or subtract imaginary numbers, the answer is always imaginary and that when we multiply or divide imaginary numbers, the answer is either real or imaginary depending on the powers of i.

But what happens when we add a real number to an imaginary number? The answer is neither a pure real number nor a pure imaginary number. Instead, we get a complex number. A complex number is simply the

sum of a real number and an imaginary number. A **complex number** is expressed in **standard form** when written as $a + bi$ where a is the real part and b is the imaginary part (see Figure 2).

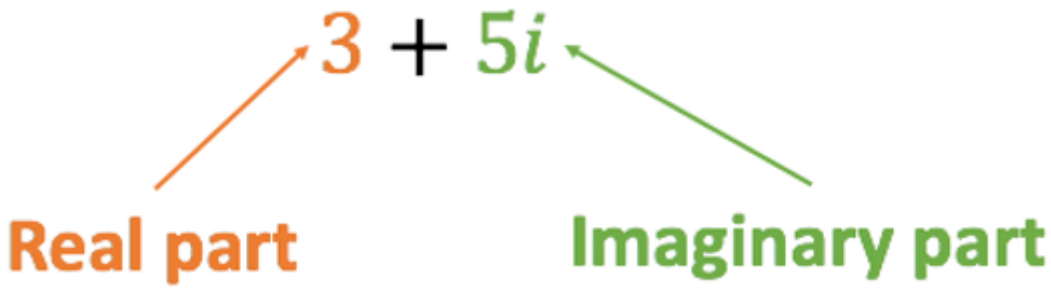


Figure 2: Standard form of complex numbers

For example, $5 + 2i$ is a complex number, as is $3 + 4\sqrt{3}i$. The entire number system is represented in Figure 3.

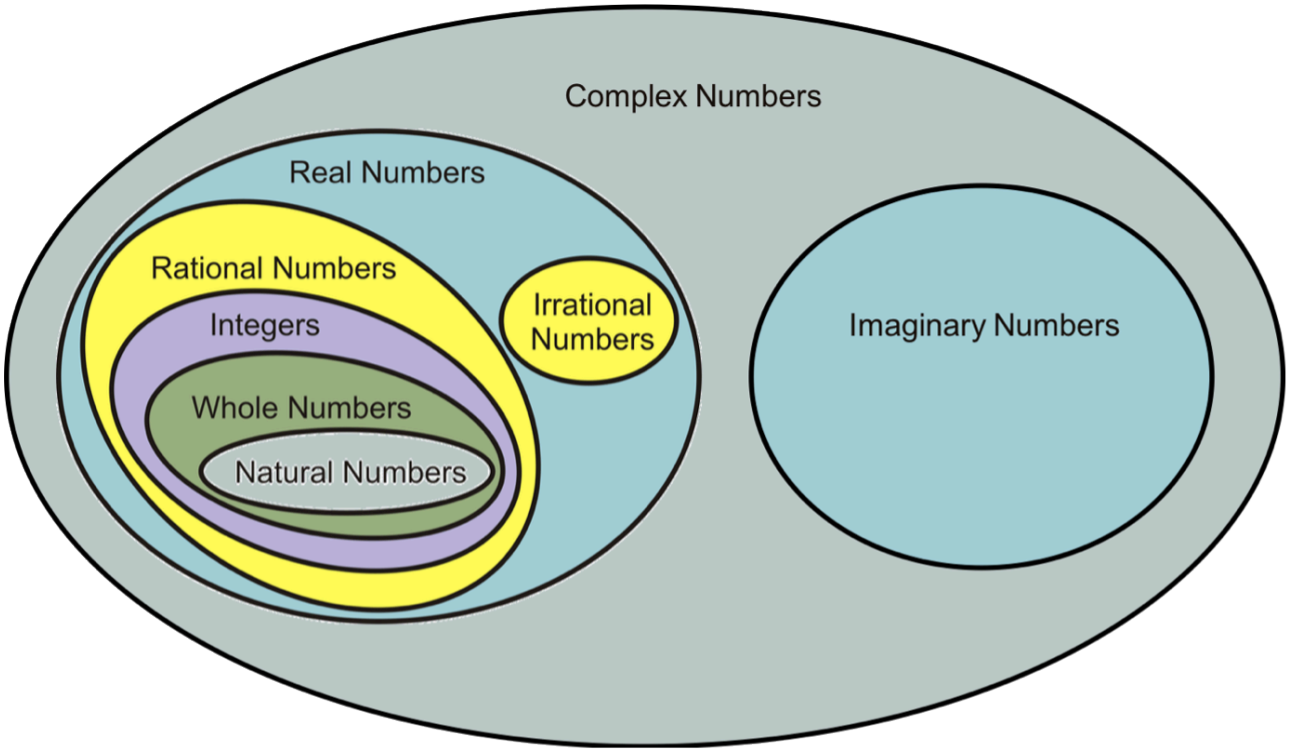


Figure 3: The complex number system

A complex number is a number of the form $a + bi$ where:

- a is the real part
- b is the imaginary part.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and $b \neq 0$, then the complex number is called a **pure imaginary number**.

Take note!

Standard form is sometimes also referred to as **rectangular form**.

Example 1.4

Express the following numbers in standard form:

1. $\sqrt{-36}$
2. $\sqrt{-42}$
3. $\sqrt{-32} - \sqrt{7}$
4. $\sqrt{80} - \sqrt{-4}$

Solutions

1. $$\begin{aligned}\sqrt{-36} &= \sqrt{(36)(-1)} \\ &= 6\sqrt{-1} \\ &= 6i\end{aligned}$$
   In standard form this is $0 + 6i$. There is no real part, just an imaginary part.
2. $$\begin{aligned}\sqrt{-42} &= \sqrt{(42)(-1)} \\ &= \sqrt{42}i\end{aligned}$$
   In standard form this is $0 + \sqrt{42}i$. There is no real part, just an imaginary part.
3. $$\begin{aligned}\sqrt{-32} - \sqrt{7} &= \sqrt{(32)(-1)} - \sqrt{(7)} \\ &= \sqrt{(16)(2)(-1)} - \sqrt{7} \\ &= 4\sqrt{2}i - \sqrt{7} \\ &= -\sqrt{7} + 4\sqrt{2}i\end{aligned}$$
4. $$\begin{aligned}\sqrt{80} - \sqrt{-4} &= \sqrt{(16)(5)} - \sqrt{(4)(-1)} \\ &= 4\sqrt{5} - 2i\end{aligned}$$

Did you know?

The German mathematician Leopold Kronecker once said “God made the integers; all else is the work of man”. Complex numbers represent an incremental journey of investigation, struggle and invention over a thousand years by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and

many others. Complex numbers answer questions that for centuries had puzzled the greatest minds in science.



Figure 4: Leopold Kronecker

Exercise 1.3

Express the following in standard form:

1. $\sqrt{-144}$
2. $\sqrt{-600}$
3. $\sqrt{12}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- $i^2 = -1$ by definition, and therefore, $\sqrt{-1} = i$.
- How to write the square roots of negative numbers as imaginary numbers.
- How to add, subtract, multiply and divide imaginary numbers.
- How to simplify powers of i by using the fact that $i^4 = 1$.
- Complex numbers are written in standard form as $a + bi$ where a is the real part and bi is the imaginary part.

Unit 1: Assessment

Suggested time to complete: 30 minutes

Simplify the following writing your answers in standard form:

1. $\dfrac{5i + 6i}{2i^2}$
2. $7i^5 \times 3i^{14} \div 7i^{12}$
3. $\dfrac{-7i^5 + 13i^5}{-3i^4 - 5i^4}$
4. $\dfrac{\left(\sqrt{-16} - \sqrt{-144}\right)}{2i}$
5. $\dfrac{\sqrt{-2}.\sqrt{-7}.\sqrt{-9}}{\sqrt{-4}.3i} - \dfrac{\sqrt{24}}{\sqrt{12}}$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $$\begin{aligned}\sqrt{-400} &= \sqrt{(400)(-1)} \\ &= 20i\end{aligned}$$

2. $$\begin{aligned}\sqrt{-49} - \sqrt{-100} &= \sqrt{(49)(-1)} - \sqrt{(100)(-1)} \\ &= 7i - 10i \\ &= -3i\end{aligned}$$

3. $$\begin{aligned}\sqrt{-4} + \sqrt{-25} - \sqrt{-3} &= \sqrt{(4)(-1)} + \sqrt{(25)(-1)} - \sqrt{(3)(-1)} \\ &= 2i + 5i - \sqrt{3}i \\ &= 7i - \sqrt{3}i \\ &= \left(7 - \sqrt{3}\right) i\end{aligned}$$

4. $$\begin{aligned}5i - \sqrt{-25} + 2\sqrt{2}i &= 5i - \sqrt{(25)(-1)} + 2\sqrt{2}i \\ &= 5i - 5i + 2\sqrt{2}i \\ &= 2\sqrt{2}i\end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1. $$\begin{aligned}\frac{7i^5}{49i^7} &= \frac{1}{7i^2} \\ &= -\frac{1}{7}\end{aligned}$$

2. $$\begin{aligned}\frac{\sqrt{6}i}{\sqrt{-24}} &= \frac{\sqrt{6}i}{\sqrt{(6)}(4)(-1)} \\ &= -\frac{\sqrt{6}i}{2\sqrt{6}i} \\ &= \frac{1}{2}\end{aligned}$$

3. $$\begin{aligned}\frac{7i^3.3i^8.2i^4}{21i^5.3i^5} &= \frac{42i^15}{63i^10} \\ &= \frac{2i^5}{3} \\ &= \frac{2 \times i^4 \times i}{3} \\ &= \frac{2}{3}i\end{aligned}$$

4.

$$\begin{aligned}\frac{\sqrt{12}i.(-4i).3i^5}{\sqrt{-24}.6i^4} &= \frac{\sqrt{(4)(3)}i(-4i).3i^5}{\sqrt{(4)(6)(-1)}.6i^4} \\ &= \frac{2\sqrt{3}i.(-12i^6)}{2\sqrt{6}i.6i^4} \\ &= \frac{-24\sqrt{3}i^7}{12\sqrt{6}i^5} \\ &= \frac{-2\sqrt{3}i^2}{\sqrt{6}} \\ &= \frac{-2\sqrt{3}i^2}{\sqrt{2}\sqrt{3}} \\ &= \frac{-2(-1)}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{2}}{2} \\ &= \sqrt{2}\end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1. $$\begin{aligned}\sqrt{-144} &= \sqrt{(144)(-1)} \\ &= 12\sqrt{-1} \\ &= 12i\end{aligned}$$
   Standard form: $0 + 12i$
2. $$\begin{aligned}&\sqrt{-600} \\ &= \sqrt{(6)(100)(-1)} \\ &= 10\sqrt{(6)(-1)} \\ &= 10\sqrt{6}i\end{aligned}$$
   Standard form: $0 + 10\sqrt{6}i$
3. $$\begin{aligned}&\sqrt{12} \\ &= \sqrt{(4)(3)} \\ &= 2\sqrt{3}\end{aligned}$$
   Standard form: $2\sqrt{3} + 0i$

Back to Exercise 1.3

Unit 1: Assessment

1.

$$\begin{aligned}\frac{5i+6i}{2i^2} &= \frac{11i}{2(-1)} \\ &= \frac{11i}{-2} \\ &= -\frac{11}{2}i\end{aligned}$$

Standard form: $0-\frac{11}{2}i$

2.

$$\begin{aligned}7i^5 \times 3i^{14} \div 7i^{12} &= \frac{7i^5 \times 3i^{14}}{7i^{12}} \\ &= \frac{21i^{19}}{7i^{12}} \\ &= 3i^7 \\ &= 3.i^4.i^3 \qquad \left(i^4=1\right) \\ &= 3.i^2.i \qquad \left(i^2=-1\right) \\ &= -3i\end{aligned}$$

Standard form: $0-3i$

3.

$$\begin{aligned}\frac{-7i^5+13i^5}{-3i^4-5i^4} &= \frac{6i^5}{-8i^4} \\ &= -\frac{3i}{4} \\ &= -\frac{3}{4}i\end{aligned}$$

Standard form: $0-\frac{3}{4}i$

4.

$$\begin{aligned}\frac{\left(\sqrt{-16}-\sqrt{-144}\right)}{2i} &= \frac{\left(\sqrt{(16)(-1)}-\sqrt{(144)(-1)}\right)}{2i} \\ &= \frac{(4i-12i)}{2i} \\ &= \frac{-8i}{2i} \\ &= -4\end{aligned}$$

Standard form: $-4+0i$

5.

$$\begin{aligned}\frac{\sqrt{-2}.\sqrt{-7}.\sqrt{-9}}{\sqrt{-4}.3i}-\frac{\sqrt{24}}{\sqrt{12}} &= \frac{\sqrt{(2)(-1)}.\sqrt{(7)(-1)}.\sqrt{(9)(-1)}}{\sqrt{(4)(-1)}.3i}-\frac{\sqrt{6.4}}{\sqrt{3.4}} \\ &= \frac{\sqrt{2}i.\sqrt{7}i.3i}{2i.3i}-\frac{2\sqrt{6}}{2\sqrt{3}} \\ &= \frac{3\sqrt{14}i^3}{6i^2}-\frac{\sqrt{2}.\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{14}i}{2}-\sqrt{2} \\ &= -\sqrt{2}+\frac{\sqrt{14}}{2}i\end{aligned}$$

Standard form: $-\sqrt{2}+\frac{\sqrt{14}}{2}i$

Back to Unit 1: Assessment

Media Attributions

- figure1 © CK-12 is licensed under a CC BY-NC (Attribution NonCommercial) license
- figure2 © DHET is licensed under a CC BY (Attribution) license
- figure3 © CK-12 is licensed under a CC BY-NC (Attribution NonCommercial) license
- figure4 © Unknown is licensed under a Public Domain license

Unit 2: Represent complex numbers using the Argand diagram

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Represent complex numbers using an Argand diagram.
- Find the modulus and argument of a complex number. $i^2 = -1$

What you should know

Before you start this unit, make sure you can:

- Define an imaginary number. Refer to unit 1 in this subject outcome if you need help with this.
- Define a complex number. Refer to unit 1 in this subject outcome if you need help with this.
- Write a complex number in standard or rectangular form. Refer to unit 1 in this subject outcome if you need help with this.
- Work with the basic trigonometric ratios sine, cosine and tangent on the Cartesian plane. Refer to unit 1 in level 2 subject outcome 3.6 if you need help with this.

Introduction

In the previous unit, we defined imaginary numbers and complex numbers. We said that a complex number is written in standard (or rectangular form) as $a + bi$. However, there is another form in which complex numbers can be written, called **polar form**.

Polar form is very useful in many situations for adding and subtracting complex numbers, especially when these numbers represent vectors.

In this unit, we are going to learn the basics about writing and working with complex numbers in polar form, and how to convert them from standard form to polar form and back again.

The complex number plane

We know that we can plot the position of any real number on a number line. Figure 1 shows a few examples.

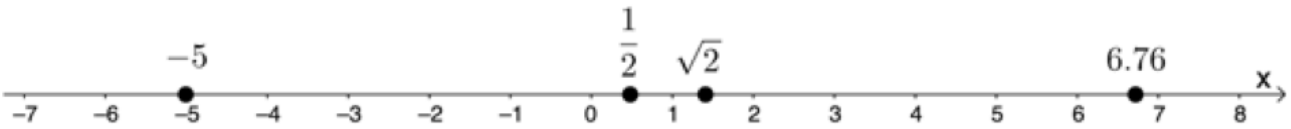


Figure 1: Real numbers on the number line.

But what do we do with complex numbers that have a real and an imaginary part? To plot complex numbers on a number line, we use two numbers lines – one for the real part and one for the imaginary part. We place these two number lines at right angles to each other to represent the **complex plane**. This is a coordinate system just like the Cartesian plane. The horizontal axis represents the real component, and the vertical axis represents the imaginary component. Complex numbers are then points on the plane, expressed as ordered pairs.

Take the complex number $2 - 3i$, for example. We can plot it on the complex plane as shown in Figure 2. It is the point $(2, -3)$.

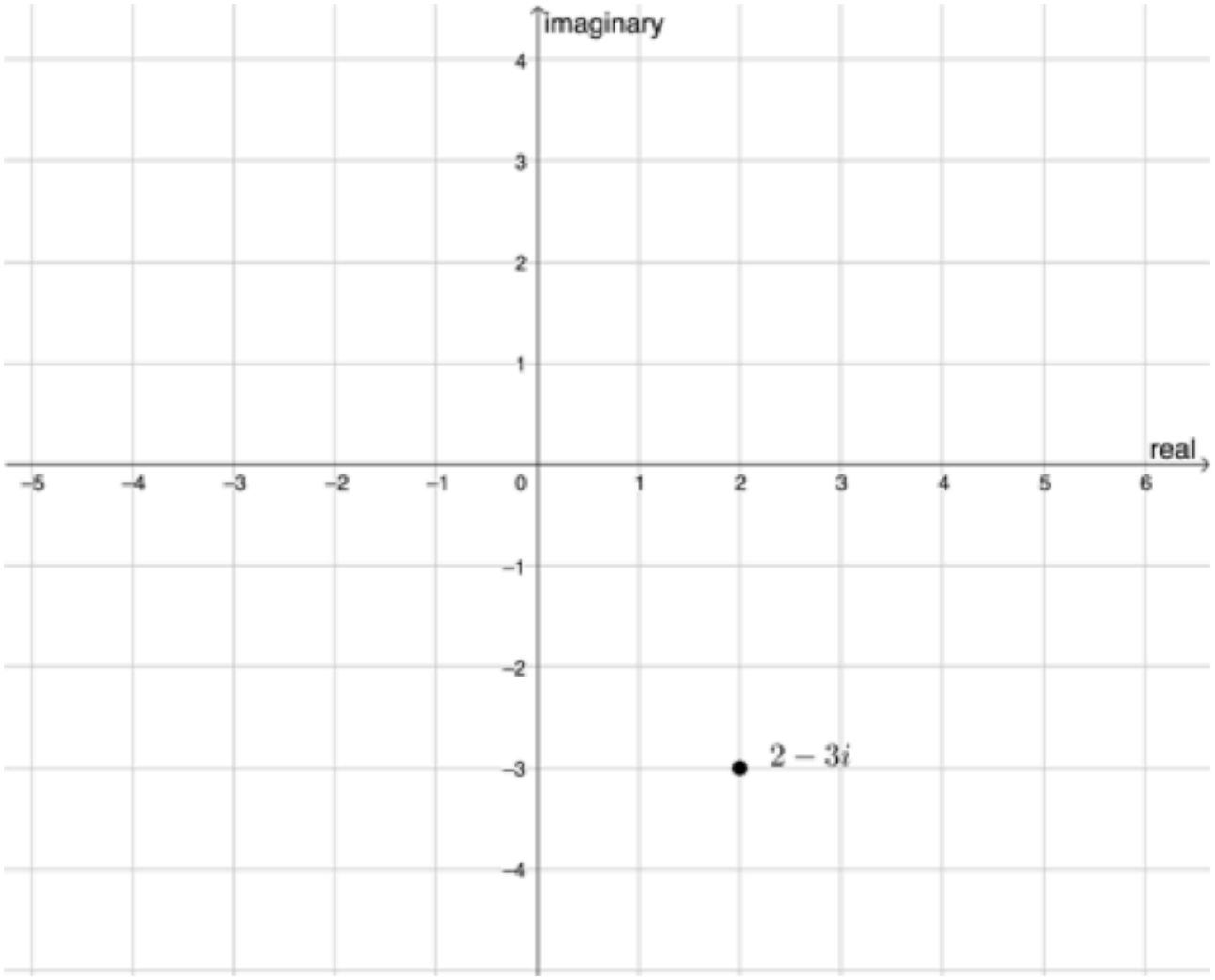


Figure 2: $2 - 3i$ plotted on the complex plane

Note

If you would like to see other worked examples, watch the video called "Plotting numbers on the complex plane".

Plotting numbers on the complex plane (Duration: 01.13)

Take note!

An **Argand diagram** is a plot of complex numbers as points in the complex plane using the x-axis as the real axis and y-axis as the imaginary axis.

Note

Watch "The Argand diagram" to see an example of an Argand diagram being drawn.

The Argand diagram (Duration: 02.52)

Did you know?

Argand diagrams are named after the French mathematician Jean-Robert Argand (1768–1822) although they were first described by the Norwegian–Danish land surveyor and mathematician Caspar Wessel (1745–1818).

Exercise 2.1

Draw Argand diagrams for the following complex numbers:

1. $4 + 3i$
2. $-4 - 7i$
3. -2
4. $3i$

The full solutions are at the end of the unit.

The absolute value or modulus of a complex number

Look at the complex number $3 + 4i$ plotted on the complex plane in Figure 3. How far is the point from the origin of the plane? In other words, how long is the line joining the point and the origin in Figure 3? See if you can work this out before reading on.

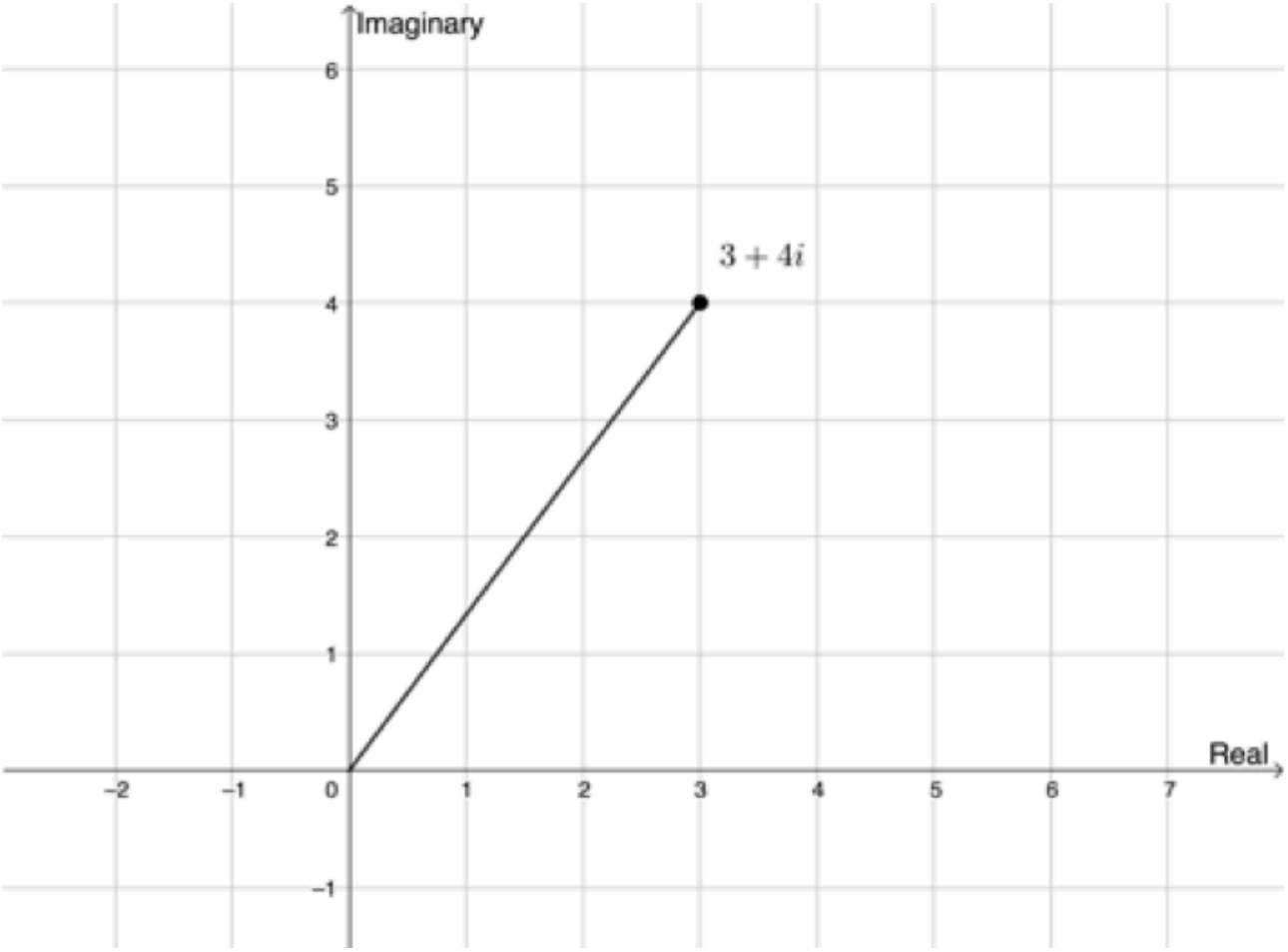


Figure 3: $3 + 4i$ plotted on the complex plane

If we drop a perpendicular line from the point representing $3 + 4i$ we will create a right-angled triangle. Now we can use Pythagoras' theorem to find the length of the line joining $3 + 4i$ to the origin as shown in Figure 4.

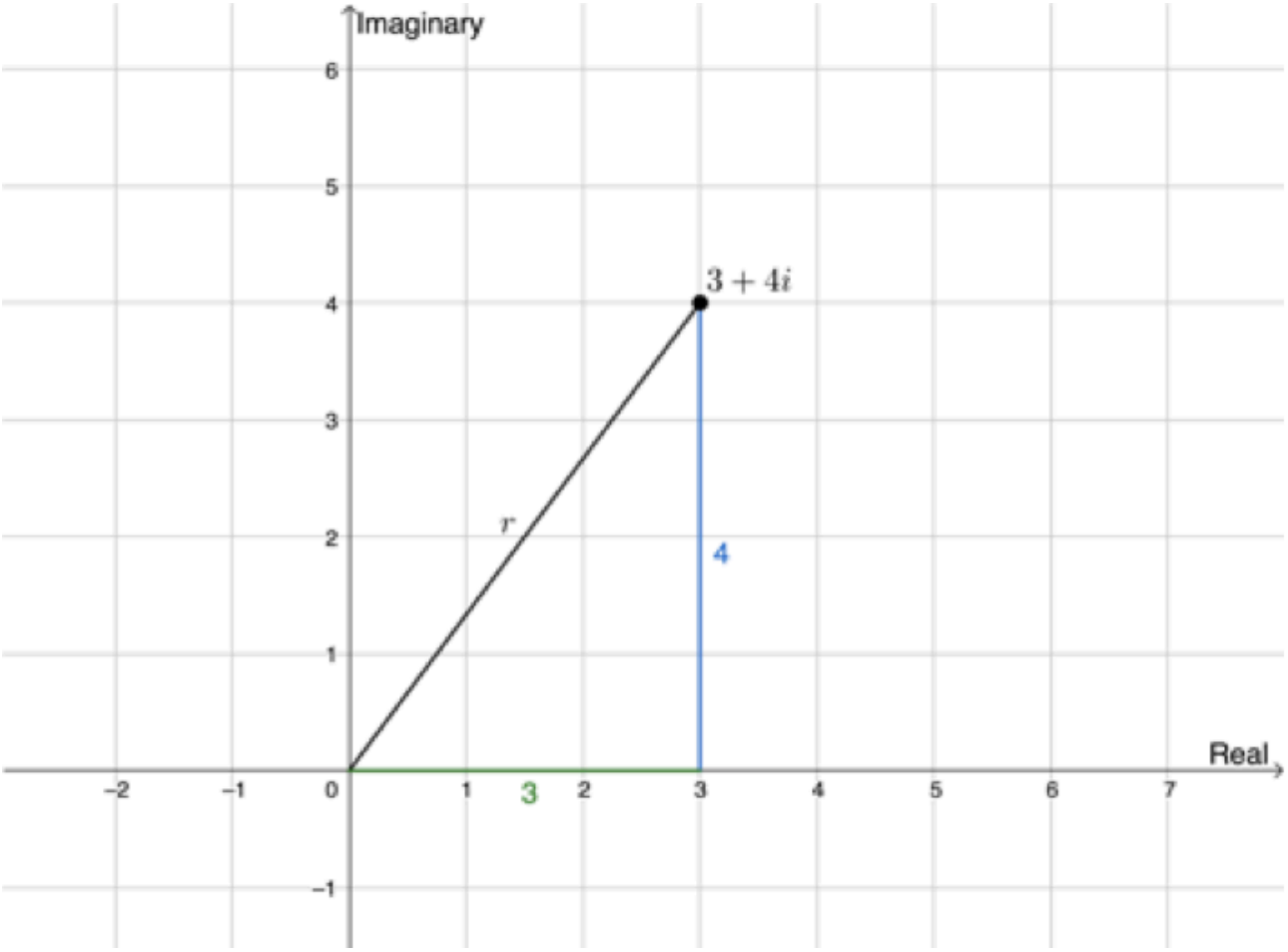


Figure 4: Calculating the length of r using Pythagoras' theorem

$$r^2 = 3^3 + 4^2$$
$$= 9 + 16 = 25$$
$$\therefore r = \sqrt{25} = 5$$

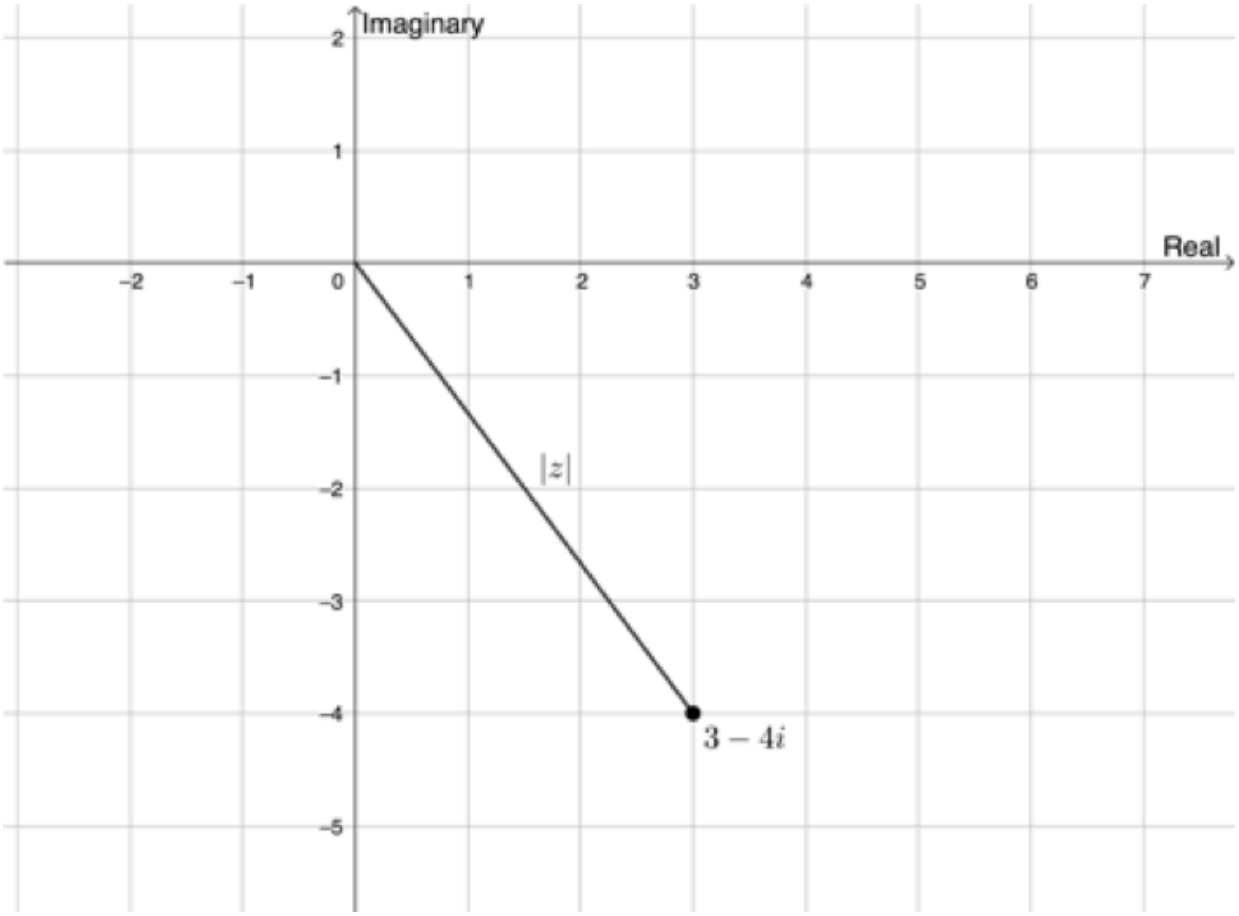
We often represent complex numbers with the letter z. In the example above $z = 3 + 4i$. We represent the absolute value of z as $|z|$. Therefore, if $z = 3 + 4i$, then $|z| = 5$. For this reason, the complex plane is also known as the **z-plane**.

Example 2.1

Given $z = 3 - 4i$, find $|z|$.

Solution

We can plot the complex number $z = 3 - 4i$ on the complex plane.



$|z| = \sqrt{x^2+y^2}$ x is the real part and y is the imaginary part

$= \sqrt{9+16}$

$= 5$

Exercise 2.2

Find $|z|$ in each case:

1. $z = 1 - 7i$
2. $z = 2 - 6i$
3. $z = -5 + 3i$
4. $z = 3 - \sqrt{-6}$

The full solutions are at the end of the unit.

Take note!

The absolute value of z, $|z|$, is also called the **modulus** of z.

The argument of complex numbers

You may have noticed in the previous section that the modulus alone is not enough to fully define a complex number on the complex plane. Have a look at Figure 5. Here we have two complex numbers plotted, $z_1 = 4 + 3i$ and $z_2 = 3 - 4i$. Work out the value of $|z|$ (or the modulus) of each number.

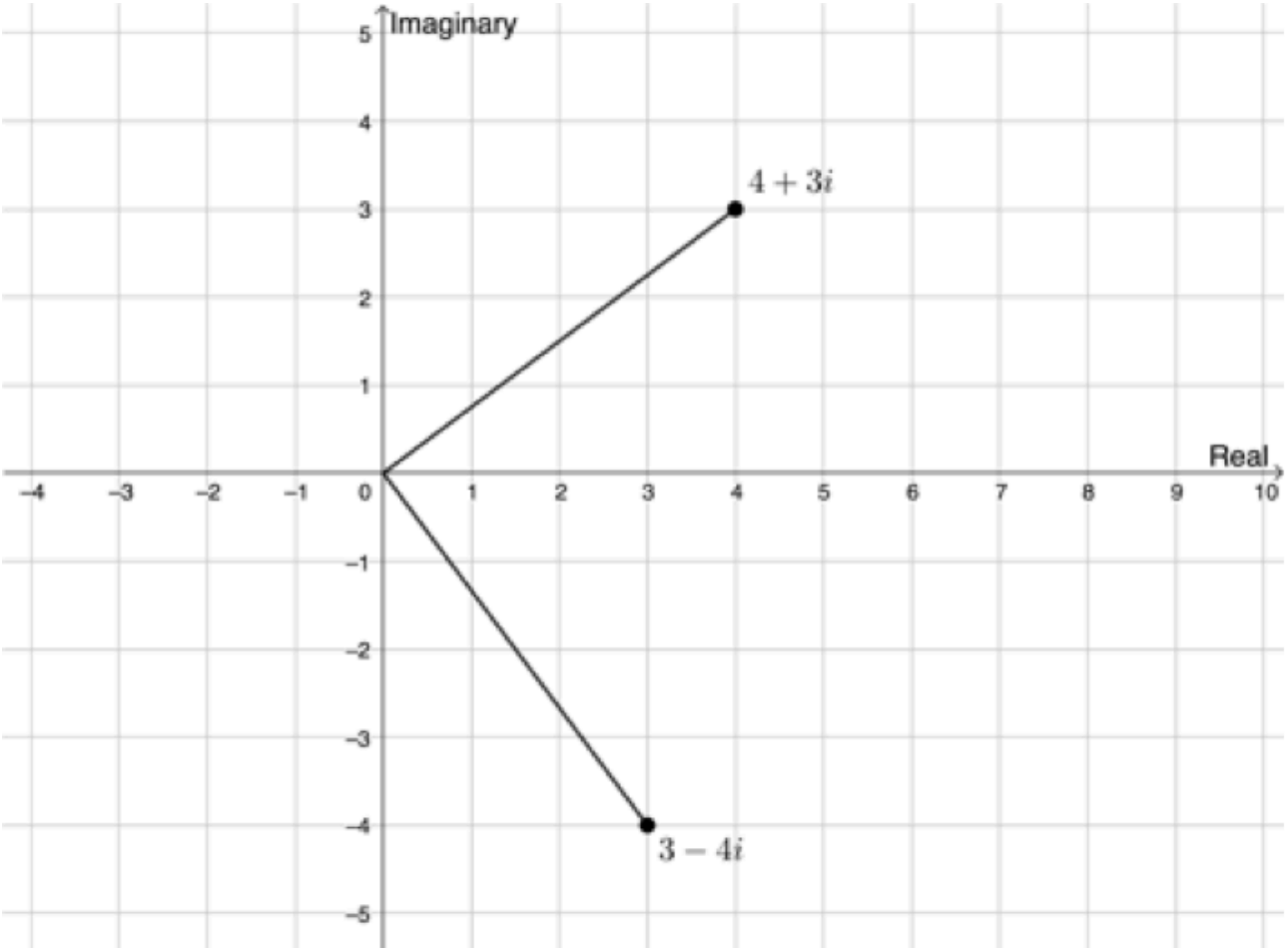


Figure 5: $z_1 = 4 + 3i$ and $z_2 = 3 - 4i$ on the complex plane

$$|z_1| = \sqrt{4^2 + 3^2} \quad |z_2| = \sqrt{3^2 + (-4)^2}$$
$$= \sqrt{25} \qquad = \sqrt{25}$$
$$= 5 \qquad = 5$$

The modulus is the same in both cases, even though the points are different. In order to differentiate these two points, we also need to know the angle that the line representing the modulus makes with the positive x-axis, as shown in Figure 6.

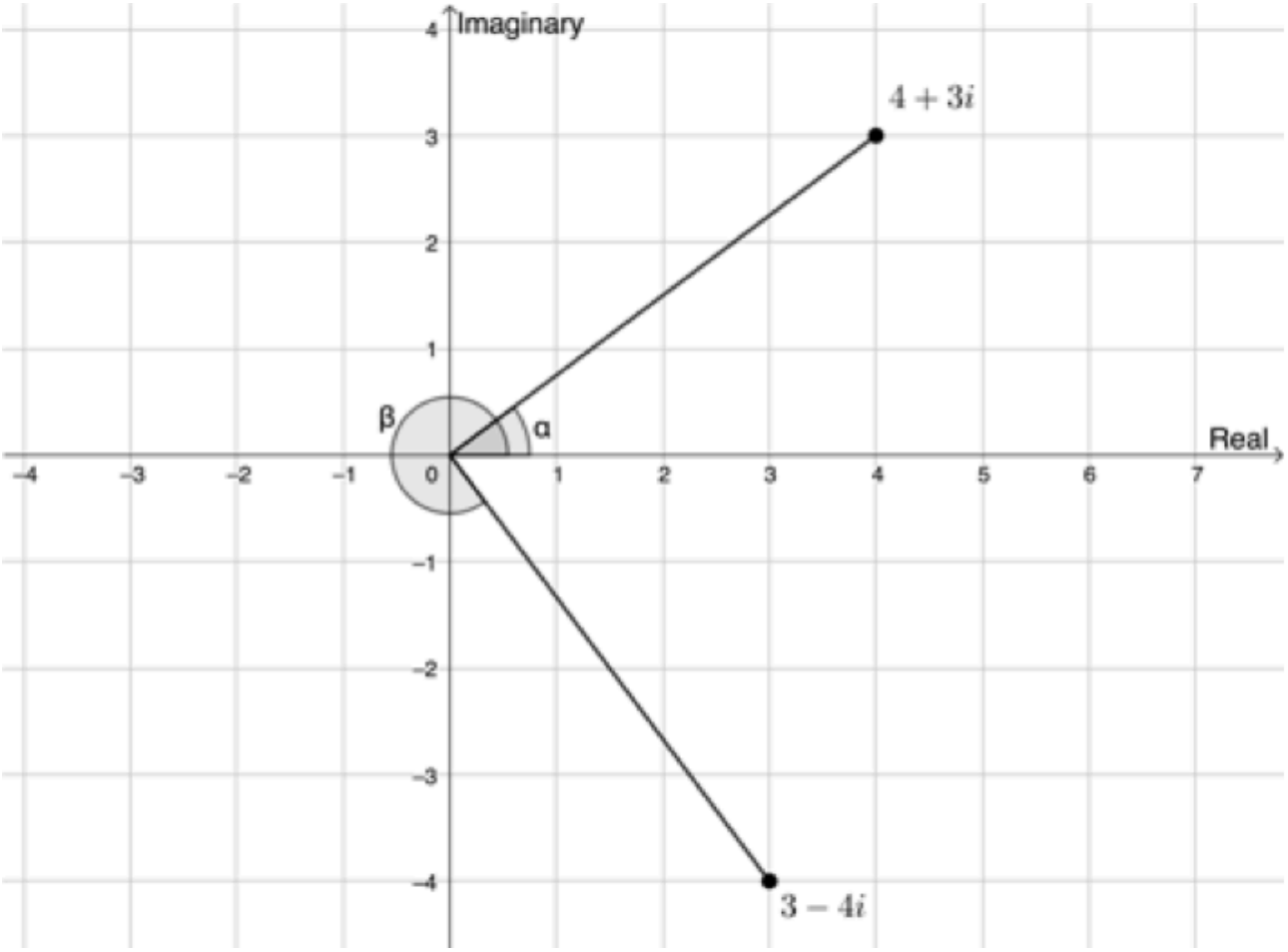


Figure 6: Angles made by the lines representing the moduli of $z_1 = 4 + 3i$ and $z_2 = 3 - 4i$ with the positive x-axis

But how can we find the size of these angles? Thankfully, we can use trigonometry to solve this problem.

In level 2 subject outcome 3.6 unit 1 we learnt that $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$ (Figure 7).

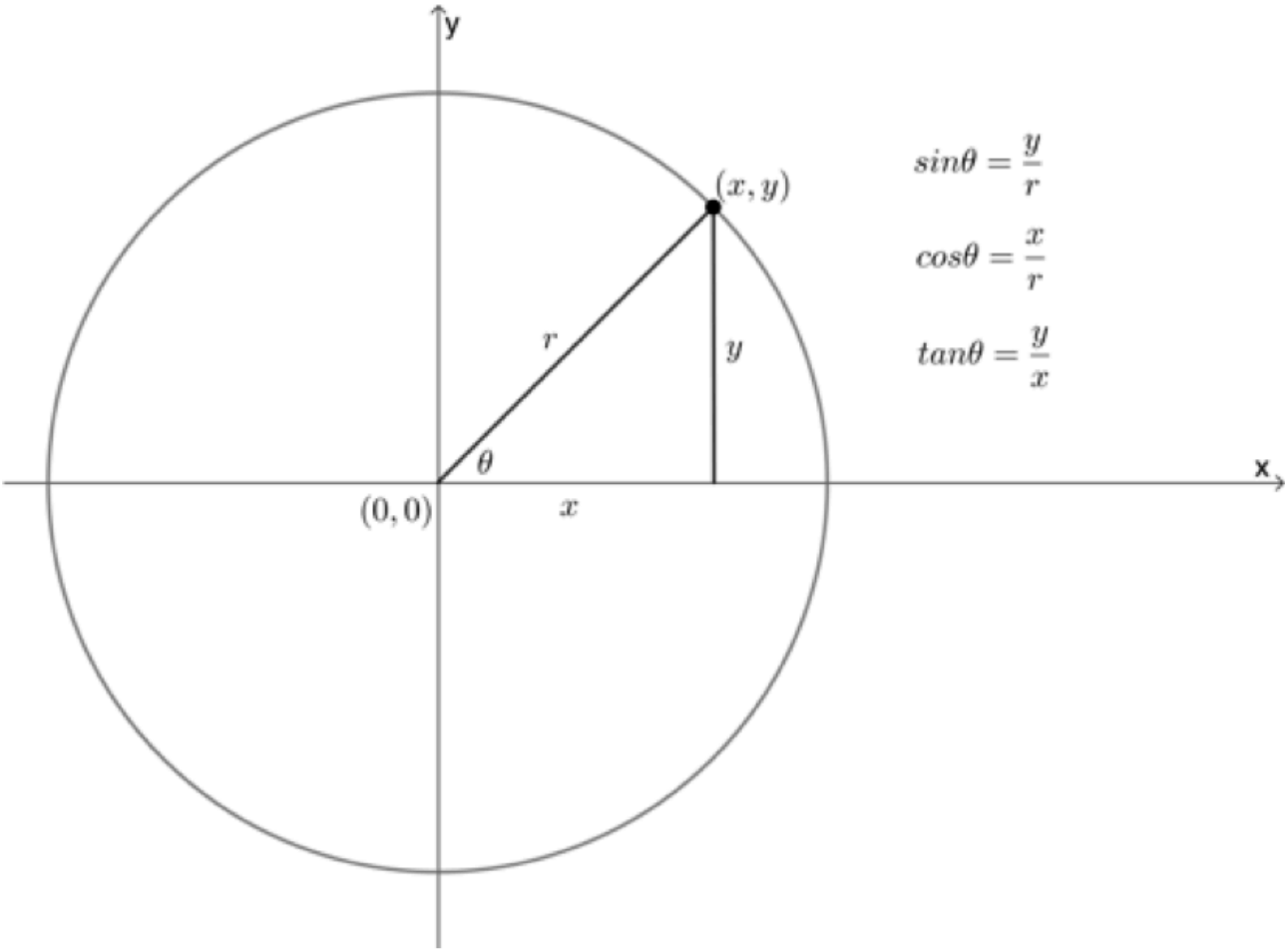


Figure 7: Definition of the trigonometric ratios on the Cartesian plane

From Figure 6 we can say that $\sin\alpha = \frac{y}{r} = \frac{3}{5}$ or that $\cos\alpha = \frac{x}{r} = \frac{4}{5}$. Therefore, we can calculate the size of α . $\alpha = 36.870°$. This answer makes sense as we can see that $z_1 = 4 + 3i$ is in the first quadrant.

Also, from Figure 6, we can say that $\sin\beta = \frac{y}{r} = \frac{-4}{5}$ or that $\cos\beta = \frac{x}{r} = \frac{3}{5}$. Therefore, we can calculate the size of β. $\beta = -53.130°$. This makes sense as we can see that $z_2 = 3 - 4i$ is in the fourth quadrant.

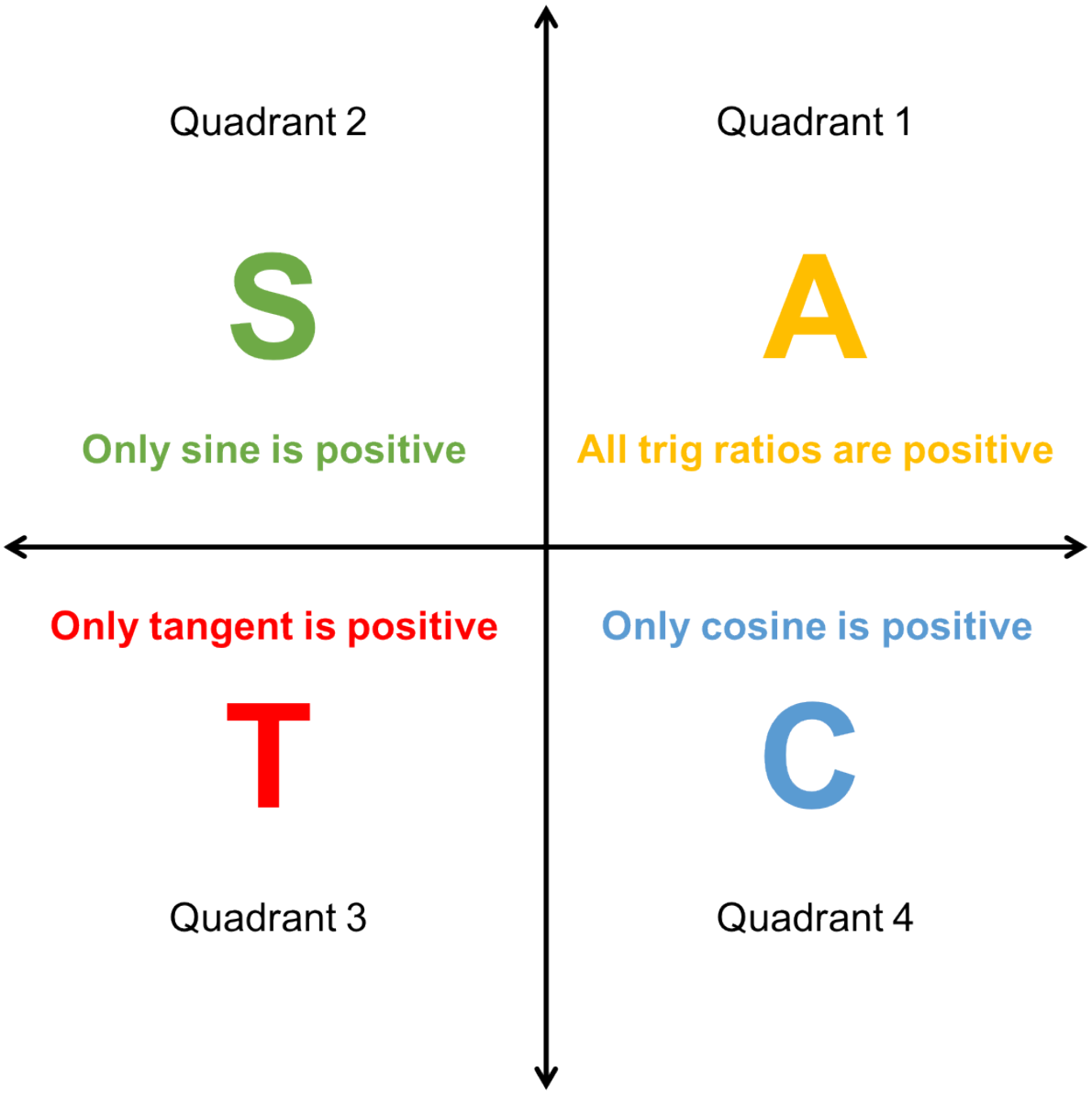
The angle between the line representing the modulus and the positive x-axis is called the **argument**. With the modulus and the argument, we can uniquely specify the position of any complex number on the complex plane.

Take note!

The value of the modulus ($|z|$) is the same as the value of r in $\sin\theta = \frac{y}{r}$ or $\cos\theta = \frac{x}{r}$.

Take note!

When calculating the argument, it is important that you draw a sketch of the position of z on the complex plane in order to determine which quadrant the point is in. The three trigonometric ratios still follow the CAST diagram on the complex plane.



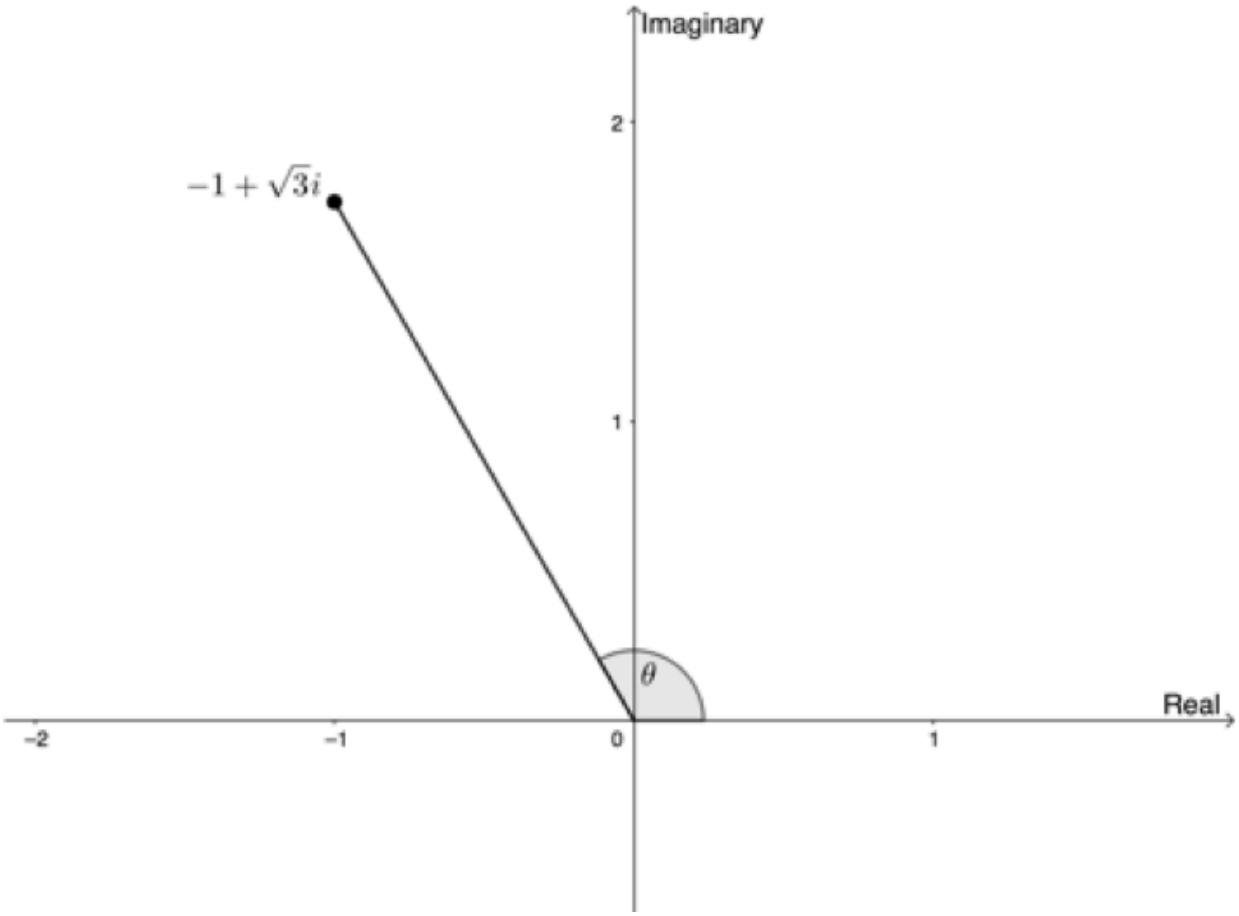
Example 2.2

Represent $z = -1 + \sqrt{-3}$ on an Argand diagram and then determine the modulus and the argument.

Solution

First write the complex number in standard form.
$z = -1 + \sqrt{-3} = -1 + \sqrt{3}i$

Now, plot the point on the complex plane to determine the quadrant in which it lies.



z lies in the second quadrant.

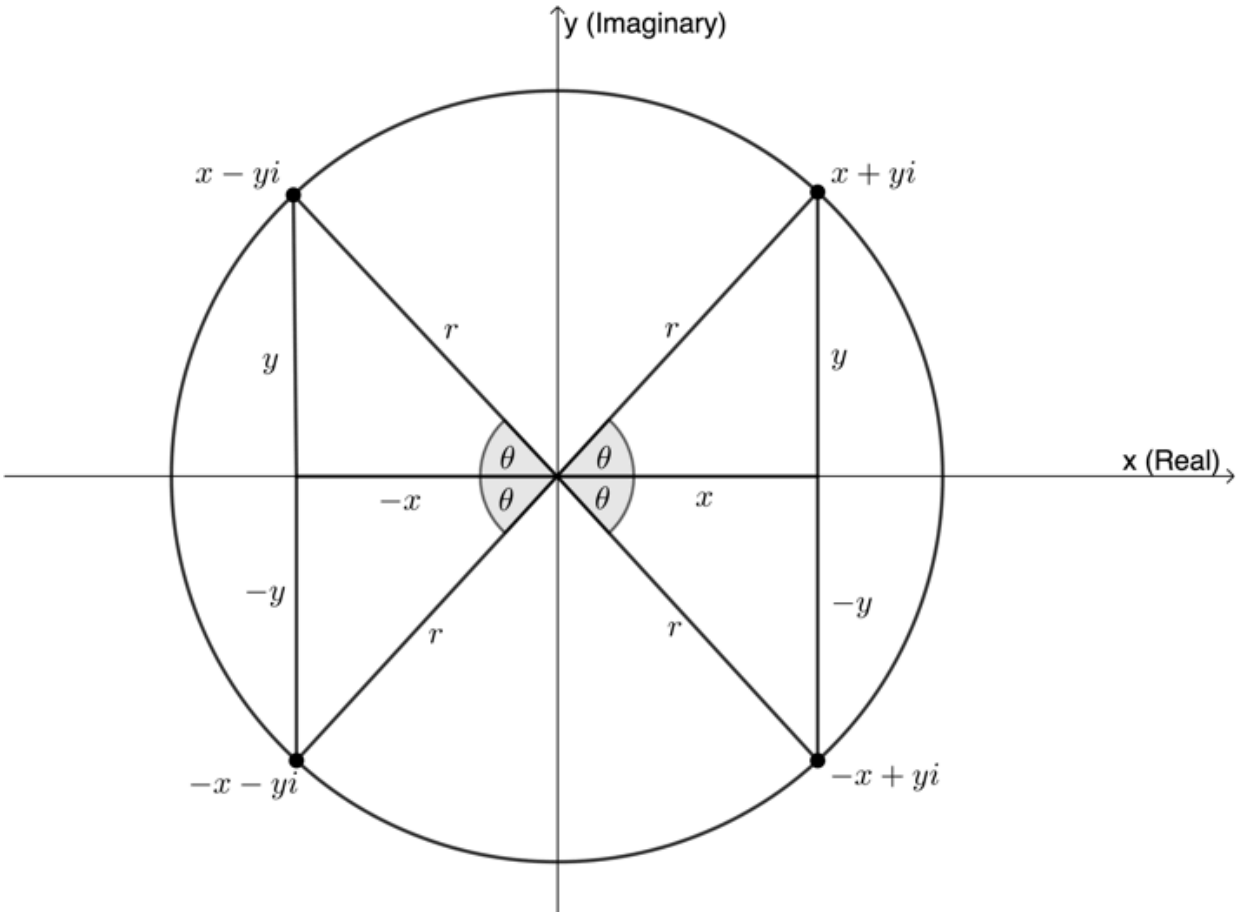
$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{(-1)^2+\left(\sqrt{3}\right)^2} \\
&= \sqrt{1+3} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta &= \frac{y}{r} = \frac{\sqrt{3}}{2} \\
\therefore \theta &= 60^\circ
\end{aligned}
$$

But z lies in the second quadrant. Therefore, $\theta = 180^\circ - 60^\circ = 120^\circ$.

Take note!

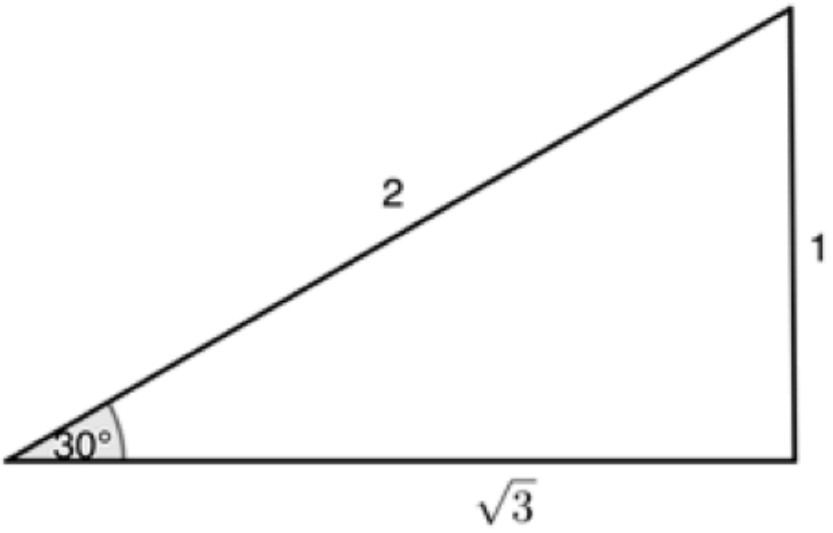
When finding the argument, remember that you need to pay attention to which quadrant the complex number is in. A useful strategy can be to make the ratio for $\sin\theta = \frac{y}{r}$ or $\cos\theta = \frac{x}{r}$ positive initially, in order to find the **reference acute angle** θ and then to transfer this angle into the necessary quadrant as follows:



- Second quadrant: $180° - \theta$
- Third quadrant: $180° + \theta$
- Fourth quadrant: $360° - \theta$

Take note!

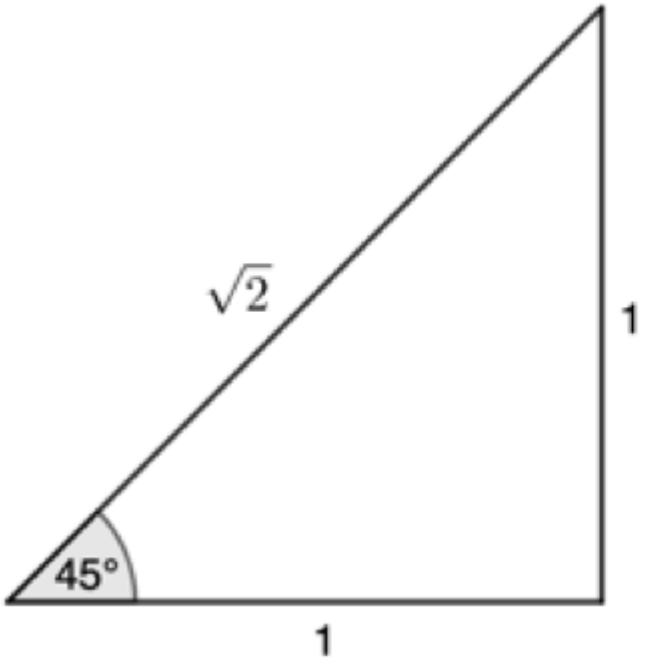
Remember the three special angles of $30°$, $45°$ and $60°$.



$$sin30° = \frac{1}{2}$$

$$cos30° = \frac{\sqrt{3}}{2}$$

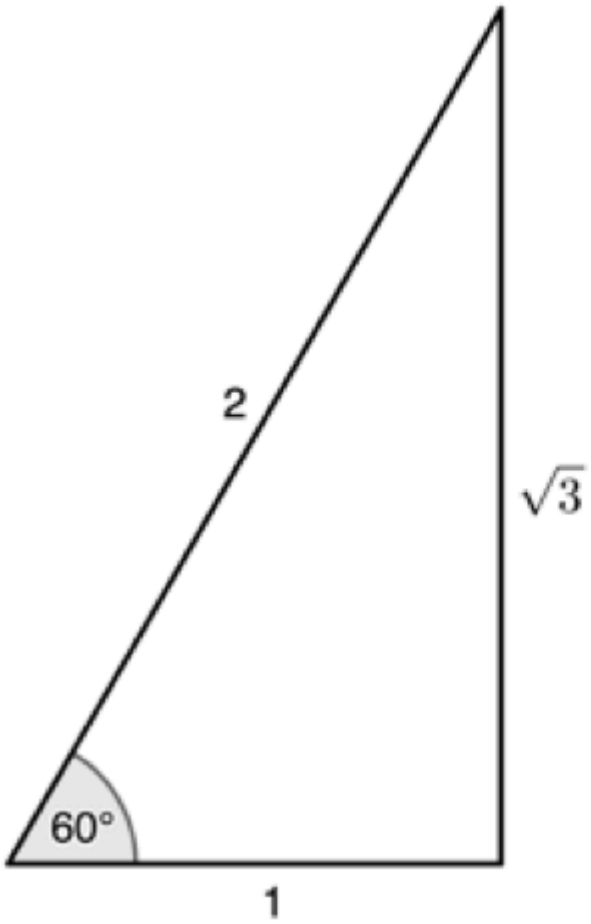
$$tan30° = \frac{\sqrt{3}}{1}$$



$$sin45° = \frac{1}{\sqrt{2}}$$

$$cos45° = \frac{1}{\sqrt{2}}$$

$$tan45° = \frac{1}{1}$$



$$sin60° = \frac{\sqrt{3}}{2}$$

$$cos60° = \frac{1}{2}$$

$$tan60° = \frac{\sqrt{3}}{1}$$

Exercise 2.3

Plot the following on an Argand diagram and then determine the modulus and the argument for each:

1. $z = -1 - \sqrt{-3}$
2. $z = 2 - 2i$
3. $z = -3 - 4i$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to represent complex numbers using an Argand diagram.
- How to find the absolute value or the modulus of a complex number.
- How to find the argument (the angle between the line representing the modulus and the positive x-axis) of a complex number.

Unit 2: Assessment

Suggested time to complete: 15 minutes

Represent the following on an Argand diagram and then determine the modulus and the argument:

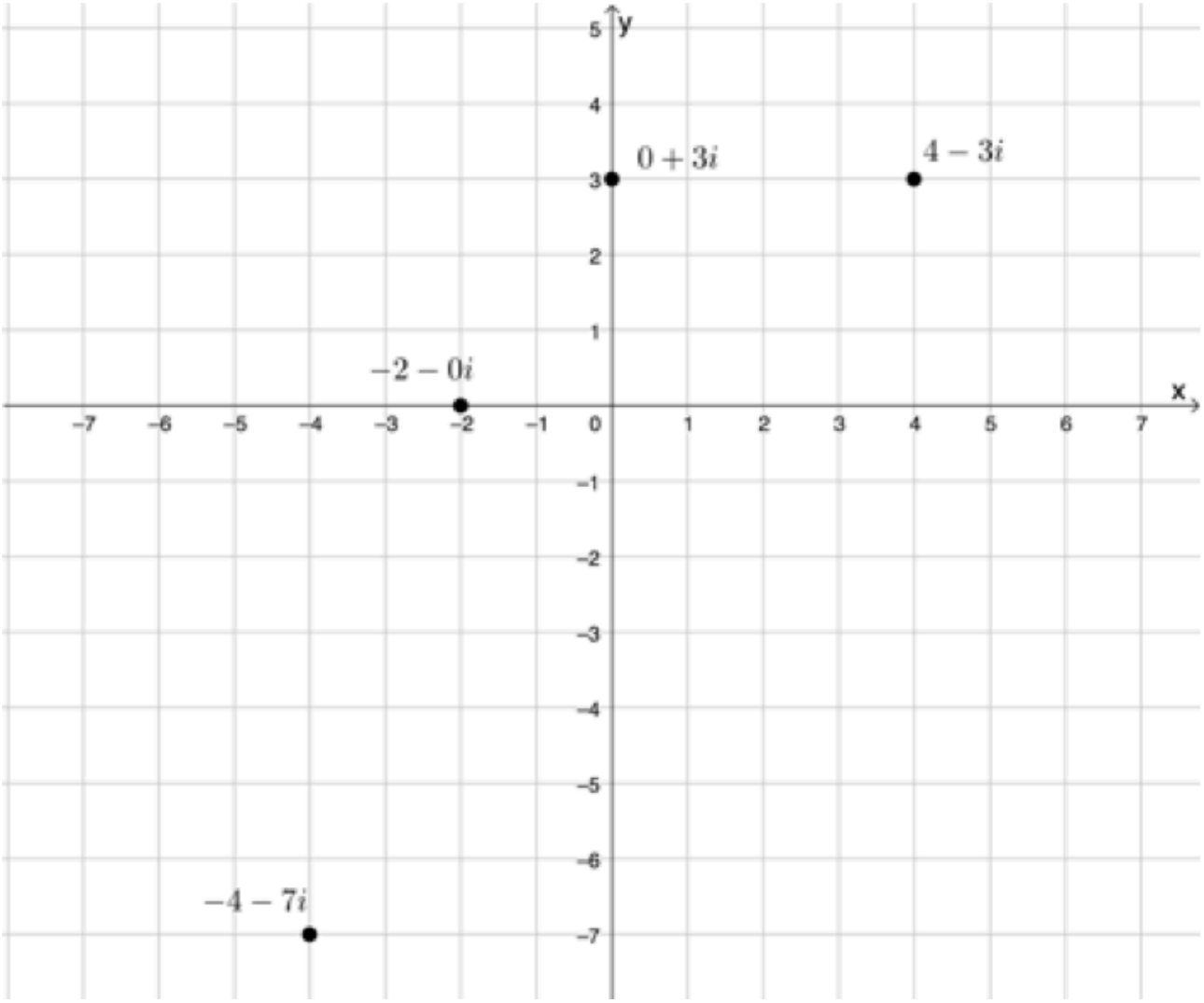
1. $z = \sqrt{2} - i$
2. $z = 5 + 12i$
3. $z = \sqrt{3} - i$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

The answers for questions 1 to 4 are shown below:



Back to Exercise 2.1

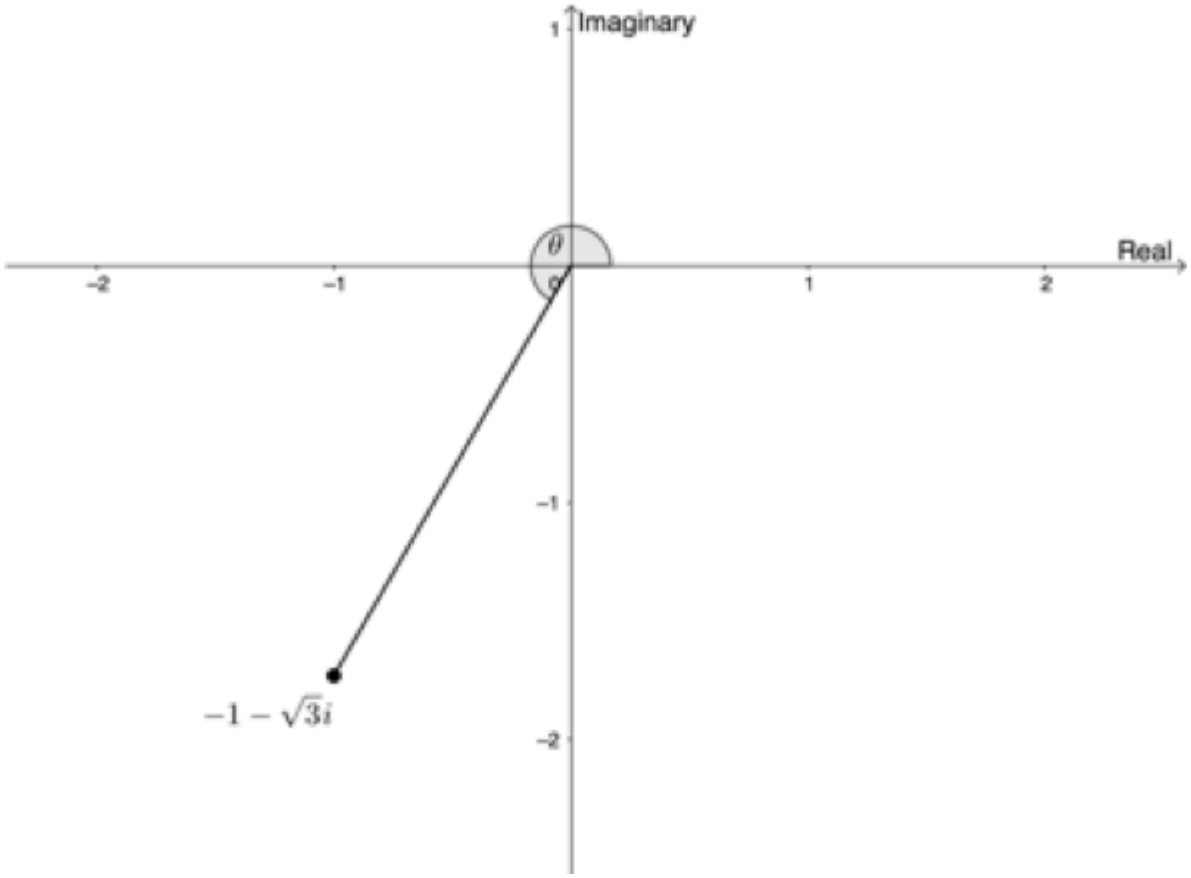
Exercise 2.2

1. $$\begin{aligned}|z| &= \sqrt{1^2 + (-7)^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} \\ &= \sqrt{(25)(2)} \\ &= 5\sqrt{2}\end{aligned}$$
2. $$\begin{aligned}|z| &= \sqrt{2^2 + (-6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= \sqrt{(4)(10)} \\ &= 2\sqrt{10}\end{aligned}$$
3. $$\begin{aligned}|z| &= \sqrt{(-5)^2 + 3^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34}\end{aligned}$$
4. $$\begin{aligned}3 - \sqrt{-6} &= 3 - \sqrt{6}i \\ \therefore |z| &= \sqrt{3^2 + (-6)^2} \\ &= \sqrt{9 + 6} \\ &= \sqrt{15}\end{aligned}$$

Back to Exercise 2.2

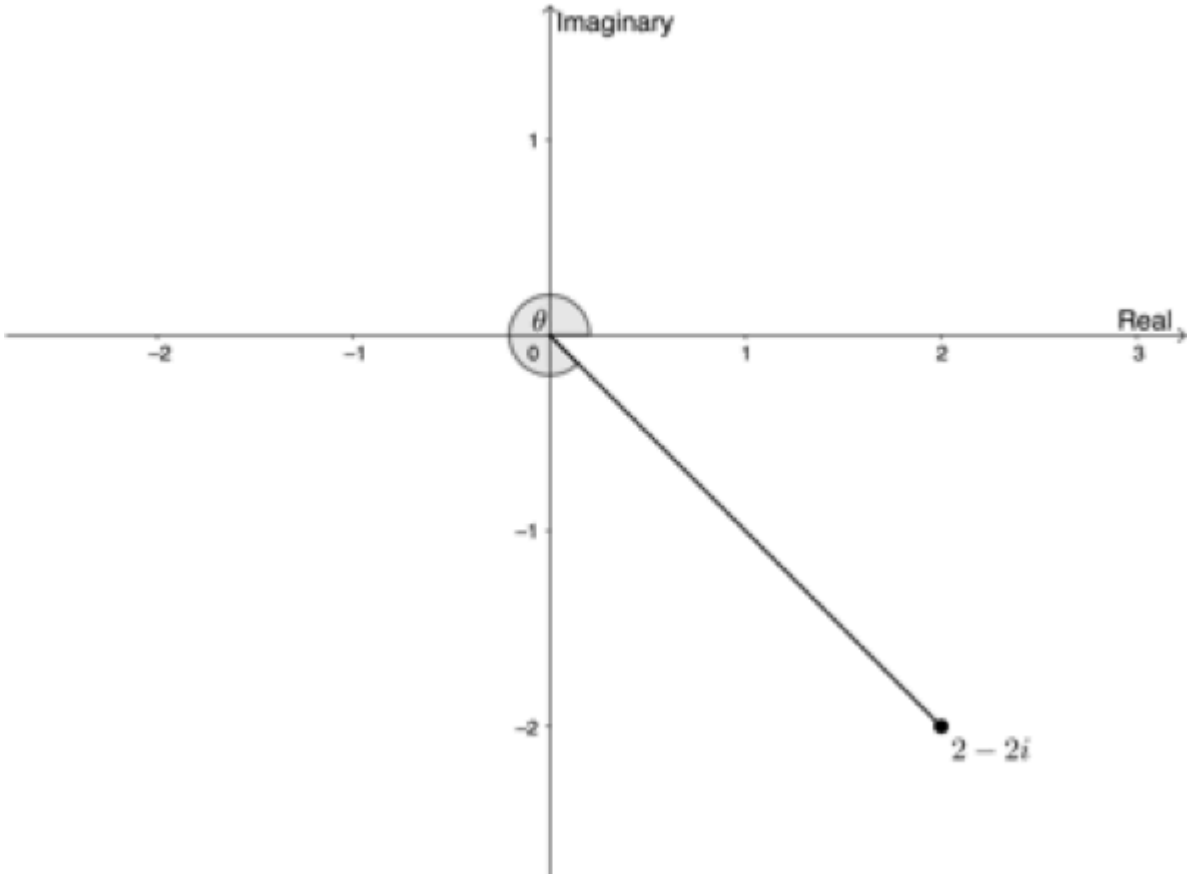
Exercise 2.3

1. $z = -1 - \sqrt{-3} = -1 - \sqrt{3}i$



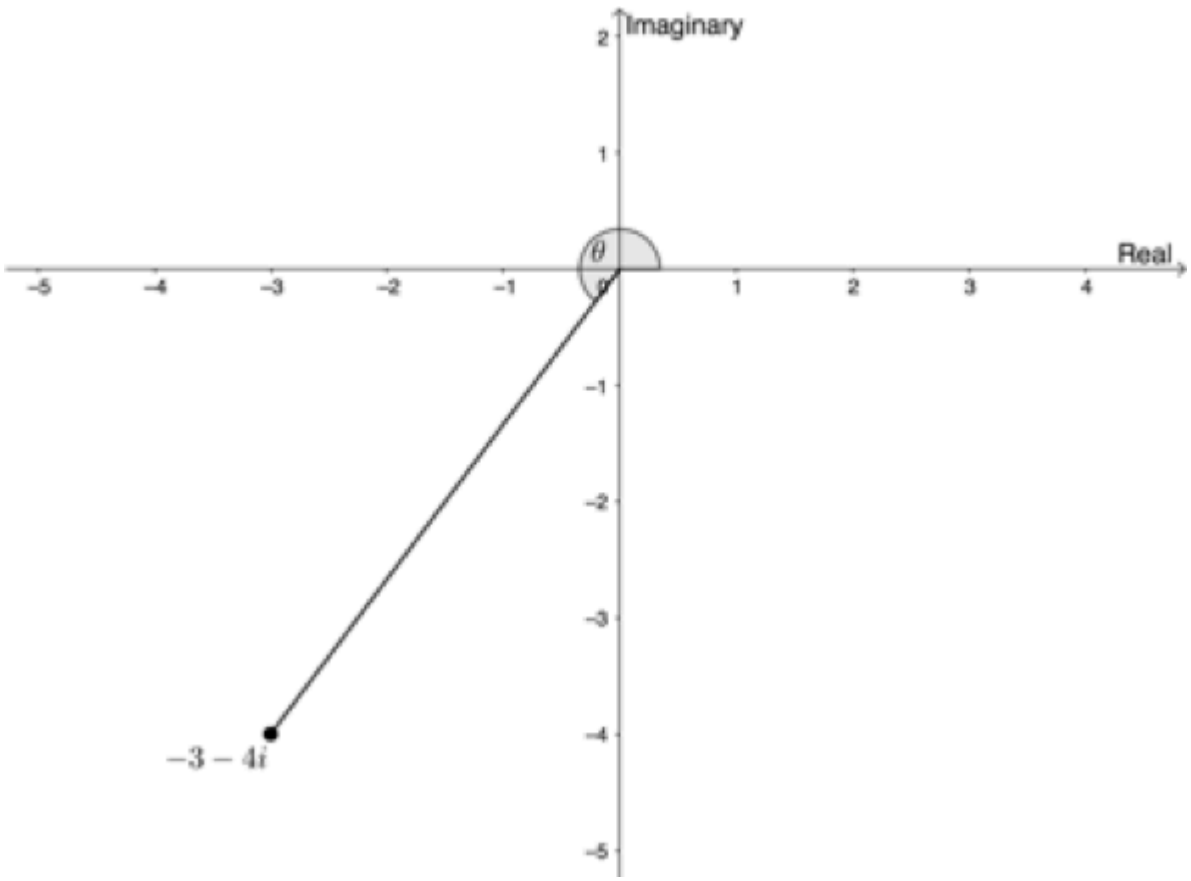
$$|z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-1)^2 + (-\sqrt{3})^2}$$
$$= \sqrt{1+3}$$
$$= \sqrt{4}$$
$$= 2$$
$$\sin\theta = \frac{-\sqrt{3}}{2}$$
$$\therefore \theta = 180^\circ + 60^\circ = 240^\circ$$

2. $z = 2 - 2i$



$$
\begin{aligned}
|z| &= \sqrt{x^2 + y^2} \\
&= \sqrt{2^2 + (-2)^2} \\
&= \sqrt{4 + 4} \\
&= \sqrt{8} \\
&= 2\sqrt{2} \\
\sin\theta &= \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} \\
\therefore \theta &= -45^\circ \quad \text{or} \quad \theta = 360^\circ - 45^\circ = 315^\circ
\end{aligned}
$$

3. $-3 - 4i$



$$
\begin{aligned}
|z| &= \sqrt{x^2 + y^2} \\
&= \sqrt{(-3)^2 + (-4)^2} \\
&= \sqrt{9 + 16} \\
&= \sqrt{25} \\
&= 5
\end{aligned}
$$

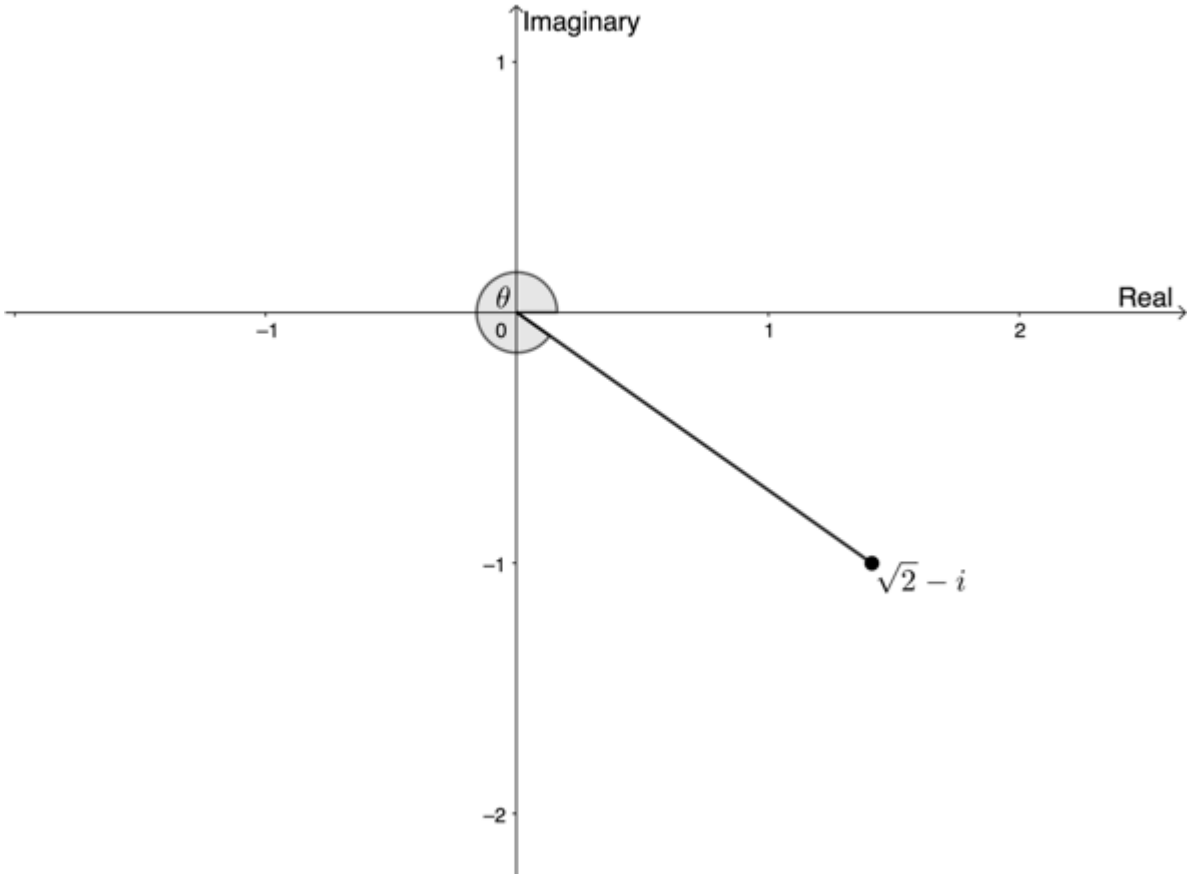
$$\sin\theta = \frac{-4}{5}$$

$$\therefore \theta = 180^\circ + 53.130^\circ = 233.13^\circ$$

Back to Exercise 2.3

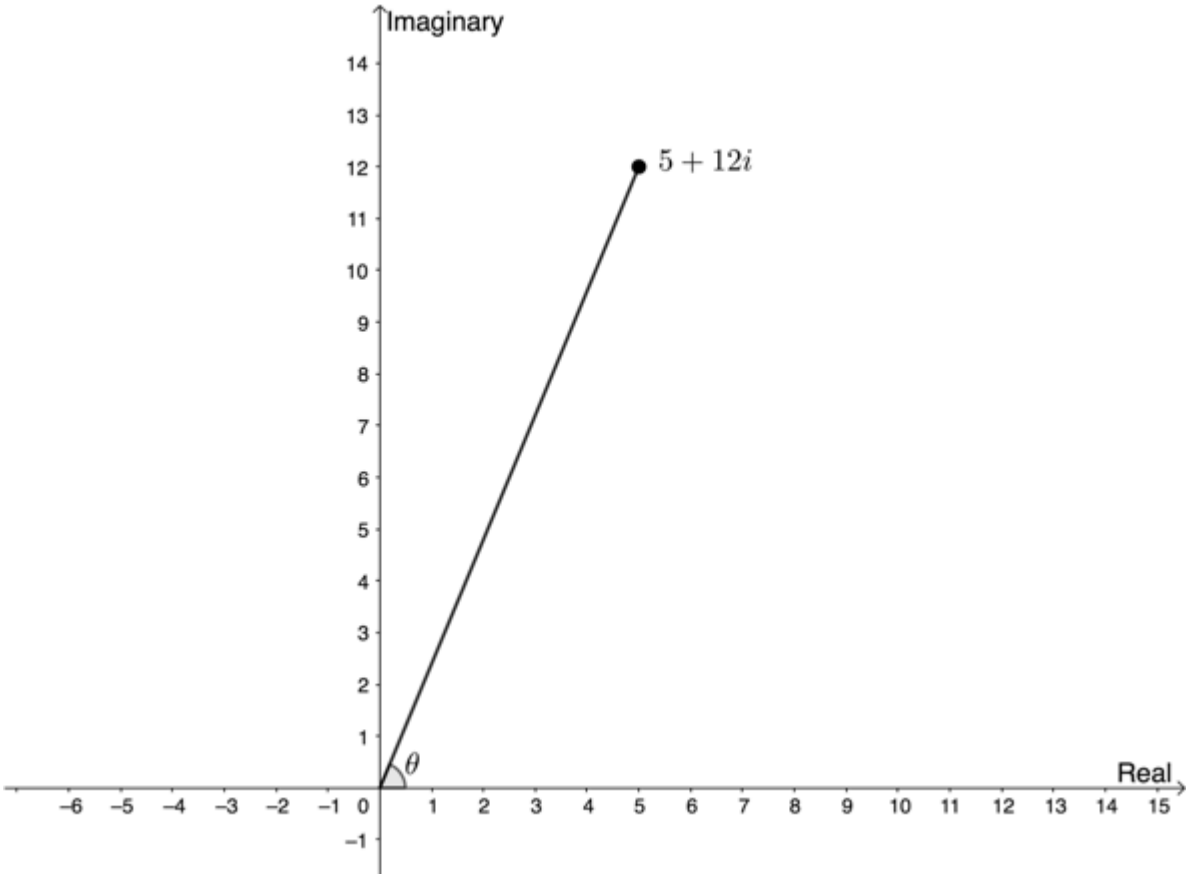
Unit 2: Assessment

1. $z = \sqrt{2} - i$



$$|z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{\left(\sqrt{2}\right)^2 + (-1)^2}$$
$$= \sqrt{2+1}$$
$$= \sqrt{3}$$
$$\sin\theta = \frac{-1}{\sqrt{3}}$$
$$\therefore \theta = -35.26^\circ \quad \text{or} \quad \theta = 360^\circ - 35.26^\circ = 324.74^\circ$$

2. $z = 5 + 12i$

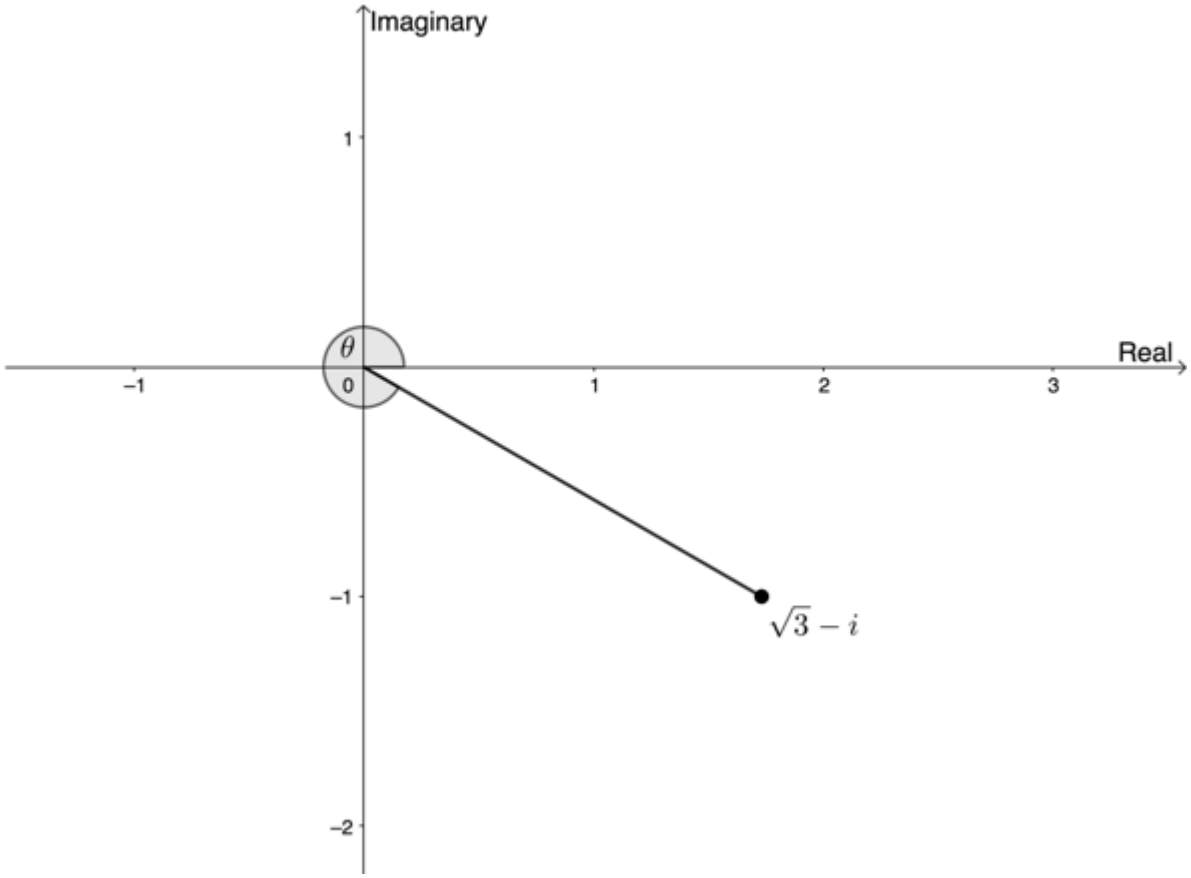


$$
\begin{aligned}
|z| &= \sqrt{x^2 + y^2} \\
&= \sqrt{(5)^2 + (12)^2} \\
&= \sqrt{25 + 144} \\
&= \sqrt{169} \\
&= 13
\end{aligned}
$$

$$\sin\theta = \frac{12}{13}$$

$$\therefore \theta = 67.38^\circ$$

3. $z = \sqrt{3} - i$



$$|z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{(\sqrt{3})^2 + (-1)^2}$$
$$= \sqrt{3+1}$$
$$= \sqrt{4}$$
$$= 2$$
$$\sin\theta = \frac{-1}{2}$$
$$\therefore \theta = -30^\circ \quad or \quad \theta = 360^\circ - 30^\circ = 330^\circ$$

Back to Unit 2: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example2.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure6 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure7 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure8 © DHET is licensed under a CC BY (Attribution) license
- example2.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- transformation © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- specialangle30 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- specialangle45 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- specialangle60 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise2.1A © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise2.3.1aA © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

- exercise2.3.1bA © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise2.3.1cA © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessment1A © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessment2A © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessment3A © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 3: Represent complex numbers in polar form

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Express complex numbers in polar form.
- Convert complex numbers from standard/rectangular form to polar form.
- Convert complex numbers from polar form to standard/rectangular form.

What you should know

Before you start this unit, make sure you can:

- Represent complex numbers using an Argand diagram. Refer to unit 2 in this subject outcome if you need help with this.
- Find the modulus of a complex number. Refer to unit 2 in this subject outcome if you need help with this.
- Find the argument of a complex number. Refer to unit 2 in this subject outcome if you need help with this.

Introduction

We saw in the previous unit that we can represent complex numbers on the complex plane. These are referred to as Argand diagrams. Figure 1 shows the complex number $z = 2 - 3i$ represented in this way, where the complex number is represented as the point $(x, y) = (2, -3)$.

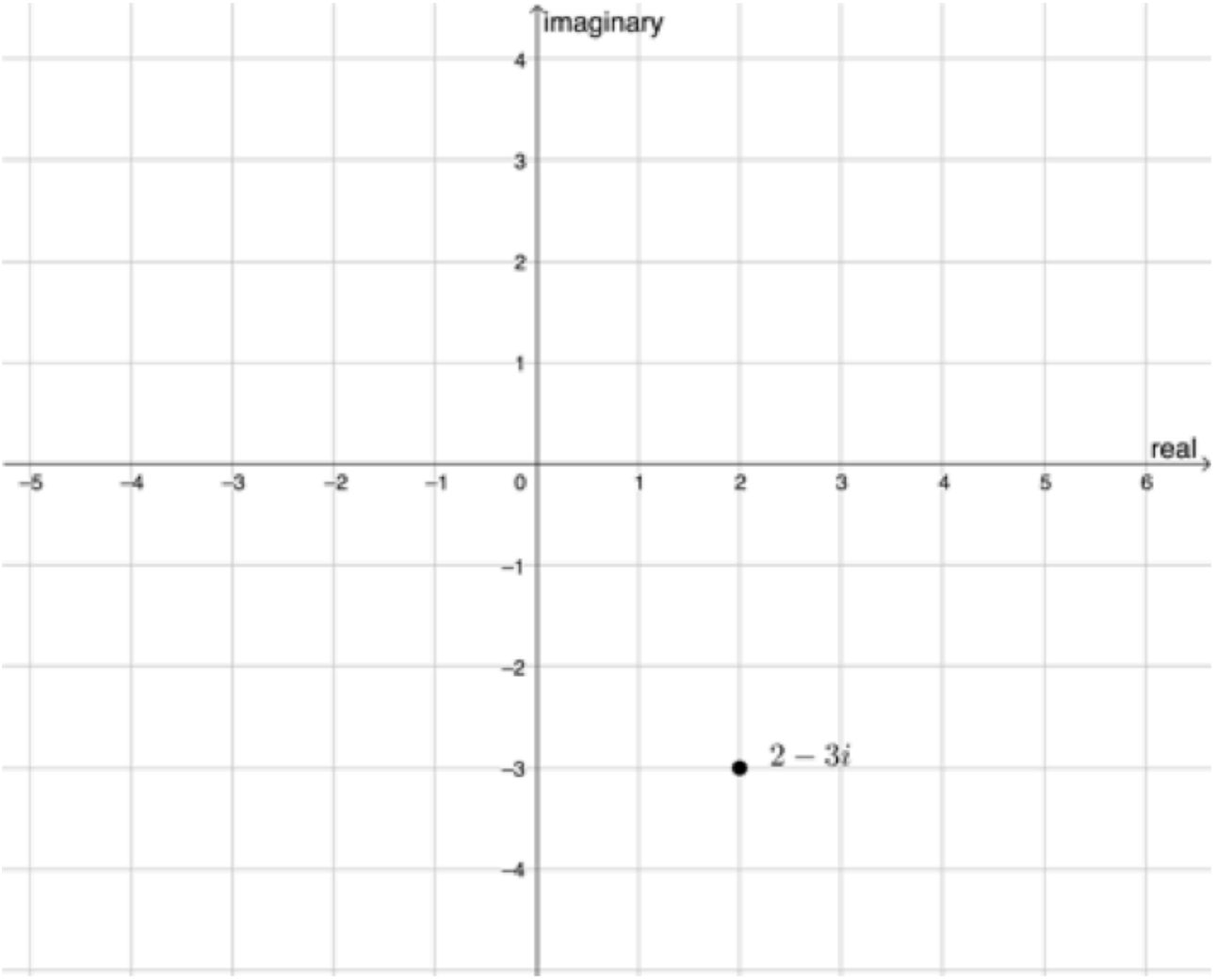


Figure 1: $z = 2 - 3i$ plotted on the complex plane

We also saw that we can calculate the length of the line joining the point representing a complex number with the origin. This is called the absolute value or **modulus** of the complex number and is represented as $|z|$. The modulus of $z = 2 - 3i$ is calculated as follows:

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13} \end{aligned}$$

We noted that it is not enough to know just the modulus of a complex number. We also need to know the angle that the line representing the modulus makes with the positive x-axis (or positive real axis). This angle is called the **argument**.

We can calculate the argument using either of the following trigonometric ratios:
$\sin\theta = \dfrac{y}{r}$ or $\cos\theta = \dfrac{x}{r}$ where r (the radius of the circle with centre the origin, passing through the point (x, y) representing the complex number) has the same value as $|z|$.

In the case above the argument is calculated as:

$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{13}}$$

$$\therefore \theta = 303.69^\circ \quad \text{The point is in the fourth quadrant}$$

With all of this knowledge, we are ready to work with complex numbers in polar form.

Complex numbers in polar form

We say that $z = 2 - 3i$ is written in standard or rectangular form where the number is expressed in terms of a real and an imaginary component. The **polar form** of a complex number expresses the number in terms of its modulus and argument.

Suppose we have a complex number $z = x + yi$. We can represent this with an Argand diagram as shown in Figure 2.

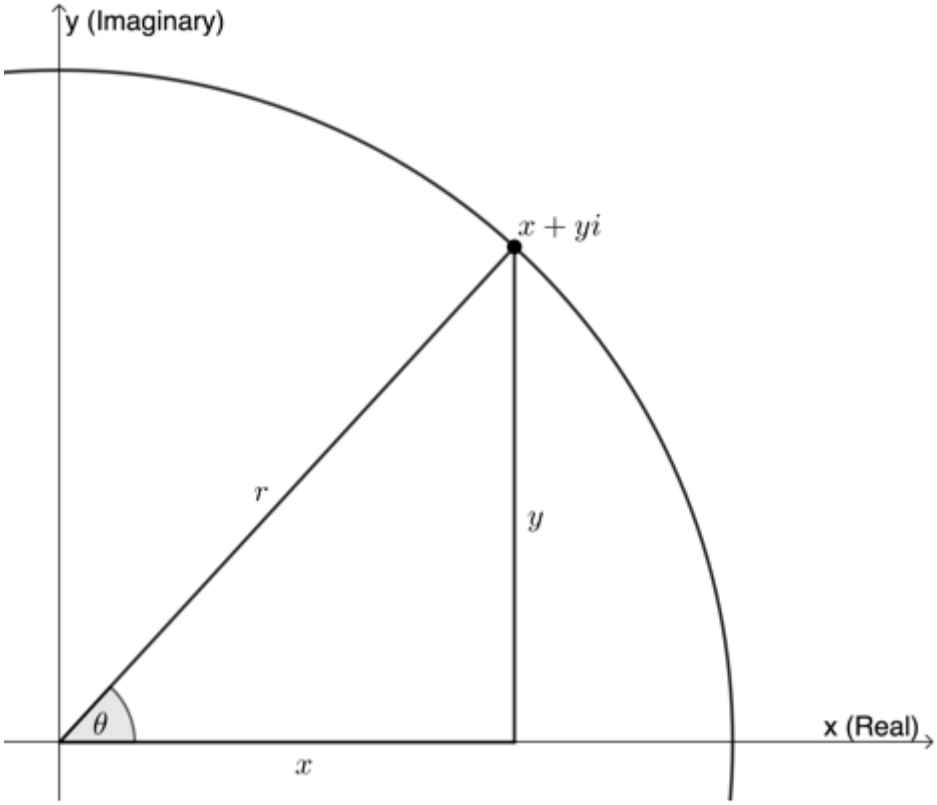


Figure 2: Argand diagram for $z = x + yi$

Now we know that $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. Therefore, $x = r\cos\theta$ and $y = r\sin\theta$.

So, the point (x, y) has coordinates given by $x = r\cos\theta$ and $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$. Therefore:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

Take note!

We often use the abbreviation $r\text{cis}\theta$ to represent $r(\cos\theta + i\sin\theta)$.

$$r(\cos\theta + i\sin\theta)$$

$$r\text{cis}\theta$$

The polar form of a complex number:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)i \\ &= r(\cos\theta + i\sin\theta) \\ &= r\text{cis}\theta \end{aligned}$$

where $r = |z|$ and θ is the argument.



Example 3.1

Find the polar form of $z = 4 + 4i$.

Solution

Step 1: Find the value of r (or $|z|$ or the modulus)

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4^2 + 4^2} \\ &= \sqrt{16 + 16} \\ &= \sqrt{32} \quad 32 = 16 \times 2 \\ &= 4\sqrt{2} \end{aligned}$$

Step 2: Find the value of θ (the argument)

$$\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}} \\ \therefore \theta &= 45^\circ \end{aligned}$$

In this case, we used the special angle of 45°. We know the point is in the first quadrant so we do not need to transform the angle to another quadrant.

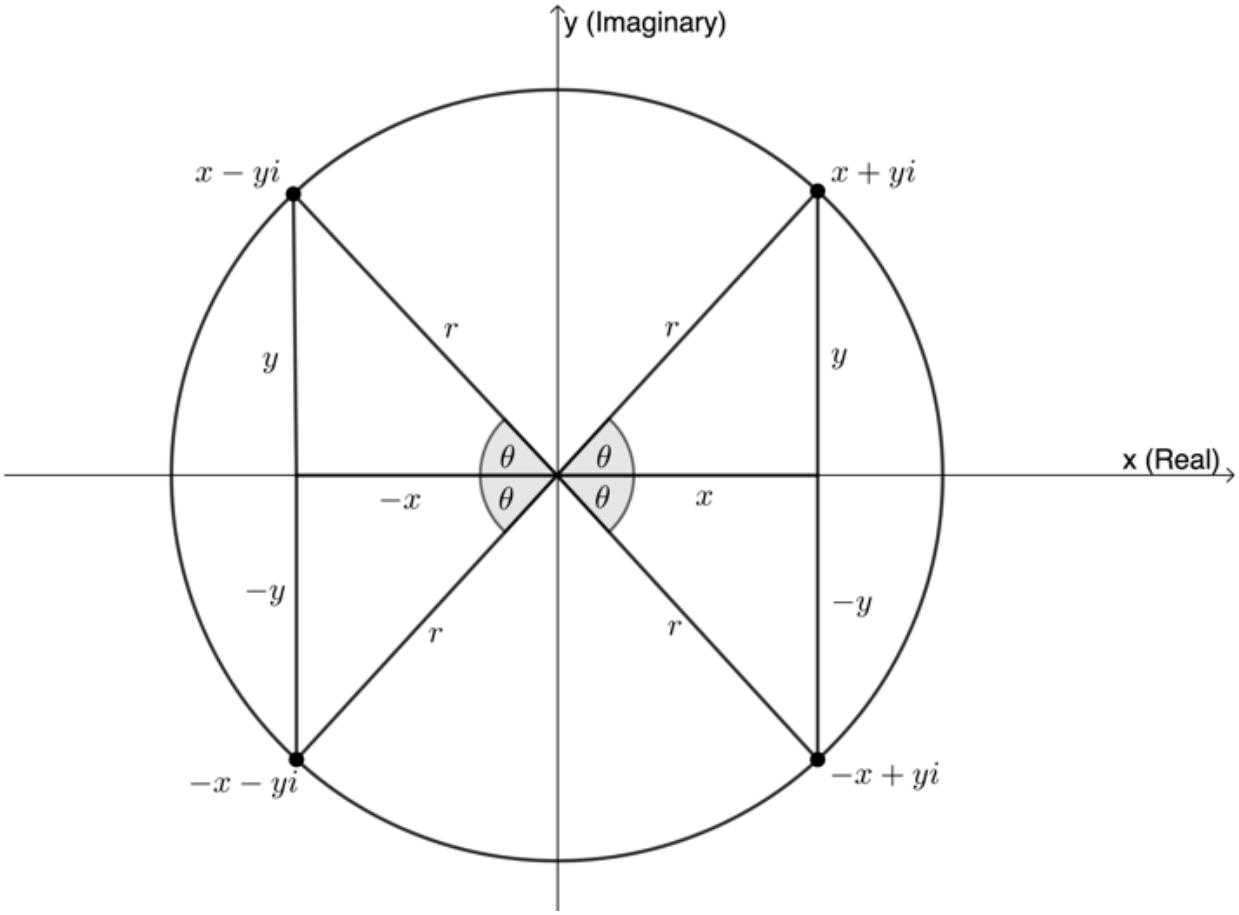
Note that we could also have used $\sin\theta = \frac{y}{r}$ to find the argument.

Step 3: Write the solution

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= 4\sqrt{2}(\cos 45^\circ + i\sin 45^\circ) \\ &= 4\sqrt{2}cis45^\circ \end{aligned}$$

Take note!

When finding the argument, remember that you need to pay attention to which quadrant the complex number is in. A useful strategy can be to make the ratio for $\cos\theta = \frac{x}{r}$ (or $\sin\theta = \frac{y}{r}$) positive, in order to find the **reference acute angle** α, and then to transfer this angle into the necessary quadrant as follows:



- Second quadrant: $180^\circ - \alpha$
- Third quadrant: $180^\circ + \alpha$
- Fourth quadrant: $360^\circ - \alpha$

Example 3.2

Find the polar form of $z = -4 - 4i$.

Solution

Step 1: Find the value of r (or $|z|$ or the modulus)

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-4)^2 + (-4)^2} \\ &= \sqrt{16 + 16} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

Step 2: Find the value of θ (the argument)

z is in the third quadrant. Find the reference angle α.

$$\cos\alpha = \frac{x}{r} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

While we can use the special angle of $45°$, we need to recognise that the point lies in the third quadrant where the ratios for cosine and sine are negative. Both the x (or real) and y (or imaginary) components are negative.

So while the reference angle is $45°$, we need to move this into the third quadrant by adding $180°$ to it. Therefore $\theta = 180° + 45° = 225°$.

Step 3: Write the solution

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= 4\sqrt{2}(\cos 225° + i\sin 225°) \\ &= 4\sqrt{2}\text{cis}225° \end{aligned}$$

Remember that this is just a shorthand expression for the full polar form of $z = 4\sqrt{2}(cos225° + i\sin 225°)$.

Exercise 3.1

Write the following complex numbers in polar form:

1. $z = 5 - 12i$
2. $z = -7 + 2i$

The full solutions are at the end of the unit.

Convert polar form to rectangular form

Sometimes it is necessary to convert a complex number in polar form into standard or rectangular form. To do this, we need to first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$, then multiply through by r.



Example 3.3

Convert $z = 12\text{cis}30^\circ$ into standard/rectangular form.

Solution

Step 1: Evaluate $\cos\theta$ and $\sin\theta$

We can use the fact that 30° is a special angle to evaluate without a calculator.

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$\sin 30^\circ = \frac{1}{2}$$

Step 2: Find the standard form

We know that the complex number is in the first quadrant where cosine and sine are both positive.

$$\begin{aligned} z &= 12(\cos 30^\circ + i\sin 30^\circ) \\ &= 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= \frac{12\sqrt{3}}{2} + 6i \\ &= 6\sqrt{3} + 6i \end{aligned}$$

The complex number in standard form is $z = 6\sqrt{3} + 6i$.



Example 3.4

Convert $z = 4\text{cis}120^\circ$ into standard/rectangular form.

Solution

Step 1: Evaluate $\cos\theta$ and $\sin\theta$

120° is not a special angle but $180^\circ - 120^\circ = 60^\circ$ is a special angle. Therefore, we can evaluate $\cos\theta$ and $\sin\theta$ without a calculator and transfer the angle into the correct quadrant, in this case, the second quadrant.

$$\cos 60^\circ = \frac{1}{2}$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

Step 2: Find the standard form

We know that the complex number is in the second quadrant where cosine is negative and sine is positive.

$$
\begin{aligned}
z &= 4(\cos 120^\circ + i \sin 120^\circ) \\
&= 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) && \text{Remember that cosine is negative in the second quadrant} \\
&= -2 + \frac{4\sqrt{3}}{2}i \\
&= -2 + 2\sqrt{3}i
\end{aligned}
$$

The complex number in standard form is $z = -2 + 2\sqrt{3}i$.

Example 3.5

Convert $z = 3\text{cis}43^\circ$ into standard/rectangular form.

Solution

Step 1: Evaluate $\cos\theta$ and $\sin\theta$

43° is not a special angle. Therefore, we have to evaluate $\cos\theta$ and $\sin\theta$ with a calculator. The complex number is in the first quadrant.

$$
\begin{aligned}
\cos 43^\circ &= 0.7314 \\
\sin 43^\circ &= 0.6820
\end{aligned}
$$

Step 2: Find the standard form

We know that the complex number is in the first quadrant where cosine and sine are both positive.

$$
\begin{aligned}
z &= 3(\cos 43^\circ + i \sin 43^\circ) \\
&= 3\left(0.7314 + 0.6820i\right) \\
&= 2.9142 + 2.046i
\end{aligned}
$$

The complex number in standard form is $z = 2.9142 + 2.046i$.

Exercise 3.2

Convert the following complex numbers into standard/rectangular form:

1. $z = 6\text{cis}45^\circ$
2. $z = \sqrt{5}\text{cis}210^\circ$
3. $z = 2\text{cis}40^\circ$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to represent complex numbers in polar form.
- How to convert between standard/rectangular form and polar form.

Unit 3: Assessment

Suggested time to complete: 15 minutes

1. Represent the following in polar form:
   a. $z = \sqrt{3} + 1i$
   b. $z = 8 - 4i$
2. Represent the following in rectangular form:
   a. $z = 3\text{cis}240^\circ$
   b. $z = 7\text{cis}25^\circ$

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1. $z = 5 - 12i$
   Find r:
   $$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{5^2 + 12^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$
   Find θ (θ is in the fourth quadrant):
   $$\cos\theta = \frac{x}{r} = \frac{5}{13}$$
   $$\therefore \text{reference angle} = 67.38^\circ$$
   $$\therefore \theta = 360^\circ - 67.38^\circ = 292.62^\circ$$
   Polar form: $z = 13\text{cis}292.62^\circ$
2. $z = -7 + 2i$
   Find r:
   $$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-7)^2 + 2^2} \\ &= \sqrt{49 + 4} \\ &= \sqrt{53} \end{aligned}$$

Find θ (θ is in the second quadrant):

$\cos\theta = \frac{x}{r} = \frac{-7}{\sqrt{53}}$

Work with $\cos\theta = \frac{7}{\sqrt{53}}$ to find a referance angle.

$\therefore \text{reference angle} = 15.95^\circ$

$\therefore \theta = 180^\circ - 15.95^\circ = 164.05^\circ$

Polar form: $z = \sqrt{53}\text{cis}164.05^\circ$

Back to Exercise 3.1

Exercise 3.2

1. $z = 6\text{cis}45^\circ$

   45° is a special angle and the complex number is in the first quadrant.

   $\cos 45^\circ = \frac{1}{\sqrt{2}}$

   $\sin 45^\circ = \frac{1}{\sqrt{2}}$

   $z = 6(\cos 45^\circ + i\sin 45^\circ)$

   $= 6\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

   $= \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}i$

   The complex number in standard form is $z = \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}i$.

2. $z = \sqrt{5}\text{cis}210^\circ$

   $180^\circ + 30^\circ = 210^\circ$. Therefore, we use the special angle of 30° in the third quadrant.

   $\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

   $\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

   $z = \sqrt{5}(\cos 210^\circ + i\sin 210^\circ)$

   $= \sqrt{5}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ Both cosine and sine are negative in the third quadrant

   $= -\frac{\sqrt{15}}{2} - \frac{\sqrt{5}}{2}i$

   The complex number in standard form is $z = -\frac{\sqrt{15}}{2} - \frac{\sqrt{5}}{2}i$.

3. $z = 2\text{cis}40^\circ$

   40° is not a special angle. The complex number is in the first quadrant.

   $\cos 40^\circ = 0.7660$

   $\sin 40^\circ = 0.6428$

   $z = 2(\cos 40^\circ + i\sin 40^\circ)$

   $= 2\,(0.7660 + 0.6428i)$ Both cosine and sine are positive in the first quadrant

   $= 1.532 + 1.2856i$

   The complex number in standard form is $z = 1.532 + 1.2856i$.

Back to Exercise 3.2

Unit 3: Assessment

1.

a. $z = \sqrt{3} + 1i$
Find r:
$r = \sqrt{x^2 + y^2}$
$= \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$
$= \sqrt{3 + 1}$
$= 2$
Find θ:
The complex number is in the first quadrant.
$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$
$\therefore \theta = 30^\circ$
Polar form: $z = 2\text{cis}30^\circ$

b. $z = 8 - 4i$
Find r:
$r = \sqrt{x^2 + y^2}$
$= \sqrt{8^2 + (-4)^2}$
$= \sqrt{64 + 16}$
$= \sqrt{80}$
$= 4\sqrt{5}$
Find θ:
The complex number is in the fourth quadrant.
$$\cos\theta = \frac{x}{r} = \frac{8}{4\sqrt{5}} = \frac{2}{\sqrt{5}}$$
$\therefore \text{Reference angle} = 26.57^\circ$
$$\therefore \theta = 360^\circ - 26.57^\circ = 333.43^\circ$$
Polar form: $z = 4\sqrt{5}\text{cis}333.43^\circ$

2.

a. $z = 3\text{cis}240^\circ$
240° is not a special angle but $240^\circ - 180^\circ = 60^\circ$ is a special angle. Therefore, we can evaluate $\cos\theta$ and $\sin\theta$ without a calculator and transfer the angle into the correct quadrant, in this case, the third quadrant.
$\cos 60^\circ = \frac{1}{2}$
$\sin 60^\circ = \frac{\sqrt{3}}{2}$
The complex number is in the third quadrant where cosine and sine are both negative.
$z = 3(\cos 240^\circ + i \sin 240^\circ)$
$= 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \quad \text{cosine and sine are negative in the third quadrant}$
$= -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$
Standard form is $z = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$.

b. $z = 7\text{cis}25^\circ$
25° is not a special angle. The complex number is in the first quadrant.
$\cos 25^\circ = 0.9063$
$\sin 25^\circ = 0.4226$The complex number is in the first quadrant where cosine and sine are positive.

$$
\begin{aligned}
z &= 7(\cos 25^\circ + i \sin 25^\circ) \\
&= 7\left(0.9063 + 0.4226i\right) \qquad \text{cosine and sine are positive in the first quadrant} \\
&= 6.3441 + 2.9582i
\end{aligned}
$$

Standard form is $z = 6.3441 + 2.9582i$.

Back to Unit 3: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- takenotecis © DHET is licensed under a CC BY (Attribution) license
- transformation © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

SUBJECT OUTCOME II

COMPLEX NUMBERS: PERFORM OPERATIONS ON COMPLEX NUMBERS

Subject outcome

Subject outcome 1.2: Perform operations on complex numbers

Learning outcomes

- Perform addition, subtraction and multiplication on complex numbers in standard/rectangular form.
- Perform division on complex numbers in standard form introducing the concept of a conjugate.
- Perform multiplication and division on complex numbers in polar form.

Unit 1 outcomes

By the end of this unit you will be able to:

- Perform addition, subtraction and multiplication on complex numbers in standard/rectangular form.
- Perform division on complex numbers in standard form introducing the concept of conjugate.

Unit 2 outcomes

By the end of this unit you will be able to:

- Perform multiplication on complex numbers in polar form.
- Perform division on complex numbers in polar form.

Unit 1: Basic complex number arithmetic

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Perform addition, subtraction and multiplication on complex numbers in standard/rectangular form.
- Perform division on complex numbers in standard form introducing the concept of conjugate.

What you should know

Before you start this unit, make sure you can:

- Simplify algebraic expressions including multiplying together binomials. Refer to level 2 subject outcome 2.2 if you need help with this.
- Perform basic arithmetic functions on imaginary numbers. Refer to unit 1 of subject outcome 1.1 if you need help with this.

Introduction

In subject outcome 1.1, we were introduced to imaginary numbers and complex numbers in standard or rectangular form. We know that complex numbers in standard form are represented as $a + bi$. But to be really useful, we need to be able to work with complex numbers in the same ways as we work with real numbers. We need to simplify expressions, solve equations and plot functions.

One example of this is when we work with capacitors and inductors in alternating current (AC) electric circuits. Components like inductors and capacitors provide a kind of resistance to the flow of AC called inductive and capacitive reactance respectively. The amount of reactance depends on the voltage (V) and current (I) and so can be described using complex numbers where $z = \mathrm{V} + i\mathrm{I}$. In order to find the total reactance in a circuit, we need to add up all these reactances.

Thankfully, we can perform the four basic arithmetic operations of addition, subtraction, multiplication and division on complex numbers just like we do with real numbers.

Adding and subtracting complex numbers

To add or subtract complex numbers, we use a technique that you will have seen before when doing algebra – collecting **like terms** together. To add or subtract complex numbers, we combine the real parts together and then combine the imaginary parts together.

Example 1.1

Add the following complex numbers:

1. $(3+4i)+(2\ +\ 5i)$
2. $(-5\ +\ 7i)+(-11\ +\ 2i)$
3. $(-3-4i)+(4-2i)$

Solutions

1. $(3+4i)+(2+5i) = 3+4i+2+5i$ Write the complex numbers without brackets
   $= (3+2)+(4+5)I$ Collect the real and imaginary parts together
   $= 5+9i$
2. $(-5+7i)+(-11+2i) = -5+7i-11+2i$
   $= (-5-11)+(7+2)i$
   $= -16+9i$
3. $(-3-4i)+(4-2i) = -3-4i+4-2i$
   $= (-3+4)+(-4-2)i$
   $= 1-6i$

Example 1.2

Subtract the following complex numbers:

1. $(3-4i)-(2+5i)$
2. $(8+3i)-(4-7i)$
3. $(7-4i)-(-3-9i)$

Solutions

1. $(3-4i)-(2+5i) = 3-4i-2-5i$ Write the complex numbers without brackets.
   Be careful of the negative sign.
   $= (3-2)+(-4-5)i$ Collect the real and imaginary parts.
   $= 1-9i$
2.

$$(8+3i)-(4-7i) = 8+3i-4+7i$$
$$= (8-4)+(3+7)i$$
$$= 4+10i$$

Be careful of the signs when writing the complex numbers without brackets.

3.
$$(7-4i)-(-3-9i) = 7-4i+3+9i$$
$$= (7+3)+(-4+9)I$$
$$= 10+5i$$

Adding complex numbers:
$(a+bi)+(c+di) = (a+c)+(b+d)i$

Subtracting complex numbers:
$(a+bi)-(c+di) = (a-c)+(b-d)i$

Note

If you have an internet connection, watch this excellent summary of adding and subtracting complex numbers called "Ex 1: Adding and Subtracting Complex Numbers".

Ex 1: Adding and Subtracting Complex Numbers (Duration: 2.32)

Exercise 1.1

Add or subtract the following complex numbers, leaving your answer in standard form:

1. $(-3+2i)-(-2+5i)$
2. $(-2+4i)+(-2-4i)$
3. $(8+6i)-(-6-6i)$
4. $(5+3i)-(2-5i)+(3+4i)-(4+2i)$

The full solutions are at the end of the unit.

Multiplying complex numbers

The process of multiplying complex numbers is very similar to multiplying two binomials. The only differences are that we always work with the real and imaginary parts separately, and we need to remember that $i^2 = -1$.



Example 1.3

Simplify the following:

1. $3(6 + 2i)$
2. $2i(5 - 3i)$
3. $(4 + 3i)(2 - 5i)$
4. $(2 - i)^3$

Solutions

1. When multiplying a complex number by a real number, we distribute the real number as we would with a binomial.
   $3(6 + 2i)$
   $= 18 + 6i$
2. When multiplying a complex number by an imaginary number, we distribute the imaginary number as we would with a binomial.
   $2i(5 - 3i)$
   $= 10i - 6(i^2)$
   $= 10i - 6(-1)$ Remember that $i^2 = -1$
   $= 10i + 6$
   $= 6 + 10i$ Standard form is written as $a + bi$
3. When we multiply two complex numbers together, we use the same process as when multiplying two binomials. Remember that FOIL is an acronym for First, Outer, Inner, and Last.
   $(4 + 3i)(2 - 5i) = 8 - 20i + 6i - 15(i^2)$
   $= 8 - 20i + 6i - 15(-1)$ Remember that $i^2 = -1$
   $= 8 - 20i + 6i + 15$ Group the real and imaginary parts
   $= 23 - 14i$
4. $(2 - i)^3 = (2 - i)(2 - i)(2 - i)$
   We start by multiplying the first two complex numbers together.
   $(2 - i)(2 - i)(2 - i) = (4 - 2i - 2i + i^2)(2 - i)$ Do the multiplication in brackets
   $= (4 - 4i - 1)(2 - i)$ $(i^2 = -1)$
   $= (3 - 4i)(2 - i)$
   Now we can multiply the product of the first two complex numbers by the third complex number.
   $(3 - 4i)(2 - i) = 6 - 3i - 8i + 4i^2$
   $= 6 - 3i - 8i - 4$ Group the real and imaginary parts
   $= 2 - 11i$

Multiplying complex numbers:

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2 \\ &= ac+adi+bci-bd \qquad (i^2=-1) \\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Note

If you have an internet connection, watch the video "Ex 2: Multiply Complex Numbers" for more on multiplying complex numbers.

Ex 2: Multiply Complex Numbers (Duration: 04.22)

Example 1.4

Simplify $(5-4i^5)(6+3i^7)$.

Solution

$$\begin{aligned}(5-4i^5)(6+3i^7) &= 30+15i^7-24i^5-12i^{12} \\ &= 30+15.i^4.i^3-24.i^4.i-12.i^{4\times 3} && \text{Factor out as many factors of } i^4 \text{ as possible} \\ &= 30+15.1.i^3-24.1.i-12.1^3 && \text{Remember that } i^4=1 \\ &= 30+15.(-1).i-24i-12 \\ &= 30-15i-24i-12 \\ &= 18-39i\end{aligned}$$

Exercise 1.2

Simplify the following, leaving your answer in standard form:

1. $(-3+7i)(-3-7i)$
2. $(2\ -\sqrt{-16})(2+\sqrt{-9})$

3. $(12\sqrt{-2})(-4\sqrt{-4}+1)$
4. $(-1+i)(3+4i)(3i)$
5. $(1-i)(1+i)(2-3i)$
6. $i(2+3i^7)+(-2-i)$

The full solutions are at the end of the unit.

Dividing complex numbers

Unfortunately, dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number. This means that any fraction must have a real-number denominator so that we can write the answer in standard form $a+bi$.

We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with only a real number as the denominator. This is called the **complex conjugate of the denominator**. Thankfully finding the complex conjugate is very easy. We just change the sign. In other words, the complex conjugate of $a+bi$ is $a-bi$.

Have a look at what happens when we multiply $a+bi$ by its complex conjugate $a-bi$:

$$\begin{aligned}&(a+bi)(a-bi)\\&=a^2-abi+abi-b^2i^2\\&=a^2-b^2i^2\\&=a^2+b^2\end{aligned}$$

We have eliminated the imaginary part of the complex number.

Take note!

We can only find the complex conjugate of a complex number if it is written in standard form. If the complex number is not in standard form, first write it in standard form.

Exercise 1.3

Find the complex conjugates of the following:

1. $2+3i$
2. $-3-4i$
3. $3+\sqrt{5}i$
4. $-\frac{1}{2}i$

The full solutions are at the end of the unit.

Take note!

The complex conjugate of the complex number $a + bi$ is $a - bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a **real number**.
- When a complex number is added to its complex conjugate, the result is a **real number**.

Example 1.5

Divide $(2 + 5i)$ by $4 - i$.

Solution

Step 1: If necessary, write the problem as a fraction, making sure that both the numerator and denominator are in standard form.

$$\frac{(2 + 5i)}{(4 - i)}$$

Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2 + 5i)}{(4 - i)} \times \frac{(4 + i)}{(4 + i)}$$

Step 3: Multiply and simplify

$$\begin{aligned}\frac{(2 + 5i)}{(4 - i)} \times \frac{(4 + i)}{(4 + i)} &= \frac{8 + 2i + 20i + 5i^2}{16 + 4i - 4i - i^2} \\ &= \frac{8 + 2i + 20i - 5}{16 - (-1)} \\ &= \frac{3 + 22i}{17}\end{aligned}$$

Step 4: Separate the real and imaginary parts to write the answer in standard form.

$$\frac{3}{17} + \frac{22}{17}i$$

Dividing complex numbers:

$$\frac{a + bi}{c + di} \quad \text{where } c \neq 0 \text{ and } d \neq 0$$

$$= \frac{(a + bi)}{(c + di)} \times \frac{(c - di)}{(c - di)}$$

Exercise 1.4

Simplify the following, writing the answer in standard form:

1. $\dfrac{(2+3i)}{(4+3i)}$
2. $\dfrac{6+4i}{i}$
3. $\dfrac{7-\sqrt{-5}}{6+\sqrt{-4}}$
4. $\dfrac{5-2i}{(3+4i)(2-i)}$
5. $\dfrac{(2i)^5}{4i}$
6. $\dfrac{(1+3i)(2-4i)}{(1+2i)}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to add, subtract and multiply complex numbers.
- The complex conjugate of $a+bi$ is $a-bi$.
- How to divide complex numbers by multiplying the numerator and denominator by the complex conjugate of the denominator.

Unit 1: Assessment

Suggested time to complete: 40 minutes

1. Simplify the following, leaving your answer in standard form:
   a. $-(10+4i)-(7+4i)$
   b. $(-3+i^3)(2-3i^3)$
   c. $(-i)(4+2i+i^2)+3i+2$
   d. $\dfrac{24i^8-36i^{16}+48i^{25}}{12i^8}$
   e. $\dfrac{(3+i)^2}{(1+2i)^2}$
   f. $\dfrac{3+2i}{1+2i}-\dfrac{2-3i}{3+i}$
2. Evaluate $2x^2+x-3$ if $x=2-3i$.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $$\begin{aligned}(-3+2i)-(-2+5i) &= -3+2i+2-5i \\ &= (-3+2)+(2-5)i \\ &= -1-3i\end{aligned}$$
2. $$\begin{aligned}(-2+4i)+(-2-4i) &= -2+4i-2-4i \\ &= (-2-2)+(4-4)i \\ &= -4+0i\end{aligned}$$
3. $$\begin{aligned}(8+6i)-(-6-6i) &= 8+6i+6+6i \\ &= (8+6)+(6+6)i \\ &= 14+12i\end{aligned}$$
4. $$\begin{aligned}(5+3i)-(2-5i)+(3+4i)-(4+2i) &= 5+3i-2+5i+3+4i-4-2i \\ &= (5-2+3-4)+(3+5+4-2)i \\ &= 2+10i\end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1. $$\begin{aligned}(-3+7i)(-3-7i) &= 9+21i-21i-49i^2 \\ &= 9+49 \\ &= 58+0i\end{aligned}$$
2. $$\begin{aligned}(2-\sqrt{-16})(2+\sqrt{-9}) &= (2-4i)(2+3i) \\ &= 4+6i-8i-12i^2 \\ &= 4-2i+12 \\ &= 16-2i\end{aligned}$$
3. $$\begin{aligned}(12\sqrt{-2})(-4\sqrt{-4}+1) &= (12\sqrt{2}i)(-4(2)i+1) \\ &= (12\sqrt{2}i)(-8i+1) \\ &= -96\sqrt{2}i^2+12\sqrt{2}i \\ &= 96\sqrt{2}+12\sqrt{2}i\end{aligned}$$
4. $$\begin{aligned}(-1+i)(3+4i)(3i) &= (-3-4i+3i+4i^2)(3i) \\ &= (-3-i-4)(3i) \\ &= (-7-i)(3i) \\ &= -21i-3i^2 \\ &= 3-21i\end{aligned}$$

5.
$$\begin{aligned}(1-i)(1+i)(2-3i) &= (1+i-i-i^2)(2-3i)\\ &= (1+1)(2-3i)\\ &= 2(2-3i)\\ &= 4-6i\end{aligned}$$

6.
$$\begin{aligned}i(2+3i^7)+(-2-i) &= 2i+3i^8-2-i\\ &= 2i+3(i^4)^2-2-i\\ &= 2i+3(1)^2-2-i\\ &= 2i+3-2-i\\ &= 1+i\end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1. Complex conjugate of $2+3i$ is $2-3i$
2. Complex conjugate of $-3-4i$ is $-3+4i$
3. Complex conjugate of $3+\sqrt{5}i$ is $3-\sqrt{5}i$
4. Write the number in standard form first: $0-\frac{1}{2}i$

   The complex conjugate of $0-\frac{1}{2}i$ is $0+\frac{1}{2}i=\frac{1}{2}i$

Back to Exercise 1.3

Exercise 1.4

1.
$$\begin{aligned}\frac{(2+3i)}{(4+3i)} &= \frac{(2+3i)}{(4+3i)}\times\frac{(4-3i)}{(4-3i)}\\ &= \frac{8-6i+12i-9i^2}{16-12i+12i-9i^2}\\ &= \frac{8-6i+12i+9}{16+9}\\ &= \frac{17+6i}{25}\\ &= \frac{17}{25}+\frac{6}{25}i\end{aligned}$$

2.
$$\begin{aligned}\frac{6+4i}{i} &= \frac{(6+4i)}{(0+i)}\\ &= \frac{(6+4i)}{(0+i)}\times\frac{0-i}{0-i}\\ &= \frac{-6i-4i^2}{-i^2}\\ &= \frac{-6i+4}{1}\\ &= 4-6i\end{aligned}$$

3.

$$\begin{aligned}\frac{7-\sqrt{-5}}{6+\sqrt{-4}} &= \frac{7-\sqrt{5}i}{6+2i}\\ &= \frac{\left(7-\sqrt{5}i\right)}{(6+2i)} \times \frac{(6-2i)}{(6-2i)}\\ &= \frac{42-14i-6\sqrt{5}i+2\sqrt{5}i^2}{36-12i+12i-4i^2}\\ &= \frac{42-14i-6\sqrt{5}i-2\sqrt{5}}{36+4}\\ &= \frac{42-\left(14+6\sqrt{5}\right)i-2\sqrt{5}}{40}\\ &= \frac{42-2\sqrt{5}}{40}-\frac{14+6\sqrt{5}}{40}i\\ &= \frac{21-\sqrt{5}}{20}-\frac{7+3\sqrt{5}}{2}i\end{aligned}$$

4.

$$\begin{aligned}\frac{5-2i}{(3+4i)(2-i)} &= \frac{(5-2i)}{(6-3i+8i-4i^2}\\ &= \frac{(5-2i)}{10+5i}\\ &= \frac{(5-2i)}{10+5i} \times \frac{10-5i}{10-5i}\\ &= \frac{50-25i-20i+10i^2}{100-50i+50i-25i^2}\\ &= \frac{40-45i}{125}\\ &= \frac{40}{125}-\frac{45}{125}i\\ &= \frac{8}{25}-\frac{9}{25}i\end{aligned}$$

5.

$$\begin{aligned}\frac{(2i)^5}{4i} &= \frac{32i^5}{4i}\\ &= 8i^4\\ &= 8\end{aligned}$$

Or

$$\begin{aligned}\frac{(2i)^5}{4i} &= \frac{32i^5}{4i}\\ &= \frac{32i^5}{0+4i}\\ &= \frac{32i^5}{0+4i} \times \frac{(0-4i)}{(0-4i)}\\ &= -\frac{-128i^6}{16i^2}\\ &= \frac{-128.i^4.i^2}{16}\\ &= \frac{128}{16} = 8\end{aligned}$$

6.

$$\begin{aligned}\frac{(1+3i)(2-4i)}{(1+2i)} &= \frac{2-4i+6i-12i^2}{(1+2i)} \\ &= \frac{(14+2i)}{(1+2i)} \times \frac{1-2i}{1-2i} \\ &= \frac{14-28i+2i-4i^2}{1-4i^2} \\ &= \frac{18-26i}{5} \\ &= \frac{18}{5} - \frac{26}{5}i\end{aligned}$$

Back to Exercise 1.4

Unit 1: Assessment

1.

a.

$$\begin{aligned}-(10+4i)-(7+4i) &= -10-4i-7-4i \\ &= -17-8i\end{aligned}$$

b.

$$\begin{aligned}(-3+i^3)(2-3i^3) &= -6+9i^3+2i^3-3i^6 \\ &= -6+9i^3+2i^3-3.i^4.i^2 \\ &= -6+9i^3+2i^3+3 \\ &= -3+11i^3 \\ &= -3-11i\end{aligned}$$

c.

$$\begin{aligned}(-i)(4+2i+i^2)+3i+2 &= -4i-2i^2-i^3+3i+2 \\ &= -4i+2+i+3i+2 \\ &= 4+0i \\ &= 4\end{aligned}$$

d.

$$\begin{aligned}\frac{24i^8-36i^{16}+48i^{25}}{12i^8} &= \frac{24(i^4)^2-36(i^4)^4+48(i^4)^6.i}{12(i^4)^2} \\ &= \frac{24-36+48i}{12} \\ &= \frac{-12+48i}{12} \\ &= \frac{-12}{12}+\frac{48}{12}i \\ &= -1+4i\end{aligned}$$

e.

$$\begin{aligned}\frac{(3+i)^2}{(1+2i)^2} &= \frac{(3+i)(3+i)}{(1+2i)(1+2i)} \\ &= \frac{9+6i-1}{1+4i-4} \\ &= \frac{(8+6i)}{(-3+4i)} \times \frac{-3-4i}{-3-4i} \\ &= \frac{-24-32i-18i-24i^2}{9-16i^2} \\ &= \frac{0-50i}{25} \\ &= 0-2i\end{aligned}$$

f.

$$\begin{aligned}\frac{3+2i}{1+2i} - \frac{2-3i}{3+i} &= \frac{(3+2i)}{(1+2i)} \times \frac{(1-2i)}{(1-2i)} - \frac{(2-3i)}{(3+i)} \times \frac{(3-i)}{(3-i)} \\ &= \frac{3-6i+2i-4i^2}{1-4i^2} - \frac{6-2i-9i+3i^2}{9-i^2} \\ &= \frac{7-4i}{5} - \frac{3-11i}{10} \\ &= \frac{2(7-4i)-(3-11i)}{10} \\ &= \frac{14-8i-3+11i}{10} \\ &= \frac{11+3i}{10} \\ &= \frac{11}{10} + \frac{3}{10}i\end{aligned}$$

2. $2x^2+x-3$ if $x = 2-3i$

$$\begin{aligned}2(2-3i)^2 + (2-3i) - 3 &= 2(2-3i)(2-3i) + 2 - 3i - 3 \\ &= 2(4-6i-6i+9i^2) - 1 - 3i \\ &= 2(-5-12i) - 1 - 3i \\ &= -10-24i-1-3i \\ &= -11-27i\end{aligned}$$

Back to Unit 1: Assessment

Unit 2: Multiply and divide complex numbers in polar form

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Perform multiplication on complex numbers in polar form.
- Perform division on complex numbers in polar form.

What you should know

Before you start this unit, make sure you can:

- Convert complex numbers from standard/rectangular form to polar form. Refer to unit 3 in subject outcome 1.1 if you need help with this.
- Convert complex numbers from polar form to standard/rectangular form. Refer to unit 3 in subject outcome 1.1 if you need help with this.
- Recognise and calculate the basic trigonometric ratios of the three special angles in all four quadrants. Refer to unit 3 in subject outcome 1.1 if you need help with this.

Introduction

In the previous unit, we saw how to multiply and divide complex numbers in standard or rectangular form. One of the benefits of polar form, however, is that it makes the multiplication and division of complex numbers even easier.

Note

In level 4 you will learn that dealing with powers and roots of complex numbers is also easier in polar form than it is in standard form.

Multiply complex numbers in polar form

Now that we can convert complex numbers to polar form, we will learn how to multiply complex numbers in polar form. The formula involved was developed by the French mathematician Abraham De Moivre (1667–1754) (shown in Figure 1).

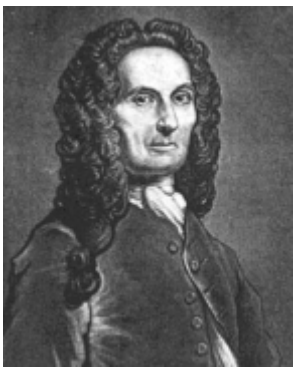


Figure 1: Abraham de Moivre

De Moivre's formula requires us to multiply the moduli and add the arguments:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $z_1 z_2 = r_1 r_2\,(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.

Can you see that we need to multiply the moduli $(r_1 \times r_2)$ and add the arguments $(\theta_1 + \theta_2)$?

We could write the product using the polar form shorthand as $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$.

Did you know?

Abraham de Moivre, on reading Isaac Newton's Principia Mathematica, realised that this book was far deeper than any he had yet read. He became so determined to read and understand it that he tore pages out of his copy in order to read them while walking around London.

Example 2.1

Find the product of z_1 and z_2 if $z_1 = 3\text{cis}40^\circ$ and $z_2 = 4\text{cis}80^\circ$.

Solution

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\,(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$z_1 = 3\text{cis}40^\circ = 3(\cos 40^\circ + i\sin 40^\circ)$ and $z_2 = 4\text{cis}80^\circ = 4(\cos 80^\circ + i\sin 80^\circ)$

$$\begin{aligned} z_1 \times z_2 &= (3 \times 4)\,(\cos(40^\circ + 80^\circ) + i\sin(40^\circ + 80^\circ)) \\ &= 12(\cos 120^\circ + i\sin 120^\circ) \\ &= 12\text{cis}120^\circ \end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\begin{aligned} 12\text{cis}120^\circ &= 12\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -6 + 6\sqrt{3}i \end{aligned}$$

Example 2.2

Find the product $z_1 z_2$, given $z_1 = 4(\cos 80^\circ + i \sin 80^\circ)$and $z_2 = 2(\cos 145^\circ + i \sin 145^\circ)$.

Solution

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

$$z_1 = 4(\cos 80^\circ + i \sin 80^\circ)\text{and } z_2 = 2(\cos 145^\circ + i \sin 145^\circ)$$

$$\begin{aligned} z_1 \times z_2 &= (4 \times 2)\left(\cos(80^\circ + 145^\circ) + i\sin(80^\circ + 145^\circ)\right) \\ &= 8(\cos 225^\circ + i \sin 225^\circ) \\ &= 8\text{cis}225^\circ \end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$$\cos 225^\circ = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\begin{aligned} 8(\cos 225^\circ + i \sin 225^\circ) &= 8\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \\ &= -\frac{8}{\sqrt{2}} - \frac{8}{\sqrt{2}}i \quad \text{Rationalise the denominators by multiplying by } \frac{\sqrt{2}}{\sqrt{2}} \\ &= -\frac{8\sqrt{2}}{2} - \frac{8\sqrt{2}}{2}i \\ &= -4\sqrt{2} - 4\sqrt{2}i \end{aligned}$$

Products of complex numbers in polar form:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $z_1 z_2 = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$
$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$

The product requires **multiplying the moduli** and **adding the angles**.

Exercise 2.1

1. Find $z_1 z_2$ leaving your answer in polar form:
   a. $z_1 = 2\sqrt{3}\text{cis}116^\circ;\ z_2 = 2\text{cis}82^\circ$
   b. $z_1 = \sqrt{2}\text{cis}205^\circ;\ z_2 = 2\sqrt{2}\text{cis}118^\circ$
2. Find $z_1 z_2$ leaving your answer in standard form:
   a. $z_1 = 3\text{cis}120^\circ;\ z_2 = \frac{1}{4}\text{cis}15^\circ$
   b. $z_1 = 3\text{cis}120^\circ;\ z_2 = 5\sqrt{2}\text{cis}60^\circ$

The full solutions are at the end of the unit.

Divide complex numbers in polar form

The quotient of two complex numbers in polar form is very similar to finding the product except that we find the quotient of the two moduli and the difference of the two arguments. This is another formula developed by Abraham De Moivre.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$. Can you see that we need to divide the moduli $\left(\frac{r_1}{r_2}\right)$ and subtract the arguments $(\theta_1 - \theta_2)$?

We could write the product using the polar form in shorthand as $\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$.

Example 2.3

Find the quotient of z_1 and z_2 if $z_1 = 2\text{cis}213^\circ$ and $z_2 = 4\text{cis}33^\circ$.

Solution

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$z_1 = 2\text{cis}213^\circ = 2(\cos 213^\circ + i\sin 213^\circ)$ and $z_2 = 4\text{cis}33^\circ = 4(\cos 33^\circ + i\sin 33^\circ)$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2}{4}(\cos(213^\circ - 33^\circ) + i\sin(213^\circ - 33^\circ)) \\ &= \frac{1}{2}(\cos 180^\circ + i\sin 180^\circ) \\ &= \frac{1}{2}\text{cis}180^\circ\end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$$\begin{aligned}\frac{1}{2}(\cos 180^\circ + i\sin 180^\circ) &= \frac{1}{2}(-1 + i0) \qquad \cos 180^\circ = -1,\ \sin 180^\circ = 0 \\ &= -\frac{1}{2} + 0i\end{aligned}$$

Example 2.4

Find the quotient $\frac{z_1}{z_2}$, given $z_1 = 2\sqrt{3}(\cos 242^\circ + i \sin 242^\circ)$ and $z_2 = 2(\cos 32^\circ + i \sin 32^\circ)$.

Solution

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$z_1 = 2\sqrt{3}(\cos 242^\circ + i \sin 242^\circ)$ and $z_2 = 2(\cos 32^\circ + i \sin 32^\circ)$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2\sqrt{3}}{2}(\cos(242^\circ - 32^\circ) + i\sin(242^\circ - 32^\circ)) \\ &= \sqrt{3}(\cos 210^\circ + i \sin 210^\circ) \\ &= \sqrt{3}\text{cis}210^\circ\end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$$\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$\begin{aligned}\sqrt{3}(\cos 210^\circ + i \sin 210^\circ) &= \sqrt{3}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= -\frac{3}{2} - \frac{\sqrt{3}}{2}i\end{aligned}$$

Quotients of complex numbers in polar form:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$$

The quotient requires **dividing the moduli** and **subtracting the angles**.

Exercise 2.2

1. Find $\frac{z_1}{z_2}$ leaving your answer in polar form:
   a. $z_1 = 21\text{cis}135^\circ;\ z_2 = 3\text{cis}65^\circ$
   b. $z_1 = \sqrt{2}\text{cis}205^\circ;\ z_2 = 2\sqrt{2}\text{cis}118^\circ$
2. Find $z_1 z_2$ leaving your answer in standard form:
   a. $z_1 = \sqrt{2}\text{cis}90^\circ;\ z_2 = 3\text{cis}60^\circ$

b. $z_1 = 15\text{cis}120^\circ;\ z_2 = 3\sqrt{2}\text{cis}40^\circ$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to multiply complex numbers in polar form.
- How to divide complex numbers in polar form.

Unit 2: Assessment

Suggested time to complete: 45 minutes

1. Simplify the following complex expressions, leaving your answer in polar form:
   a. $(4\text{cis}60^\circ)\,(6\text{cis}40^\circ)\left(\frac{1}{2}\text{cis}(-20^\circ)\right)$
   b. $\dfrac{1.25\text{cis}35^\circ \times 3.1\text{cis}45^\circ}{2\text{cis}60^\circ}$
2. Simplify the following complex expressions, leaving your answer in standard form:
   a. $6(\cos 50^\circ + i\sin 50^\circ) \times 3(\cos 10^\circ + i\sin 10^\circ)$
   b. $(6\text{cis}60^\circ)\,(4\text{cis}40^\circ)\,(2\text{cis}20^\circ)$
   c. $\dfrac{5\text{cis}175^\circ \times 3\sqrt{2}\text{cis}145^\circ}{\sqrt{6}\text{cis}5^\circ}$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
   a.
   $$z_1 = 2\sqrt{3}\text{cis}116^\circ; z_2 = 2\text{cis}82^\circ$$
   $$\therefore z_1 z_2 = 4\sqrt{3}\text{cis}198^\circ$$
   b.
   $$z_1 = \sqrt{2}\text{cis}205^\circ; z_2 = 2\sqrt{2}\text{cis}118^\circ$$
   $$\therefore z_1 z_2 = 4\text{cis}323^\circ$$
2.

a. $z_1 = 3\text{cis}120^\circ;\ z_2 = \frac{1}{4}\text{cis}15^\circ$

$$\begin{aligned}
\therefore z_1 z_2 &= \frac{3}{4}\text{cis}135^\circ \\
&= \frac{3}{4}(\cos 135^\circ + i\sin 135^\circ) && (180^\circ - 45^\circ = 135^\circ) \\
&= \frac{3}{4}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) && \cos 135^\circ < 0 \text{ (second quadrant)},\ \sin 135^\circ > 0 \text{ (second quadrant)} \\
&= -\frac{3}{4\sqrt{2}} + \frac{3}{4\sqrt{2}}i && \text{Rationalise the deonominators by multiplying by } \frac{\sqrt{2}}{\sqrt{2}} \\
&= -\frac{3\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}i
\end{aligned}$$

b. $z_1 = 3\text{cis}(120^\circ);\ z_2 = 5\sqrt{2}\text{cis}(60^\circ)$

$$\begin{aligned}
\therefore z_1 z_2 &= 15\sqrt{2}\text{cis}180^\circ \\
&= 15\sqrt{2}\,(\cos 180^\circ + i\sin 180^\circ) \\
&= 15\sqrt{2}\,(-1 + 0i) && \cos 180^\circ = -1,\ \sin 180^\circ = 0 \\
&= -15\sqrt{2} + 0i
\end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.

a.

$$\begin{aligned}
z_1 &= 21\text{cis}135^\circ; z_2 = 3\text{cis}65^\circ \\
\therefore \frac{z_1}{z_2} &= 7\text{cis}70^\circ
\end{aligned}$$

b.

$$\begin{aligned}
z_1 &= \sqrt{2}\text{cis}205^\circ; z_2 = 2\sqrt{2}\text{cis}118^\circ \\
\therefore \frac{z_1}{z_2} &= \frac{\sqrt{2}}{2\sqrt{2}}\text{cis}87^\circ \\
&= \frac{1}{2}87^\circ
\end{aligned}$$

2.

a. $z_1 = \sqrt{2}\text{cis}90^\circ;\ z_2 = 3\text{cis}60^\circ$

$$\begin{aligned}
\therefore \frac{z_1}{z_2} &= \frac{\sqrt{2}}{3}\text{cis}30^\circ \\
&= \frac{\sqrt{2}}{3}(\cos 30^\circ + i\sin 30^\circ) \\
&= \frac{\sqrt{2}}{3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\
&= \frac{\sqrt{6}}{6} + \frac{\sqrt{2}}{6}i
\end{aligned}$$

b. $z_1 = 15\text{cis}120^\circ;\ z_2 = 3\sqrt{2}\text{cis}40^\circ$

$$\therefore \frac{z_1}{z_2} = \frac{15}{3\sqrt{2}}\text{cis}(120^\circ - 40^\circ)$$
$$= \frac{15}{3\sqrt{2}}\text{cis}80^\circ$$
$$= \frac{5}{\sqrt{2}}(\cos 80^\circ + i\sin 80^\circ)$$
$$= \frac{5}{\sqrt{2}}(0.1736 + 0.9848i)$$
$$= 0.6139 + 4.818i$$

Back to Exercise 2.2

Unit 2: Assessment

1.

a.

$$(4\text{cis}60^\circ)\,(6\text{cis}40^\circ)\left(\frac{1}{2}\text{cis}(-20^\circ)\right) = \left(4 \times 6 \times \frac{1}{2}\right)\text{cis}\,(60^\circ + 40^\circ + (-20^\circ))$$
$$= 12\text{cis}80^\circ$$

b.

$$\frac{1.25\text{cis}35^\circ \times 3.1\text{cis}45^\circ}{2\text{cis}60^\circ} = \frac{(1.25 \times 3.1\text{cis}80^\circ)}{2\text{cis}60^\circ}$$
$$= \frac{3.875\text{cis}80^\circ}{2\text{cis}60^\circ}$$
$$= \left(\frac{3.875}{2}\right)\text{cis}20^\circ$$
$$= 1.9375\text{cis}20^\circ$$

2.

a.

$$6(\cos 50^\circ + i\sin 50^\circ) \times 3(\cos 10^\circ + i\sin 10^\circ) = 18(\cos 60^\circ + i\sin 60^\circ)$$
$$= 18\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= 9 + 9\sqrt{3}i$$

b.

$$(6\text{cis}60^\circ)(4\text{cis}40^\circ)(2\text{cis}20^\circ) = (6 \times 4 \times 2)\text{cis}(60^\circ + 40^\circ + 20^\circ)$$
$$= 48\text{cis}120^\circ \qquad (120^\circ = 180^\circ - 60^\circ)$$
$$= 48\,(\cos 120^\circ + i\sin 120^\circ)$$
$$= 48\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \qquad \cos 120^\circ < 0 \text{ (second quadrant)},\ \sin 120^\circ > 0 \text{ (second quadrant)}$$
$$= -24 + 24\sqrt{3}i$$

c.

$$
\begin{aligned}
\frac{5\text{cis}175^\circ \times 3\sqrt{2}\text{cis}145^\circ}{\sqrt{6}\text{cis}5^\circ} &= \frac{15\sqrt{2}\text{cis}320^\circ}{\sqrt{6}\text{cis}5^\circ} \\
&= \frac{15\sqrt{2}}{\sqrt{2\times 3}}\text{cis}315^\circ && (315^\circ = 360^\circ - 45^\circ) \\
&= \frac{15}{\sqrt{3}}(\cos 315^\circ + i sin 315^\circ) \\
&= \frac{15}{\sqrt{3}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) && \cos 315^\circ > 0 \text{ (fourth quadrant)}, \sin 315^\circ < 0 \text{ (fourth quadrant)} \\
&= \frac{15}{\sqrt{6}} - i\frac{15}{\sqrt{6}} && \text{Rationalise the denominators by multiplying each by } \frac{\sqrt{6}}{\sqrt{6}} \\
&= \frac{15\sqrt{6}}{6} - \frac{15\sqrt{6}}{6}i \\
&= \frac{5\sqrt{6}}{2} - \frac{5\sqrt{6}}{2}i
\end{aligned}
$$

Back to Unit 2: Assessment

Media Attributions

- figure1 © Cristian.gerard is licensed under a CC BY-SA (Attribution ShareAlike) license

SUBJECT OUTCOME III

FUNCTIONS AND ALGEBRA: MANIPULATE AND SIMPLIFY ALGEBRAIC EXPRESSIONS

Subject outcome

Subject outcome 2.2: Manipulate and simplify algebraic expressions

Learning outcomes

- Simplify algebraic fractions with monomial, binomial and trinomial denominators where two or more fractions are added, subtracted, divided or multiplied by each other.
- Manipulate and simplify algebraic expressions by completing the square.

Unit 1 outcomes

By the end of this unit you will be able to:

- Multiply algebraic fractions using factorisation.
- Divide algebraic fractions using factorisation.

Unit 2 outcomes

By the end of this unit you will be able to:

- Add algebraic fractions
- Subtract algebraic fractions with monomial, binomial and trinomial denominators.

Unit 3 outcomes

By the end of this unit you will be able to:

- Manipulate and simplify algebraic expressions by completing the square.

Unit 1: Multiplying algebraic fractions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Multiply algebraic fractions using factorisation.
- Divide algebraic fractions using factorisation.

What you should know

Before you start this unit, make sure you can:

- Manipulate and simplify algebraic expressions. Refer to level 2 subject outcome 2.2 units 1 and 3 if you need help with this.
- Simplify algebraic expressions by means of factorisation. Refer to level 2 subject outcome 2.2 unit 2 if you need help with this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit:

1. Simplify $2(3a+b)(3a-b)-(3a-b)^2$.
2. Factorise $(x+y)^2-16y^2$.
3. Simplify $\dfrac{9a^2-4b^2}{18ab}\times\dfrac{6a}{3a-2b}$.

Solutions

1. $$\begin{aligned}2(3a+b)(3a-b)-(3a-b)^2 &= 2(9a^2-b^2)-(9a^2-6ab+b^2)\\ &= 18a^2-2b^2-9a^2+6ab-b^2\\ &= 9a^2+6ab-3b^2\end{aligned}$$

2. $$\begin{aligned}(x+y)^2-16y^2 &= \left(\sqrt{(x+y)^2}-\sqrt{16y^2}\right)\left(\sqrt{(x+y)^2}+\sqrt{16y^2}\right)\\ &= (x+y-4y)(x+y+4y)\\ &= (x-3y)(x+5y)\end{aligned}$$

3. $$\begin{aligned}\frac{9a^2-4b^2}{18ab}\times\frac{6a}{3a-2b} &= \frac{(3a-2b)(3a+2b)}{18ab}\times\frac{6a}{(3a-2b)}\\ &= \frac{(3a+2b)}{3\cancel{1}\cancel{8}\cancel{a}b}\times\frac{\cancel{6}\cancel{a}}{1} && \text{Factors of } (3a-2b) \text{ cancel}\\ &= \frac{3a+2b}{3b}\end{aligned}$$

Introduction

Algebraic expressions are the bread and butter of much of mathematics. To mix metaphors, they are the nuts and bolts of equations. In order to solve equations, you need to be able to simplify them and the algebraic expressions from which they are made.

Remember that an algebraic expression is made up of constants and variables joined together by addition, subtraction, multiplication and division.
In the expression $3x + 2$:

- the 2 is a **constant** because its value does not change
- the x is a **variable** because its value can change
- the 3 in front of the variable x is called the **coefficient** of x and is being multiplied by the x
- the $3x$ and 2 are called the **terms** of the algebraic expression. Terms are separated by addition or subtraction. In this algebraic expression we have two terms.

We usually have to simplify algebraic expressions to make them easier to work with or calculate with or use in some other way.

Factorisation recap

Before we dive in, let's spend some time revising what we know about factorisation.

In the same way that we can split any number into its factors (numbers that multiply together to produce the original number) we can also factorise algebraic expressions. For example, the factors of 12, in their pairs, are $1 \text{ and } 12$; $2 \text{ and } 6$; $3 \text{ and } 4$. These are just the whole number factor pairs. $\frac{1}{2} \text{ and } 24$ are also factors of 12 because $\frac{1}{2} \times 24 = 12$.

Similarly, we can say that the factors of $4x^2$ are $4 \text{ and } x \text{ and } x$ because $4 \times x \times x = 4x^2$.

Factorising by taking out the highest common factor

Taking out the (usually) highest common factor is the first step of factorising algebraic expressions. It is the simplest to do but is also the first thing you should try when you need to split an algebraic fraction into its factors (i.e. factorise it). Here is an example:

$4x^3 + 2x^2 + 6x$

We can see the common factors in each term more easily if we expand each term.

$(2 \times 2 \times x \times x \times x) + (2 \times x \times x) + (2 \times 3 \times x)$

The highest common factor we can take out of each term is $2 \times x$.

$(2 \times 2 \times x \times x \times x) + (2 \times x \times x) + (2 \times 3 \times x)$

If we take this highest common factor out of each term we get:

$$\begin{aligned} &4x^3 + 2x^2 + 6x \\ &= 2x\left[(2 \times x \times x) + (x) + (3)\right] \\ &= 2x\left(2x^2 + x + 3\right) \end{aligned}$$

Factorising by grouping

Factorising by grouping is another way to create highest common factors in different parts of an algebraic expression. This is done by grouping these different parts together. Here is an example:

$3ax + bx - 3ay - by - 9a - 3b = (3ax + bx) - (3ay + by) - (9a + 3b)$

You need to be careful of the signs inside the brackets. If you multiply the brackets out, you must always be left with the original expression.

Now we can take a common factor out of each bracket:

$$\begin{aligned} 3ax + bx - 3ay - by - 9a - 3b &= (3ax + bx) - (3ay + by) - (9a + 3b) \\ &= x(3a + b) - y(3a + b) - 3(3a + b) \end{aligned}$$

Can you see that we now have a common factor in each term of $(3a + b)$? Let's take it out:

$$\begin{aligned} 3ax + bx - 3ay - by - 9a - 3b &= (3ax + bx) - (3ay + by) - (9a + 3b) \\ &= x(3a + b) - y(3a + b) - 3(3a + b) \\ &= (3a + b)(x - y - 3) \end{aligned}$$

Factorising the difference of two squares

This factorisation technique only applies to algebraic expressions where there are two terms subtracted from each other. The pattern is always the same.

$x^2 - y^2 = (x + y)(x - y)$

Sometimes, as in the next example, you need to do some extra work before you can get to the difference of two squares.

$$\begin{aligned} &2a(a^2 - 9) - 7(a^2 - 9) && \text{Take out the common factor of } (a^2 - 9) \\ &= (a^2 - 9)(2a - 7) && \text{Now factorise the difference of two squares} \\ &= (a + 3)(a - 3)(2a - 7) && \text{Don't forget the original common factor of } (2a - 7) \end{aligned}$$

You don't need to have perfect squares to factorise using this method. Just follow the basic pattern:

$x - y = \left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right)$

Factorising a quadratic trinomial

A trinomial is an algebraic expression with three terms. A quadratic trinomial is an algebraic expression of the form $ax^2 + bx + c$.

In level 2, we saw how to factorise quadratic trinomials using trial and error or a method called grouping. You can use whichever method you find easiest. Her are two examples:

$$\begin{aligned} &2y^2 + 5y - 3 && \text{This quadratic trinomial is in the standard form aleady} \\ &= (2y + 1)(y - 3) && \text{Multiply the binomial together to make sure you get the original} \end{aligned}$$

$$\begin{aligned} &7x - 6 + 5x^2 && \text{This quadratic trinomial is not yet in standard form} \\ &= 5x^2 + 7x - 6 \\ &= (5x - 3)(x + 2) && \text{Multiply the binomial together to make sure you get the original expression} \end{aligned}$$

Multiply algebraic fractions using factorisation

Multiplying algebraic fractions together works the same way as multiplying any other fractions together. We multiply the numerators to find the numerator of the product, and then we multiply the denominators to find the denominator of the product.

However, if we factorise the numerators and denominators of the fractions, we may find factors that we can 'cancel', making the multiplication of numerators and denominators simpler.

Remember, we never actually cancel anything. If we have the same factor in the numerator and the denominator, we recognise that the value of this is equal to 1 and therefore it has no influence on the value of the final product.

Example 1.1

Multiply the algebraic fractions and simplify:

$$\frac{x^2+4x-5}{3x+18} \times \frac{x+6}{x+5}$$

Solution

The first thing we need to do is factorise as many of the numerators and denominators as we can:

$$\frac{x^2+4x-5}{3x+18} \times \frac{x+6}{x+5}$$
$$= \frac{(x-1)(x+5)}{3(x+6)} \times \frac{x+6}{x+5}$$

Now you can either multiply the numerators and denominators and then 'cancel', or 'cancel' and then multiply the numerators and denominators.

$$\frac{(x-1)(x+5)}{3(x+6)} \times \frac{x+6}{x+5}$$
$$= \frac{(x-1)\cancel{(x+5)}\ \cancel{(x+6)}}{3\cancel{(x+6)}\ \cancel{(x+5)}}$$
$$= \frac{(x-1)}{3}$$

OR

$$\frac{(x-1)\cancel{(x+5)}}{3\cancel{(x+6)}} \times \frac{\cancel{x+6}}{\cancel{x+5}}$$
$$= \frac{(x-1)}{3}$$

Example 1.2

Simplify the algebraic expression:

$$\frac{x^2+11x+30}{x^2+5x+6} \cdot \frac{x^2+7x+12}{x^2+8x+16}$$

Solution

$$\frac{x^2+11x+30}{x^2+5x+6} \cdot \frac{x^2+7x+12}{x^2+8x+16}$$
$$= \frac{(x+5)(x+6)}{(x+2)\cancel{(x+3)}} \cdot \frac{\cancel{(x+4)}\ \cancel{(x+3)}}{\cancel{(x+4)}\ (x+4)}$$
$$= \frac{(x+5)(x+6)}{(x+2)(x+4)}$$

Note: While we could multiply the numerator and denominator out, we tend to leave them in factorised form. This is usually considered the simplest form of an expression.

Take note!

If you have an expression such as $\frac{x+3}{x+3}$, you can 'cancel' the **whole** factor of $(x+3)$ in the numerator with the **whole** factor of $(x+3)$ in the denominator.

$$\frac{x+3}{x+3} = \frac{\cancel{(x+3)}}{\cancel{(x+3)}} = 1$$

If you have an expression such as $\frac{x+3}{x-3}$, you **cannot** 'cancel' the x's or the 3's because these are separate terms separated by $+$ and $-$ signs.

$$\frac{x+3}{x-3} \neq \frac{\cancel{x}+\cancel{3}}{\cancel{x}-\cancel{3}}$$

Convince yourself that this is true by substituting different values for x.

Exercise 1.1

Simplify the following algebraic expressions:

1. $\frac{n-12}{n^2-144}$
2. $\frac{x^2-x-6}{x^2-9} \cdot \frac{2x^2+7x-15}{2x^2+x-6}$
3. $\frac{2}{(x-y)^2} \times \frac{x^2-y^2}{6}$

4. $\dfrac{t^2-1}{t^2+4t+3} \cdot \dfrac{t^2+2t-15}{t^2-4t+3}$

5. $\dfrac{36x^2-25}{5x^2+70x+120} \cdot \dfrac{3x^2+36x}{2x^2-18} \cdot \dfrac{x^2-2x-3}{6x^2+11x+5}$

The full solutions are at the end of the unit.

Divide algebraic fractions using factorisation

Dividing algebraic fractions works the same way as the division of other fractions. To divide one algebraic fraction by another one, multiply the first expression by the reciprocal of the second. In other words:

$$\frac{1}{x} \div \frac{x^2}{3} = \frac{1}{x} \times \frac{3}{x^2}$$

Once the division expression has been rewritten as a multiplication expression, we can multiply the fractions as normal.

Example 1.3

Divide the following algebraic expression and simplify:

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

Solution

The first step is to rewrite the division expression as a multiplication expression.

$$\begin{aligned}&\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}\\ &= \frac{2x^2+x-6}{x^2-1} \times \frac{x^2+2x+1}{x^2-4}\end{aligned}$$

Now we can proceed as normal.

$$\begin{aligned}&\frac{2x^2+x-6}{x^2-1} \times \frac{x^2+2x+1}{x^2-4}\\ &= \frac{(2x-3)\cancel{(x+2)}}{\cancel{(x+1)}(x-1)} \times \frac{\cancel{(x+1)}(x+1)}{\cancel{(x+2)}(x-2)}\\ &= \frac{(2x-3)(x+1)}{(x-1)(x-2)}\end{aligned}$$

Example 1.4

Divide the following algebraic expression and simplify:

$$\frac{9x^2-16}{3x^2+25x+28} \div \frac{3x^2-10x+8}{x^2+5x-14}$$

Solution

Rewrite the division expression as a multiplication expression by multiplying by the reciprocal of the second fraction.

$$\begin{aligned}&\frac{9x^2-16}{3x^2+25x+28} \div \frac{3x^2-10x+8}{x^2+5x-14}\\ &= \frac{9x^2-16}{3x^2+25x+28} \times \frac{x^2+5x-14}{3x^2-10x+8}\end{aligned}$$

Now proceed as normal to factorise the numerators and denominators.

$$\begin{aligned}&\frac{9x^2-16}{3x^2+25x+28} \times \frac{x^2+5x-14}{3x^2-10x+8}\\ &= \frac{\cancel{(3x+4)}\ \cancel{(3x-4)}}{\cancel{(3x+4)}\ \cancel{(x+7)}} \times \frac{\cancel{(x-2)}\ \cancel{(x+7)}}{\cancel{(3x-4)}\ \cancel{(x-2)}}\\ &= 1\end{aligned}$$

Note

The reciprocal of a number is its multiplicative inverse. If x is any non-zero number, then its reciprocal is $\frac{1}{x}$ such that $x \times \frac{1}{x} = 1$.



Exercise 1.2

Divide the following algebraic expressions and simplify.

1. $\frac{2x^2+x-3}{7x^2-7} \div \frac{2x^2-x-6}{3x+3}$
2. $\frac{q^2-9}{q^2+6q+9} \div \frac{q^2-2q-3}{q^2+2q-3}$
3. $\frac{144b^2-25}{72b^2-6b-10} \div \frac{24b^2+22b+5}{36b^2-9}$

The full solutions are at the end of the unit.

Simplify complex fractions

Let's now look at how to simplify complex fractions.

Example 1.5

Simplify the algebraic expression:

$$\frac{x^2+x-6}{x^2-2x-3} \cdot \frac{2x^2-3x-9}{x^2-x-2} \div \frac{10x^2+27x+18}{x^2+2x+1}$$

Solution

In this case, we can either multiply the first two fractions and then deal with the division of the third fraction or we can rewrite the division as a multiplication straight away. Rewriting it as a multiplication straight away will allow us to 'cancel' in one step.

$$\begin{aligned}
&\frac{x^2+x-6}{x^2-2x-3} \cdot \frac{2x^2-3x-9}{x^2-x-2} \div \frac{10x^2+27x+18}{x^2+2x+1} \\
&= \frac{x^2+x-6}{x^2-2x-3} \cdot \frac{2x^2-3x-9}{x^2-x-2} \times \frac{x^2+2x+1}{10x^2+27x+18} \\
&= \frac{\cancel{(x-2)}\,(x+3)}{(x+1)\,\cancel{(x-3)}} \times \frac{\cancel{(2x+3)}\,\cancel{(x-3)}}{(x+1)\,\cancel{(x-2)}} \times \frac{\cancel{(x+1)}\,\cancel{(x+1)}}{(5x+6)\,\cancel{(2x+3)}} \\
&= \frac{x+3}{5x+6}
\end{aligned}$$

Example 1.6

Simplify the algebraic expression:

$$\frac{x^2+7x+12}{x^2+x-6} \div \frac{x^2-16}{x^2+4x+3} \div \frac{3x+9}{3x^2-12}$$

Solution

When you have an expression with two division signs, you can rewrite both of them by multiplying by the reciprocal of each fraction.

$$\begin{aligned}
&\frac{x^2+7x+12}{x^2+x-6} \div \frac{x^2-16}{x^2+4x+3} \div \frac{3x+9}{3x^2-12} \\
&= \frac{x^2+7x+12}{x^2+x-6} \times \frac{x^2+4x+3}{x^2-16} \times \frac{3x^2-12}{3x+9} \\
&= \frac{(x+3)(x+4)}{(x-2)(x+3)} \times \frac{(x+1)(x+3)}{(x+4)(x-4)} \times \frac{3(x^2-4)}{3(x+3)} \\
&= \frac{\cancel{(x+3)}\,\cancel{(x+4)}}{(x-2)\,\cancel{(x+3)}} \times \frac{(x+1)\,\cancel{(x+3)}}{\cancel{(x+4)}\,(x-4)} \times \frac{\cancel{3}\,(x+2)\,\cancel{(x-2)}}{\cancel{3}\,\cancel{(x+3)}} \\
&= \frac{(x+1)(x+2)}{(x-4)}
\end{aligned}$$

Exercise 1.3

Simplify the following:

1. $\dfrac{x^2+6x-7}{x+1} \cdot \dfrac{x^2-x-2}{2x-14} \div \dfrac{5x^3-20x}{6}$
2. $\left(\dfrac{2x^2-3x-9}{x^2-9}\right)\left(\dfrac{2x^3+8x^2+6x}{16x^3+24x^2}\right) \div \dfrac{x^2-x-2}{x^2+3x-10}$
3. $\dfrac{\left(\dfrac{3y^2-10y+3}{3y^2+5y-2} \cdot \dfrac{2y^2-3y-20}{2y^2-y-15}\right)}{y-4}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to multiply algebraic fractions by first factorising the numerators and denominators in order to 'cancel' like factors.
- How to divide algebraic fractions by multiplying by the reciprocal of the fraction.

Unit 1: Assessment

Suggested time to complete: 25 minutes

Simplify the following:

1. $\dfrac{a^2+9a+18}{a^2+3a-18}$
2. $\dfrac{c^2+2c-24}{c^2+12c+36} \cdot \dfrac{c^2-10c+24}{c^2-8c+16}$
3. $\dfrac{16a^2-24a+9}{4a^2+17a-15} \div \dfrac{16a^2-9}{4a^2+11a+6}$
4. $\dfrac{2x^2+x}{2x-1} \times \dfrac{4x^2-1}{x^2} \div \dfrac{2x+1}{x}$
5. $\dfrac{3x^2-21x}{18x^2+24x} \div \dfrac{x-7}{4x+5} \cdot \dfrac{18x^3-32x}{4x^2+x-5} \div \dfrac{3x-4}{x-1}$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $$\begin{aligned}&\frac{n-12}{n^2-144}\\&=\frac{\cancel{n-12}}{(n+12)\,\cancel{(n-12)}}\\&=\frac{1}{n+12}\end{aligned}$$

2. $$\begin{aligned}&\frac{x^2-x-6}{x^2-9}\cdot\frac{2x^2+7x-15}{2x^2+x-6}\\&=\frac{\cancel{(x+2)}\,\cancel{(x-3)}}{(x+3)\,\cancel{(x-3)}}\cdot\frac{\cancel{(2x-3)}\,(x+5)}{\cancel{(2x-3)}\,\cancel{(x+2)}}\\&=\frac{(x+5)}{(x+3)}\end{aligned}$$

3. $$\begin{aligned}&\frac{2}{(x-y)^2}\times\frac{x^2-y^2}{6}\\&=\frac{\cancel{2}}{\cancel{(x-y)}\,(x-y)}\times\frac{(x+y)\,\cancel{(x-y)}}{\cancel{6}\,3}\\&=\frac{(x+y)}{3(x-y)}\end{aligned}$$

4. $$\begin{aligned}&\frac{t^2-1}{t^2+4t+3}\cdot\frac{t^2+2t-15}{t^2-4t+3}\\&=\frac{\cancel{(t+1)}\,\cancel{(t-1)}}{\cancel{(t+1)}\,(t+3)}\cdot\frac{\cancel{(t-3)}\,(t+5)}{\cancel{(t-1)}\,\cancel{(t-3)}}\\&=\frac{(t+5)}{(t+3)}\end{aligned}$$

5. $$\begin{aligned}&\frac{36x^2-25}{5x^2+70x+120}\cdot\frac{3x^2+36x}{2x^2-18}\cdot\frac{x^2-2x-3}{6x^2+11x+5}\\&=\frac{\cancel{(6x+5)}\,(6x-5)}{5\,\cancel{(x+12)}\,(x+2)}\cdot\frac{3x\,\cancel{(x+12)}}{2(x+3)\,\cancel{(x-3)}}\cdot\frac{\cancel{(x+1)}\,\cancel{(x-3)}}{\cancel{(6x+5)}\,\cancel{(x+1)}}\\&=\frac{3x(6x-5)}{10(x+2)(x+3)}\end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1. $$\begin{aligned}&\frac{2x^2+x-3}{7x^2-7} \div \frac{2x^2-x-6}{3x+3} \\ &= \frac{2x^2+x-3}{7x^2-7} \times \frac{3x+3}{2x^2-x-6} \\ &= \frac{\cancel{(2x+3)}\ \cancel{(x-1)}}{7\ \cancel{(x+1)}\ \cancel{(x-1)}} \times \frac{3\ \cancel{(x+1)}}{\cancel{(2x+3)}\ (x-2)} \\ &= \frac{3}{7(x-2)}\end{aligned}$$

2. $$\begin{aligned}&\frac{q^2-9}{q^2+6q+9} \div \frac{q^2-2q-3}{q^2+2q-3} \\ &= \frac{q^2-9}{q^2+6q+9} \times \frac{q^2+2q-3}{q^2-2q-3} \\ &= \frac{\cancel{(q+3)}\ \cancel{(q-3)}}{\cancel{(q+3)}\ \cancel{(q+3)}} \times \frac{(q-1)\ \cancel{(q+3)}}{(q+1)\ \cancel{(q-3)}} \\ &= \frac{(q-1)}{(q+1)}\end{aligned}$$

3. $$\begin{aligned}&\frac{144b^2-25}{72b^2-6b-10} \div \frac{24b^2+22b+5}{36b^2-9} \\ &= \frac{144b^2-25}{72b^2-6b-10} \times \frac{36b^2-9}{24b^2+22b+5} \\ &= \frac{\cancel{(12b+5)}\ \cancel{(12b-5)}}{2\ \cancel{(12b-5)}\ (3b+1)} \times \frac{9\ \cancel{(2b+1)}\ (2b-1)}{\cancel{(12b+5)}\ \cancel{(2b+1)}} \\ &= \frac{9(2b-1)}{2(3b+1)}\end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1.

$$\frac{x^2+6x-7}{x+1} \cdot \frac{x^2-x-2}{2x-14} \div \frac{5x^3-20x}{6}$$
$$= \frac{x^2+6x-7}{x+1} \cdot \frac{x^2-x-2}{2x-14} \times \frac{6}{5x^3-20x}$$
$$= \frac{x^2+6x-7}{x+1} \cdot \frac{x^2-x-2}{2x-14} \times \frac{6}{5x(x^2-4)}$$
$$= \frac{\cancel{(x+1)}\ \cancel{(x-7)}}{\cancel{(x+1)}} \cdot \frac{(x+1)\ \cancel{(x-2)}}{2\ \cancel{(x-7)}} \times \frac{6}{5x(x+2)\ \cancel{(x-2)}}$$
$$= \frac{6(x+1)}{10x(x+2)}$$
$$= \frac{3(x+1)}{5x(x+2)}$$

2.

$$\frac{2x^2-3x-9}{x^2-9} \times \frac{2x^3+8x^2+6x}{16x^3+24x^2} \div \frac{x^2-x-2}{x^2+3x-10}$$
$$= \frac{2x^2-3x-9}{x^2-9} \times \frac{2x^3+8x^2+6x}{16x^3+24x^2} \times \frac{x^2+3x-10}{x^2-x-2}$$
$$= \frac{2x^2-3x-9}{x^2-9} \times \frac{2x(x^2+4x+3)}{16x^3+24x^2} \times \frac{x^2+3x-10}{x^2-x-2}$$
$$= \frac{\cancel{(2x+3)}\ \cancel{(x-3)}}{\cancel{(x+3)}\ \cancel{(x-3)}} \times \frac{2x\ \cancel{(x+3)}\ \cancel{(x+1)}}{8x^2\ \cancel{(2x+3)}} \times \frac{\cancel{(x-2)}\ (x+5)}{\cancel{(x+1)}\ \cancel{(x-2)}}$$
$$= \frac{2x(x+5)}{8x^2}$$
$$= \frac{(x+5)}{4x}$$

3.

$$\frac{\left(\frac{3y^2-10y+3}{3y^2+5y-2} \cdot \frac{2y^2-3y-20}{2y^2-y-15}\right)}{y-4}$$
$$= \left(\frac{3y^2-10y+3}{3y^2+5y-2} \cdot \frac{2y^2-3y-20}{2y^2-y-15}\right) \div \frac{y-4}{1}$$
$$= \left(\frac{3y^2-10y+3}{3y^2+5y-2} \cdot \frac{2y^2-3y-20}{2y^2-y-15}\right) \times \frac{1}{y-4}$$
$$= \frac{\cancel{(3y-1)}\ \cancel{(y-3)}}{\cancel{(3y-1)}\ (y+2)} \cdot \frac{\cancel{(2y+5)}\ \cancel{(y-4)}}{\cancel{(2y+5)}\ \cancel{(y-3)}} \times \frac{1}{\cancel{(y-4)}}$$
$$= \frac{1}{(y+2)}$$

Back to Exercise 1.3

Unit 1: Assessment

1.

$$\frac{a^2+9a+18}{a^2+3a-18}$$
$$=\frac{(a+3)\cancel{(a+6)}}{(a-3)\cancel{(a+6)}}$$
$$=\frac{a+3}{a-3}$$

2.
$$\frac{c^2+2c-24}{c^2+12c+36}\cdot\frac{c^2-10c+24}{c^2-8c+16}$$
$$=\frac{\cancel{(c-4)}\cancel{(c+6)}}{(c+6)\cancel{(c+6)}}\cdot\frac{(c-6)\cancel{(c-4)}}{\cancel{(c-4)}\cancel{(c-4)}}$$
$$=\frac{(c-6)}{(c+6)}$$

3.
$$\frac{16a^2-24a+9}{4a^2+17a-15}\div\frac{16a^2-9}{4a^2+11a+6}$$
$$=\frac{16a^2-24a+9}{4a^2+17a-15}\times\frac{4a^2+11a+6}{16a^2-9}$$
$$=\frac{\cancel{(4a-3)}\cancel{(4a-3)}}{\cancel{(4a-3)}(a+5)}\times\frac{\cancel{(4a+3)}(a+2)}{\cancel{(4a+3)}\cancel{(4a-3)}}$$
$$=\frac{(a+2)}{(a+5)}$$

4.
$$\frac{2x^2+x}{2x-1}\times\frac{4x^2-1}{x^2}\div\frac{2x+1}{x}$$
$$=\frac{2x^2+x}{2x-1}\times\frac{4x^2-1}{x^2}\times\frac{2}{2x+1}$$
$$=\frac{x\cancel{(2x+1)}}{\cancel{(2x-1)}}\times\frac{\cancel{(2x+1)}\cancel{(2x-1)}}{x^2}\times\frac{2}{\cancel{(2x+1)}}$$
$$=\frac{2x}{x^2}$$
$$=\frac{2}{x}$$

5.
$$\frac{3x^2-21x}{18x^2+24x}\div\frac{x-7}{4x+5}\cdot\frac{18x^3-32x}{4x^2+x-5}\div\frac{3x-4}{x-1}$$
$$=\left(\frac{3x^2-21x}{18x^2+24x}\div\frac{x-7}{4x+5}\right)\cdot\left(\frac{18x^3-32x}{4x^2+x-5}\div\frac{3x-4}{x-1}\right)$$

Placing the fractions being divided in brackets makes things clearer

$$=\left(\frac{3x^2-21x}{18x^2+24x}\times\frac{4x+5}{x-7}\right)\cdot\left(\frac{18x^3-32x}{4x^2+x-5}\times\frac{x-1}{3x-4}\right)$$
$$=\frac{3x^2-21x}{18x^2+24x}\times\frac{4x+5}{x-7}\cdot\frac{2x(9x^2-16)}{4x^2+x-5}\times\frac{x-1}{3x-4}$$
$$=\frac{3x\cancel{(x-7)}}{6x\cancel{(3x+4)}}\times\frac{\cancel{(4x+5)}}{\cancel{(x-7)}}\cdot\frac{2x\cancel{(3x+4)}(3x-4)}{\cancel{(4x+5)}(x-1)}\times\frac{(x-1)}{(3x-4)}$$
$$=\frac{6x^2}{6x}$$
$$=x$$

Back to Unit 1: Assessment

Unit 2: Simplifying algebraic fractions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Add algebraic fractions
- Subtract algebraic fractions with monomial, binomial and trinomial denominators.

What you should know

Before you start this unit, make sure you can:

- Manipulate and simplify algebraic expressions. Refer to level 2 subject outcome 2.2 unit 1 if you need help with this.
- Simplify algebraic expressions by means of factorisation. Refer to level 2 subject outcome 2.2 unit 2 and unit 1 in this subject outcome if you need help with this.
- Add and subtract algebraic fractions. Refer to level 2 subject outcome 2.2 unit 3 if you need help with this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Add or subtract the following fractions:

1. $\frac{x}{y}+\frac{2}{x}$
2. $\frac{a}{2b}-\frac{2b}{9a}$

Solutions

1. $$\frac{x}{y}+\frac{2}{x}=\frac{x\cdot x+2\cdot y}{xy}$$ $$=\frac{x^2+2y}{xy}$$
2. $$\frac{a}{2b}-\frac{2b}{9a}=\frac{9\cdot a\cdot a}{18ab}-\frac{2b\cdot(2b)}{18ab}$$ $$=\frac{9a^2-4b^2}{18ab}$$

Introduction

We saw in the previous unit that we can make the multiplication (and division) of algebraic fractions easier if we first factorise the numerators and denominators and then 'cancel' like factors in the numerator and denominator. Remember that we are not really cancelling anything. We are simply recognising that having the same factor in the numerator and denominator is equivalent to 1.

In this unit, we are going to learn how first factorising, especially the denominators, of algebraic fractions allows us to identify the lowest common denominator so that we can add and subtract more complicated algebraic fractions.

Adding and subtracting algebraic fractions

You know that the only way to add (or subtract) fractions is to make the fractions' denominators 'look the same'.

$$\frac{1}{2}+\frac{1}{3}-\frac{1}{5}$$

The **lowest common denominator** of all three of these denominators is 30. Therefore, we have to change all the denominators to 30. This means multiplying the denominator of 2 by 15, the denominator of 3 by 10 and the denominator of 5 by 6.

But if we just changed the denominators without also changing the numerators, we would change the value of each fraction. So, we multiply the whole fraction of $\frac{1}{2}$, for example, by $\frac{15}{15}$. $\frac{15}{15}$ is really just another way to write 1. Hence, because we are really multiplying $\frac{1}{2}$ by 1, we are not changing its value.

$$\begin{aligned}\frac{1}{2}+\frac{1}{3}-\frac{1}{5} &= \left(\frac{1}{2}\times\frac{15}{15}\right)+\left(\frac{1}{3}\times\frac{10}{10}\right)+\left(\frac{1}{5}\times\frac{6}{6}\right)\\ &= \frac{15}{30}+\frac{10}{30}-\frac{6}{30}\end{aligned}$$

Now that all the denominators are the same, we can add and subtract the numerators.

$$\begin{aligned}\frac{15}{30}+\frac{10}{30}-\frac{6}{30} &= \frac{15+10-6}{30}\\ &= \frac{19}{30}\end{aligned}$$

Factorising the denominators of algebraic fractions is often an essential step in being able to identify the lowest common denominator (LCD).



Example 2.1

Simplify the following:

$$\frac{x}{x+y}+\frac{x^2}{x^2-y^2}$$

Solution

The first thing we need to do is factorise the denominators so that we can identify the LCD.

$$\frac{x}{x+y}+\frac{x^2}{x^2-y^2}=\frac{x}{x+y}+\frac{x^2}{(x+y)(x-y)}$$

Now we need to identify the LCD. In simple terms, each factor in the denominators needs to be represented in the LCD. So, the LCD is $(x+y)(x-y)$.

Next, we need to multiply each fraction by the form of 1 that will change its denominator into the LCD. This basically means multiplying the numerator and denominator by the factors of the LCD that are missing from each fraction's denominator.

$$\begin{aligned}\frac{x}{x+y}+\frac{x^2}{(x+y)(x-y)}&=\left(\frac{x}{(x+y)}\times\frac{(x-y)}{(x-y)}\right)+\frac{x^2}{(x+y)(x-y)}\\&=\frac{x(x-y)}{(x+y)(x-y)}+\frac{x^2}{(x+y)(x-y)}\end{aligned}$$

Now we can collect all the fractions over the LCD, add and subtract the numerators, and simplify.

$$\begin{aligned}\frac{x(x-y)}{(x+y)(x-y)}+\frac{x^2}{(x+y)(x-y)}&=\frac{x(x-y)+x^2}{(x+y)(x-y)}\\&=\frac{x^2-xy+x^2}{(x+y)(x-y)}\\&=\frac{2x^2-xy}{(x+y)(x-y)}\\&=\frac{x(2x-y)}{(x+y)(x-y)}\end{aligned}$$

None of the factors in the numerator cancel with the denominator, so this is the expression's simplest form.



Exercise 2.1

Simplify the following:

1. $\frac{2}{xy}+\frac{4}{xz}+\frac{3}{yz}$
2. $\frac{4}{a+1}-\frac{5}{a-3}$
3. $\frac{x-1}{x+1}-\frac{2x+3}{2x+2}$
4. $\frac{4p}{p+1}-\frac{p+1}{4p}$

The full solutions are at the end of the unit.

This next example is a bit more complicated but the same principles and procedure applies.

Example 2.2

Simplify the following:

$$\frac{x}{xy+y^2}+\frac{2x}{x^2-y^2}+\frac{y}{x^2-xy}$$

Solution

The first thing we need to do is factorise the denominators so that we can identify the LCD.

$$\frac{x}{xy+y^2}+\frac{2x}{x^2-y^2}+\frac{y}{x^2-xy}$$
$$=\frac{x}{y(x+y)}+\frac{2x}{(x+y)(x-y)}+\frac{y}{x(x-y)}$$

Remember, each factor in the denominators needs to be represented in the LCD. So, the LCD is $xy(x+y)(x-y)$.

Now, we need to multiply each fraction by that form of 1 that will match each denominator to the LCD by multiplying the numerator and denominator by all the factors in the LCD that are missing in each fraction's denominator.

$$\frac{x}{y(x+y)}-\frac{2x}{(x+y)(x-y)}+\frac{y}{x(x-y)}$$
$$=\left(\frac{x}{y(x+y)}\times\frac{x(x-y)}{x(x-y)}\right)+\left(\frac{2x}{(x+y)(x-y)}\times\frac{xy}{xy}\right)+\left(\frac{y}{x(x-y)}\times\frac{y(x+y)}{y(x+y)}\right)$$
$$=\frac{x^2(x-y)}{xy(x+y)(x-y)}+\frac{2x^2y}{xy(x+y)(x-y)}+\frac{y^2(x+y)}{xy(x+y)(x-y)}$$

Now we can collect all the fractions over the LCD, add and subtract the numerators, and simplify.

$$\frac{x^2(x-y)}{xy(x+y)(x-y)}+\frac{2x^2y}{xy(x+y)(x-y)}+\frac{y^2(x+y)}{xy(x+y)(x-y)}$$
$$=\frac{x^2(x-y)+2x^2y+y^2(x+y)}{xy(x+y)(x-y)}$$
$$=\frac{x^3-x^2y+2x^2y+xy^2+y^3}{xy(x+y)(x-y)}$$ Expand the numerator
$$=\frac{x^3+x^2y+xy^2+y^3}{xy(x+y)(x-y)}$$ Collect like terms
$$=\frac{\left(x^3+x^2y\right)+\left(xy^2+y^3\right)}{xy(x+y)(x-y)}$$ Factorise the numerator (in this case by grouping)
$$=\frac{x^2\,(x+y)+y^2\,(x+y)}{xy(x+y)(x-y)}$$
$$=\frac{\cancel{(x+y)}\,(x^2+y^2)}{xy\,\cancel{(x+y)}\,(x-y)}$$ Cancel like factors
$$=\frac{(x^2+y^2)}{(x-y)}$$

Exercise 2.2

Simplify the following:

1. $\dfrac{3}{p^2-4}+\dfrac{2}{(p-2)^2}$
2. $\dfrac{2}{(x-y)^2}-\dfrac{3}{y^2-x^2}$
3. $\dfrac{2}{t^2+t-6}+\dfrac{2}{t^2-9}+\dfrac{1}{t-3}$

The full solutions are at the end of the unit.

Simplify more complex algebraic expressions

Sometimes you will need to add and subtract algebraic fractions that are themselves in the numerator or denominator of other fractions. When faced with these more complicated questions, it is important to simplify any algebraic fractions in these numerators and denominators first before tackling the overall fraction. Have a look at the next example.

Example 2.3

Simplify the following:

$$\frac{y+\dfrac{1}{x}}{\dfrac{y}{x}}$$

Solution

In this case, we need to add the two fractions in the overall numerator first. So, let's just focus on $y+\dfrac{1}{x}$:

$$\begin{aligned} & y+\frac{1}{x} \\ & \text{LCD: } x \\ & =\frac{xy}{x}+\frac{1}{x} \\ & =\frac{xy+1}{x} \end{aligned}$$

Now that we have simplified the numerator of the overall fraction, we can start to simplify the fraction. We know from unit 1 [insert link] that we can deal with a fraction divided by a fraction by multiplying

by the reciprocal of the fraction in the denominator.

$$\frac{y+\frac{1}{x}}{\frac{y}{x}} = \frac{\frac{xy+1}{x}}{\frac{y}{x}}$$

$$= \frac{xy+1}{\cancel{x}} \times \frac{\cancel{x}}{y} \qquad \text{'Cancel' the factors}$$

$$= \frac{(xy+1)}{y}$$

Example 2.4

Simplify the following:

$$\frac{\frac{2}{x+1}+\frac{2}{x-1}}{\frac{x-1}{x+1}}$$

Solution

We will start by adding the fractions in the numerator.

$$\frac{2}{x+1}+\frac{2}{x-1}$$

$$\text{LCD: } (x+1)(x-1)$$

$$= \frac{2(x-1)}{(x+1)(x-1)} + \frac{2(x+1)}{(x+1)(x-1)}$$

$$= \frac{2x-2+2x+2}{(x+1)(x-1)}$$

$$= \frac{4x}{(x+1)(x-1)}$$

Now, we can continue simplifying the overall fraction.

$$\frac{\frac{2}{x+1}+\frac{2}{x-1}}{\frac{x-1}{x+1}} = \frac{\frac{4x}{(x+1)(x-1)}}{\frac{(x-1)}{(x+1)}}$$

$$= \frac{4x}{\cancel{(x+1)}\,(x-1)} \times \frac{\cancel{(x+1)}}{(x-1)}$$

$$= \frac{4x}{(x-1)^2}$$

Exercise 2.3

Simplify the following:

1. $\dfrac{\dfrac{6}{y}-\dfrac{4}{x}}{y}$
2. $\dfrac{\dfrac{3}{a}+\dfrac{b}{6}}{\dfrac{2b}{3a}}$
3. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a+b}{ab}}$
4. $\dfrac{\dfrac{c}{c+2}+\dfrac{c-1}{c+1}}{\dfrac{2c-1}{c+1}}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to add and subtract algebraic fractions by factorising the denominators in order to find the LCD.
- How to simplify more complex algebraic fractions with other algebraic fractions in the numerator and/or denominator.

Unit 2: Assessment

Suggested time to complete: 40 minutes

1. Simplify the following:
   a. $\dfrac{x}{x+1}-\dfrac{y}{y+1}$
   b. $\dfrac{3z}{z+1}-\dfrac{2z+5}{z-2}$
   c. $\dfrac{4x}{4x^2-25}+\dfrac{3x}{2x^2+11x+15}$
   d. $\dfrac{8}{a^2-1}-\dfrac{6}{a^2+2a+1}+\dfrac{3}{a^2-2a+1}$
2. Simplify the following:

a. $\dfrac{\dfrac{2x}{3}+\dfrac{4x}{7}}{\dfrac{x}{2}}$

b. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}}$

c. $\dfrac{\dfrac{4a+1}{2a-3}-\dfrac{2a-3}{2a+3}}{\dfrac{a}{4a^2-9}}$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1. $\dfrac{2}{xy}+\dfrac{4}{xz}+\dfrac{3}{yz}$
   $\text{LCD: } xyz$
   $=\dfrac{2z}{xyz}+\dfrac{4y}{xyz}+\dfrac{3x}{xyz}$
   $=\dfrac{2z+4y+3x}{xyz}$
2. $\dfrac{4}{a+1}-\dfrac{5}{a-3}$
   $\text{LCD: } (a+1)(a-3)$
   $=\dfrac{4(a-3)}{(a+1)(a-3)}-\dfrac{5(a+1)}{(a+1)(a-3)}$
   $=\dfrac{4(a-3)-5(a+1)}{(a+1)(a-3)}$
   $=\dfrac{4a-12-5a-5}{(a+1)(a-3)}$
   $=\dfrac{-a-17}{(a+1)(a-3)}$
   $=\dfrac{-(a+17)}{(a+1)(a-3)}$
3.

$$\frac{x-1}{x+1}-\frac{2x+3}{2x+2}$$
$$=\frac{x-1}{x+1}-\frac{2x+3}{2(x+1)}$$
LCD: $2(x+1)$
$$=\frac{2(x-1)}{2(x+1)}-\frac{(2x+3)}{2(x+1)}$$
$$=\frac{2(x-1)-(2x+3)}{2(x+1)}$$

Write 2nd numerator in brackets so you don't make an error with the $-$ sign

$$=\frac{2x-1-2x-3}{2(x+1)}$$
$$=\frac{-4}{2(x+1)}$$
$$=\frac{-2}{(x+1)}$$

4.

$$\frac{4p}{p+1}-\frac{p+1}{4p}$$
LCD: $4p(p+1)$
$$=\frac{4p(4p)}{4p(p+1)}-\frac{(p+1)(p+1)}{4p(p+1)}$$
$$=\frac{16p^2-(p^2+2p+1)}{4p(p+1)}$$
$$=\frac{16p^2-p^2-2p-1}{4p(p+1)}$$
$$=\frac{15p^2-2p-1}{4p(p+1)}$$
$$=\frac{(5p+1)(3p-1)}{4p(p+1)}$$

Back to Exercise 2.1

Exercise 2.2

1.

$$\frac{3}{p^2-4}+\frac{2}{(p-2)^2}$$
$$=\frac{3}{(p+2)(p-2)}+\frac{2}{(p-2)^2}$$
LCD: $(p+2)(p-2)^2$
$$=\frac{3(p-2)}{(p+2)(p-2)^2}+\frac{2(p+2)}{(p+2)(p-2)^2}$$
$$=\frac{3(p-2)+2(p+2)}{(p+2)(p-2)^2}$$
$$=\frac{3p-6+2p+4}{(p+2)(p-2)^2}$$
$$=\frac{5p-2}{(p+2)(p-2)^2}$$

2.

$$\frac{2}{(x-y)^2}-\frac{3}{y^2-x^2}$$
$$=\frac{2}{(x-y)^2}-\frac{3}{(y+x)(y-x)}$$
$$=\frac{2}{(x-y)^2}-\frac{3}{(x+y)\left(-(-y+x)\right)}$$ Change $(y-x)$ into $(x-y)$ by taking out a factor of -1
$$=\frac{2}{(x-y)^2}+\frac{3}{(x+y)(x-y)}$$

LCD: $(x+y)(x-y)^2$

$$=\frac{2(x+y)}{(x-y)^2(x+y)}+\frac{3(x-y)}{(x+y)(x-y)^2}$$
$$=\frac{2(x+y)+3(x-y)}{(x-y)^2(x+y)}$$
$$=\frac{2x+2y+3x-3y}{(x-y)^2(x+y)}$$
$$=\frac{5x-y}{(x-y)^2(x+y)}$$

3.

$$\frac{2}{t^2+t-6}+\frac{2}{t^2-9}+\frac{1}{t-3}$$
$$=\frac{3}{(t-2)(t+3)}+\frac{2}{(t+3)(t-3)}+\frac{1}{(t-3)}$$

LCD: $(t-2)(t+3)(t-3)$

$$=\frac{2(t-3)}{(t-2)(t+3)(t-3)}+\frac{2(t-2)}{(t-2)(t+3)(t-3)}+\frac{1(t-2)(t+3)}{(t-2)(t+3)(t-3)}$$
$$=\frac{2t-6+2t-4+(t^2+t-6)}{(t-2)(t+3)(t-3)}$$
$$=\frac{2t-6+2t-4+t^2+t-6}{(t-2)(t+3)(t-3)}$$
$$=\frac{t^2+5t-16}{(t-2)(t+3)(t-3)}$$ Quadratic cannot be factorised with current techniques

Back to Exercise 2.2

Exercise 2.3

1.

$$\frac{6}{y}-\frac{4}{x}$$

LCD: xy

$$=\frac{6x}{xy}-\frac{4y}{xy}$$
$$=\frac{6x-4y}{xy}$$
$$=\frac{2(3x-2y)}{xy}$$

Therefore:

$$\frac{\frac{6}{y}-\frac{4}{x}}{y}=\frac{\frac{2(3x-2y)}{xy}}{y}$$
$$=\frac{2(3x-2y)}{xy}\times\frac{1}{y}$$
$$=\frac{2(3x-2y)}{xy^2}$$

2.

$$\frac{3}{a}+\frac{b}{6}$$
LCD: $6a$
$$=\frac{18+ab}{ab}$$
Therefore:
$$\frac{\frac{3}{a}+\frac{b}{6}}{\frac{2b}{3a}}=\frac{\frac{18+ab}{ab}}{\frac{2a}{3b}}$$
$$=\frac{18+ab}{a\cancel{b}}\times\frac{3\cancel{b}}{2a}$$
$$=\frac{3(18+ab)}{2a^2}$$

3.

$$\frac{a}{b}-\frac{b}{a}$$
LCD: ab
$$=\frac{a^2}{ab}-\frac{b^2}{ab}$$
$$=\frac{(a+b)(a-b)}{ab}$$
Therefore:
$$\frac{\frac{a}{b}-\frac{b}{a}}{\frac{a+b}{ab}}=\frac{\frac{(a+b)(a-b)}{ab}}{\frac{a+b}{ab}}$$
$$=\frac{\cancel{(a+b)}\,(a-b)}{\cancel{ab}}\times\frac{\cancel{ab}}{\cancel{(a+b)}}$$
$$=a-b$$

4.

$$\frac{c}{c+2}-\frac{c-1}{c+1}$$
LCD: $(c+2)(c+1)$
$$=\frac{c(c+1)}{(c+2)(c+1)}-\frac{(c-1)(c+2)}{(c+2)(c+1)}$$
$$=\frac{c^2+c-c^2-c+2}{(c+2)(c+1)}$$
$$=\frac{2}{(c+2)(c+1)}$$
Therefore:

$$\frac{\frac{c}{c+2}+\frac{c-1}{c+1}}{\frac{2c-1}{c+1}} = \frac{\frac{2}{(c+2)(c+1)}}{\frac{2c-1}{c+1}}$$
$$= \frac{2}{(c+2)\cancel{(c+1)}} \times \frac{\cancel{(c+1)}}{(2c-1)}$$
$$= \frac{2}{(c+2)(2c-1)}$$

Back to Exercise 2.3

Unit 2: Assessment

1.

a.

$$\frac{x}{x+1}-\frac{y}{y+1}$$
$$\text{LCD: } (x+1)(y+1)$$
$$= \frac{x(y+1)}{(x+1)(y+1)} - \frac{y(x+1)}{(x+1)(y+1)}$$
$$= \frac{xy+x-xy-y}{(x+1)(y+1)}$$
$$= \frac{x-y}{(x+1)(y+1)}$$

b.

$$\frac{3z}{z+1}-\frac{2z+5}{z-2}$$
$$\text{LCD: } (z+1)(z-2)$$
$$= \frac{3z(z-2)}{(z+1)(z-2)} - \frac{(2z+5)(z+1)}{(z+1)(z-2)}$$
$$= \frac{3z^2-6z-(2z^2+7z+5)}{(z+1)(z-2)}$$
$$= \frac{3z^2-6z-2z^2-7z-5}{(z+1)(z-2)}$$
$$= \frac{z^2-13z-5}{(z+1)(z-2)}$$

c.

$$\frac{4x}{4x^2-25}+\frac{3x}{2x^2+11x+15}$$
$$=\frac{4x}{(2x+5)(2x-5)}+\frac{3x}{(2x+5)(x+3)}$$
LCD: $(2x+5)(2x-5)(x+3)$
$$=\frac{4x(x+3)}{(2x+5)(2x-5)(x+3)}+\frac{3x(2x-5)}{(2x+5)(2x-5)(x+3)}$$
$$=\frac{4x^2+12x+6x^2-15x}{(2x+5)(2x-5)(x+3)}$$
$$=\frac{10x^2-3x}{(2x+5)(2x-5)(x+3)}$$
$$=\frac{x(10x-3)}{(2x+5)(2x-5)(x+3)}$$

d.

$$\frac{8}{a^2-1}-\frac{6}{a^2+2a+1}+\frac{3}{a^2-2a+1}$$
$$=\frac{8}{(a+1)(a-1)}-\frac{6}{(a+1)(a+1)}+\frac{3}{(a-1)(a-1)}$$
LCD: $(a+1)^2(a-1)^2$
$$=\frac{8(a+1)(a-1)}{(a+1)^2(a-1)^2}-\frac{6(a-1)^2}{(a+1)^2(a-1)^2}+\frac{3(a+1)^2}{(a+1)^2(a-1)^2}$$
$$=\frac{8(a^2-1)-6(a^2-2a+1)+3(a^2+2a+1)}{(a+1)^2(a-1)^2}$$
$$=\frac{8a^2-8-6a^2+12a-6+3a^2+6a+3}{(a+1)^2(a-1)^2}$$
$$=\frac{5a^2+18a-11}{(a+1)^2(a-1)^2}$$

2.

a.

$$\frac{2x}{3}+\frac{4x}{7}$$
LCD: 21
$$=\frac{14x}{21}+\frac{12x}{21}$$
$$=\frac{12x}{21}$$
Therefore:
$$\frac{\frac{2x}{3}+\frac{4x}{7}}{\frac{x}{2}}=\frac{\frac{12x}{21}}{\frac{x}{2}}$$
$$=\frac{12\cancel{x}}{21}\times\frac{2}{\cancel{x}}$$
$$=\frac{24}{21}$$
$$=\frac{8}{7}$$

b.

$$\frac{x}{y}-\frac{y}{x}$$

LCD: xy

$$=\frac{x^2}{xy}-\frac{y^2}{xy}$$

$$=\frac{(x+y)(x-y)}{xy}$$

$$\frac{x}{y}+\frac{y}{x}$$

LCD: xy

$$=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$=\frac{x^2+y^2}{xy}$$

Therefore:

$$\frac{\frac{x}{y}-\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}=\frac{\frac{(x+y)(x-y)}{xy}}{\frac{x^2+y^2}{xy}}$$

$$=\frac{(x+y)(x-y)}{\cancel{xy}}\times\frac{\cancel{xy}}{x^2+y^2}$$

$$=\frac{(x+y)(x-y)}{x^2+y^2}$$

c.

$$\frac{4a+1}{2a-3}-\frac{2a-3}{2a+3}$$

LCD: $(2a-3)(2a+3)$

$$=\frac{(4a+1)(2a+3)}{(2a-3)(2a+3)}-\frac{(2a-3)(2a-3)}{(2a-3)(2a+3)}$$

$$=\frac{8a^2+14a+3-4a^2+12a-9}{(2a-3)(2a+3)}$$

$$=\frac{4a^2+26a-6}{(2a-3)(2a+3)}$$

$$=\frac{2(2a^2+13a-3)}{(2a-3)(2a+3)}$$

Therefore:

$$\frac{\frac{4a+1}{2a-3}+\frac{2a-3}{2a+3}}{\frac{a}{4a^2-9}}=\frac{\frac{2(2a^2+13a-3)}{(2a-3)(2a+3)}}{\frac{a}{4a^2-9}}$$

$$=\frac{2(2a^2+13a-3)}{(2a-3)(2a+3)}\times\frac{4a^2-9}{a}$$

$$=\frac{2(2a^2+13a-3)}{\cancel{(2a-3)}\ \cancel{(2a+3)}}\times\frac{\cancel{(2a+3)}\ \cancel{(2a-3)}}{a}$$

$$=\frac{2(2a^2+13a-3)}{a}$$

Back to Unit 2: Assessment

Unit 3: Completing the square

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Manipulate and simplify algebraic expressions by completing the square.

What you should know

Before you start this unit, make sure you can:

- Manipulate and simplify algebraic expressions. Refer to level 2 subject outcome 2.2 unit 1 if you need help with this.
- Simplify algebraic expressions by means of factorisation. Refer to level 2 subject outcome 2.2 unit 2 and unit 1 in this subject outcome if you need help with this.
- Add and subtract algebraic fractions. Refer to level 2 subject outcome 2.2 unit 3 if you need help with this.

Introduction

Completing the square is a powerful technique that has many useful applications. It can be used to:

- simplify algebraic expressions
- solve quadratic equations, especially when the roots are either irrational or non-real
- find the turning point of a quadratic function by converting the quadratic function from standard form $f(x) = ax^2 + bx + c$ into the turning point form $f(x) = a(x + p)^2 + q$.

The technique was first developed by the great Persian mathematician and 'Father of Algebra' Muḥammad ibn Mūsā al-Khwārizmī as a way to solve quadratic equations that did not factorise easily.

The technique is called **completing the square** because it literally completes the square.

Completing the square

Suppose we have the quadratic expression $x^2 + 6x + 12$. You will notice that this quadratic does not factorise nicely at all. But we can break it up into its components and represent these visually (see Figure 1).

$$x^2 + 6x + 12$$

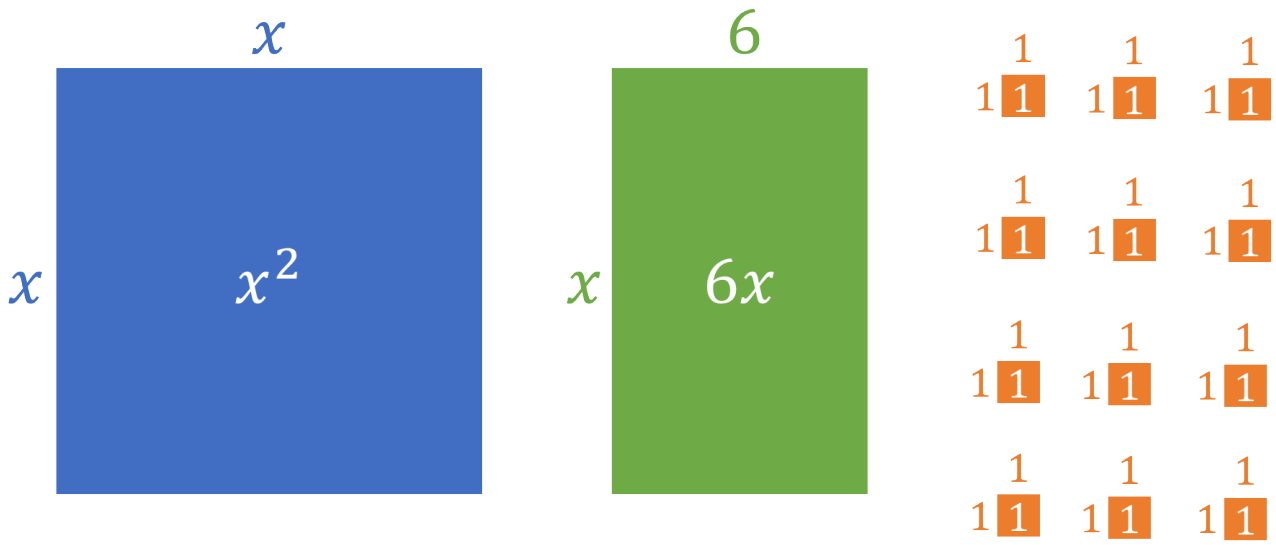


Figure 1: Visual representation of the components of the algebraic expression $x^2 + 6x + 12$

Now, if we cut the green $6x$ rectangle in **half** to make two $3x$ rectangles, we can lay these pieces as shown in Figure 2.

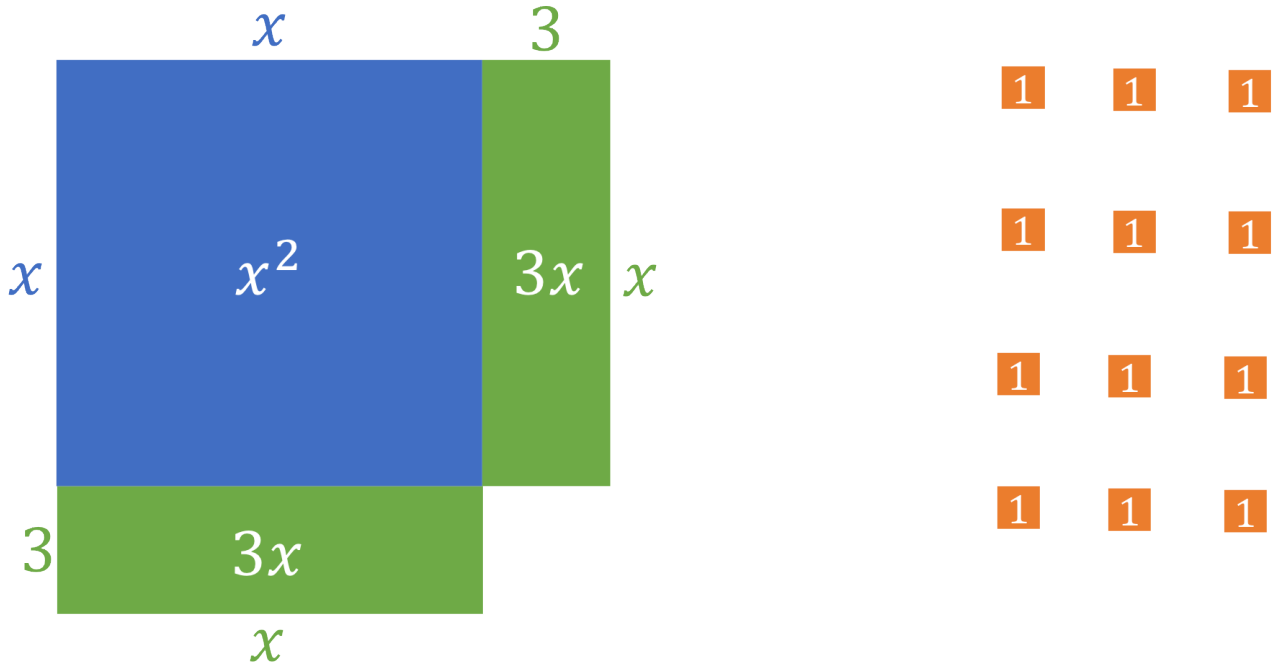


Figure 2: Arranging the $6x$ *term to help us 'complete the square'*

But we have a missing piece of the square that is a 3×3 hole in the bottom right-hand corner. We can fill it with nine of the 1×1 blocks (see Figure 3).

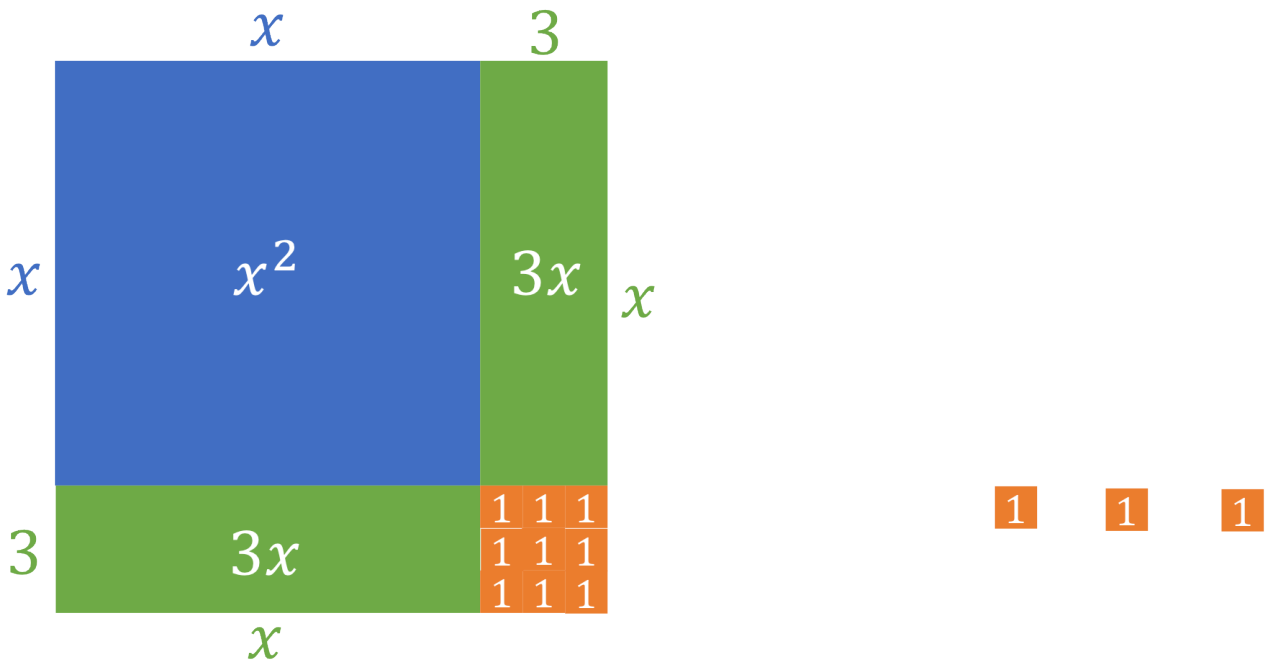


Figure 3: Filling the hole with nine unit blocks

In this way we have 'completed the square'. But what are the dimensions of this square? Well, they are $(x+3) \times (x+3)$ or $(x+3)^2$.

So, our original expression $x^2+6x+12$ can be written as $(x+3)^2+3$ (don't forget the three 1×1 blocks).

Let's have another look at what we did, but this time algebraically.

Step 1: Identify the coefficient of the x term.
This is 6.

Step 2: Halve the coefficient of the x term.
$\frac{6}{2}=3$

Step 3: Square the halved coefficient of the x term.
$\left(\frac{6}{2}\right)^2=3^2=9$

Step 4: Add this new term to the expression.
$x^2+6x+9+12$

Step 5: To keep the value of the expression the same, subtract this new term from the expression as well.
$x^2+6x+9-9+12$

Step 6: Factorise the newly created perfect square quadratic expression.
$$\begin{aligned} & x^2+6x+9-9+12 \\ &= (x+3)(x+3)-9+12 \\ &= (x+3)^2-9+12 \end{aligned}$$

In general, the constant in the squared binomial (the 3 in $(x+3)^2$ in this case) will always be half the coefficient of the x term ($\frac{6}{2}=3$ in this case). You should also expand this squared binomial to make sure you get back to the original quadratic – $(x+3)^2=(x+3)(x+3)=x^2+6x+9$.

Step 7: Collect the constants.

$$
\begin{aligned}
&x^2 + 6x + 9 - 9 + 12 \\
&= (x + 3)(x + 3) - 9 + 12 \\
&= (x + 3)^2 - 9 + 12 \\
&= (x + 3)^2 + 3
\end{aligned}
$$

Note

If you have an internet connection watch these two excellent videos about the process of completing the square:

Completing the Square – A geometric Perspective (Duration: 7.20)

Beautiful Visual Explanation of Completing the Square (Duration: 3.33)

Example 3.1

Complete the square for $x^2 + 4x + 5$.

Solution

Step 1: Identify the coefficient of the x term.
This is 4.

Step 2: Halve the coefficient of the x term.

$$\frac{4}{2} = 2$$

Step 3: Square the halved coefficient of the x term.

$$\left(\frac{4}{2}\right)^2 = 2^2 = 4$$

Step 4: Add this new term to the expression and, to keep the value of the expression the same, subtract

this new term from the expression as well.
$x^2 + 4x + 4 - 4 + 5$

Step 5: Factorise the newly created perfect square quadratic expression.
$x^2 + 4x + 4 - 4 + 5$
$= (x + 2)(x + 2) - 4 + 5$
$= (x + 2)^2 - 4 + 5$

Step 6: Collect the constants.
$(x + 2)^2 - 4 + 5$
$= (x + 2)^2 + 1$



Example 3.2

Complete the square for $x^2 - 4x + 5$.

Solution

Step 1: Identify the coefficient of the x term.
This is -4.

Step 2: Halve the coefficient of the x term.
$\frac{-4}{2} = -2$

Step 3: Square the halved coefficient of the x term.
$\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$

Step 4: Add this new term to the expression and, to keep the value of the expression the same, subtract this new term from the expression as well.
$x^2 - 4x + 4 - 4 + 5$

Step 5: Factorise the newly created perfect square quadratic expression.
$x^2 - 4x + 4 - 4 + 5$
$= (x - 2)(x - 2) - 4 + 5$ Note how the perfect square quadratic factorises
$= (x - 2)^2 - 4 + 5$

Step 6: Collect the constants.
$(x - 2)^2 - 4 + 5$
$= (x - 2)^2 + 1$



Example 3.3

Complete the square for $x^2 - 3x - 5$.

Solution

Step 1: Identify the coefficient of the x term.
This is -3.

Step 2: Halve the coefficient of the x term.
$\frac{-3}{2}$

Step 3: Square the halved coefficient of the x term.
$\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$

Step 4: Add this new term to the expression and, to keep the value of the expression the same, subtract this new term from the expression as well.
$x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 5$

Step 5: Factorise the newly created perfect square quadratic expression.
$x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 5 = \left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) - \frac{9}{4} - 5$
$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 5$

Note how the perfect square quadratic factorises

Step 6: Collect the constants.
$\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 5 = \left(x - \frac{3}{2}\right)^2 - \frac{9 - 20}{4}$
$= \left(x - \frac{3}{2}\right)^2 - \frac{29}{4}$

Example 3.4

Complete the square for $2x^2 - 3x + 4$.

Solution

Before we can start to complete the square, we need to get the coefficient of x^2 to be 1. We need to take out a common factor of 2 from each term.
$2x^2 - 3x + 4$
$= 2\left(x^2 - \frac{3}{2}x + 2\right)$

Step 1: Identify the coefficient of the x term.
This is $-\frac{3}{2}$.

Step 2: Halve the coefficient of the x term.
$\frac{\frac{-3}{2}}{2} = -\frac{3}{2} \times \frac{1}{2} = -\frac{3}{4}$

Step 3: Square the halved coefficient of the x term.
$\left(\frac{-3}{4}\right)^2 = \frac{9}{16}$

Step 4: Add this new term to the expression and, to keep the value of the expression the same, subtract this new term from the expression as well.

$$2\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+2\right)$$

Step 5: Before we factorise the perfect square we have created, we need to take the $-\frac{9}{16}$ and 2 terms out of the brackets. But when we do so, we need to remember to multiply them by 2.

$$2\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+2\right)=2\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)-\frac{9}{8}+4$$

Step 6: Factorise the newly created perfect square quadratic expression.

$$2\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)-\frac{9}{8}+4=2\left(x-\frac{3}{4}\right)\left(x-\frac{3}{4}\right)-\frac{9}{8}+4$$

Note how the perfect square quadratic factorises

$$=2\left(x-\frac{3}{4}\right)^2-\frac{9}{8}+4$$

Step 7: Collect the constants.

$$2\left(x-\frac{3}{4}\right)^2-\frac{9}{8}+4=2\left(x-\frac{3}{2}\right)^2-\frac{9+32}{8}$$

$$=2\left(x-\frac{3}{2}\right)^2+\frac{23}{8}$$

Exercise 3.1

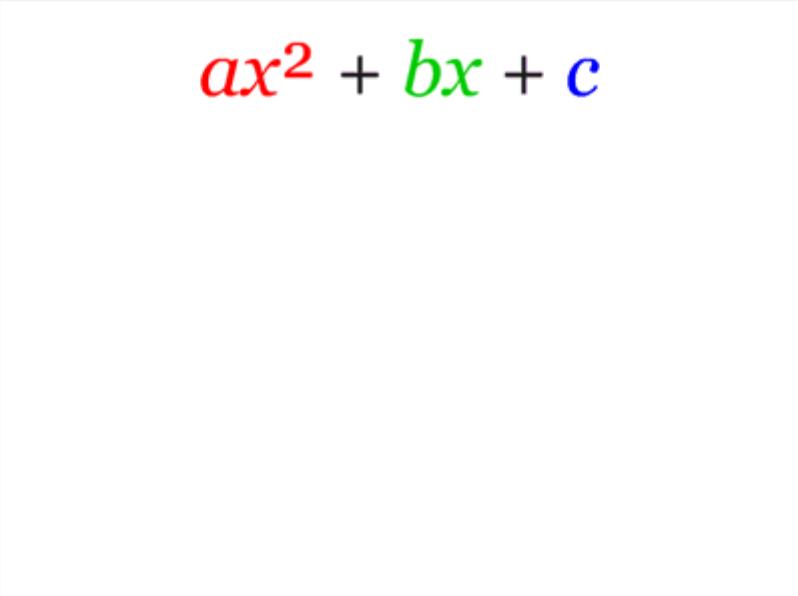
Complete the square for the following expressions:

1. x^2+6x
2. $x^2+\frac{5}{7}x$
3. $x^2-\frac{5}{7}x-1$
4. x^2-2x+6
5. $5x^2+10x+15$
6. $9+6x-3x^2$

The full solutions are at the end of the unit.

Note

For a generalised summary of the process of completing the square of any quadratic expression, watch this animation.



https://en.wikipedia.org/wiki/File:Completing_the_square.gif – Public domain

Summary

In this unit you have learnt the following:

- How to complete the square of any quadratic expression $ax^2 + bx + c$.

Unit 3: Assessment

Suggested time to complete: 25 minutes

Complete the square for the following expressions:

1. $x^2 + 5x - 6$
2. $x^2 - 4x - 8$
3. $x^2 + 6x - 4$
4. $10x^2 - 7x - 12$
5. $2 - 12x - 3x^2$
6. $ax^2 + bx + c$

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.

$$x^2 + 6x \qquad \left[\left(\frac{b}{2}\right)^2 = 9\right]$$
$$= x^2 + 6x + 9 - 9$$
$$= (x+3)^2 - 9$$

2.

$$x^2 + \frac{5}{7}x \qquad \left[\left(\frac{b}{2}\right)^2 = \left(\frac{5}{14}\right)^2 = \frac{25}{196}\right]$$
$$= x^2 + \frac{5}{7}x + \frac{25}{196} - \frac{25}{196}$$
$$= \left(x + \frac{5}{14}\right)^2 - \frac{25}{196}$$

3.

$$x^2 - \frac{5}{7}x - 1 \qquad \left[\left(\frac{b}{2}\right)^2 = \left(-\frac{5}{14}\right)^2 = \frac{25}{196}\right]$$
$$= x^2 - \frac{5}{7}x + \frac{25}{196} - \frac{25}{196} - 1$$
$$= \left(x - \frac{5}{14}\right)^2 - \frac{25 - 196}{196}$$
$$= \left(x - \frac{5}{14}\right)^2 - \frac{221}{196}$$

4.

$$x^2 - 2x + 6 \qquad \left[\left(\frac{b}{2}\right)^2 = (-1)^2 = 1\right]$$
$$= x^2 - 2x + 1 - 1 + 6$$
$$= (x-1)^2 + 5$$

5.

$$5x^2 + 10x + 15$$
$$= 5(x^2 + 2x + 3) \qquad \left[\left(\frac{b}{2}\right)^2 = 1\right]$$
$$= 5(x^2 + 2x + 1 - 1 + 3)$$
$$= 5(x^2 + 2x + 1) - 5 + 15$$ Remember to multiply the constants by the common factor when you take them out of the bracket
$$= 5(x+1)^2 + 10$$

6.

$$9 + 6x - 3x^2$$
$$= -3(-3 - 2x + x^2)$$
$$= -3(x^2 - 2x - 3) \qquad \left[\left(\frac{b}{2}\right)^2 = 1\right]$$
$$= -3(x^2 - 2x + 1 - 1 - 3)$$
$$= -3(x^2 - 2x + 1) + 3 + 9$$ Remember to multiply the constants by -3
$$= -3(x-1)^2 + 12$$

Back to Exercise 3.1

Unit 3: Assessment

1.

$x^2 + 5x - 6 \qquad \left[\left(\frac{b}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}\right]$

$= x^2 + 5x + \frac{25}{4} - \frac{25}{4} - 6$

$= \left(x + \frac{5}{2}\right)^2 - \frac{25 - 24}{4}$

$= \left(x + \frac{5}{2}\right)^2 - \frac{49}{4}$

2.

$x^2 - 4x - 8 \qquad \left[\left(\frac{b}{2}\right)^2 = (-2)^2 = 4\right]$

$= x^2 - 4x + 4 - 4 - 8$

$= (x - 2)^2 - 12$

3.

$x^2 + 6x - 4 \qquad \left[\left(\frac{b}{2}\right)^2 = 9\right]$

$= x^2 + 6x + 9 - 9 - 4$

$= (x + 3)^2 - 13$

4.

$10x^2 - 7x - 12$

$= 10\left(x^2 - \frac{7}{10}x\right) - 12 \qquad \left[\left(\frac{b}{2}\right)^2 = \left(\frac{7}{20}\right)^2 = \frac{49}{400}\right]$

Instead of taking the common factor from -12 we can leave it outside the bracket

$= 10\left(x - \frac{7}{10}x + \frac{49}{400} - \frac{49}{400}\right) - 12$

$= 10\left(x - \frac{7}{10}x + \frac{49}{400}\right) - \left(\frac{49}{40\cancel{0}} \times \frac{\cancel{10}}{1}\right) - 12$

$= 10\left(x - \frac{7}{10}x + \frac{49}{400}\right) - \frac{49}{10} - 12$

$= 10\left(x - \frac{7}{20}\right)^2 - \frac{49 - 480}{40}$

$= 10\left(x - \frac{7}{20}\right)^2 - \frac{529}{40}$

5.

$2 - 12x - 3x^2 \qquad$ Take out a common factor of -3

$= 2 - 3(4x + x^2)$

$= -3(x^2 + 4x) + 2 \qquad \left[\left(\frac{b}{2}\right)^2 = 4\right]$

$= -3(x^2 + 4x + 4 - 4) + 2$

$= -3(x^2 + 4x + 4) + 12 + 2$

$= -3(x + 2)^2 + 14$

6.

$$
\begin{aligned}
& ax^2 + bx + c \\
& = a\left(x^2 + \frac{b}{a}x\right) + c \qquad \left[\left(\frac{b}{2}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}\right] \\
& = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
& = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c \\
& = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
& = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\
& = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{-b^2 + 4ac}{4a}\right)
\end{aligned}
$$

Back to Unit 3: Assessment

Media Attributions

- figure1 © DHET is licensed under a CC BY (Attribution) license
- figure2 © DHET is licensed under a CC BY (Attribution) license
- figure3 © DHET is licensed under a CC BY (Attribution) license
- Completing_the_square

SUBJECT OUTCOME IV

FUNCTIONS AND ALGEBRA: SOLVE ALGEBRAIC EQUATIONS AND INEQUALITIES

Subject outcome

Subject outcome 2.3: Solve algebraic equations and inequalities

Learning outcomes

- Solve quadratic equations by means of:
  - Factorisation
  - Completing the square
  - Using the quadratic formula.
- Solve simultaneous equations with two unknowns algebraically and graphically, where the one equation is linear and the other equation is quadratic.
- Solve quadratic inequalities in one variable and represent the solution in the following ways:
  - In set builder notation
  - Interval notation
  - On the number line.

Unit 1 outcomes

By the end of this unit you will be able to:

- Apply the zero-product law.
- Solve rational quadratic equations which have fractions with the unknown in the denominator.
- Factorise quadratic expressions to solve for the unknown variable.

Unit 2 outcomes

By the end of this unit you will be able to:

- Solve a quadratic equation by completing the square.



Unit 3 outcomes

By the end of this unit you will be able to:

- Understand when to use the quadratic formula to solve quadratic equations.
- Apply the quadratic formula when necessary.



Unit 4 outcomes

By the end of this unit you will be able to:

- Solve simultaneous equations algebraically where the one equation is linear and the other equation is quadratic.
- Solve simultaneous equations graphically where the one equation is linear and the other equation is quadratic.



Unit 5 outcomes

By the end of this unit you will be able to:

- Find the critical values.
- Solve quadratic inequalities using the table and graphical methods.
- Represent the solution using set builder notation, interval notation and on a number line.

Unit 1: Solve quadratic equations by factorisation

DYLAN BUSA



Unit outcomes

By the end of this unit you will be able to:

- Apply the zero-product law.
- Solve rational quadratic equations which have fractions with the unknown in the denominator.
- Factorise quadratic expressions to solve for the unknown variable.

What you should know

Before you start this unit, make sure you can:

- Solve simple quadratic equations by factorisation. Refer to level 2 subject outcome 2.3 unit 1 if you need help with this.

Introduction

In this unit we expand on what you learnt about quadratic equations in level 2 by looking at some slightly more complex quadratic equations.

Remember a quadratic equation is an equation which can be written as $ax^2 + bx + c = 0$ where a, b and c are constants and the highest power on x is 2. $ax^2 + bx + c = 0$ is called the **standard form** of the quadratic equation.

Here are some examples of quadratic equations:

- $2x^2 + 5x + 2 = 0$ This is in standard form
- $a(a - 3) = 10$ This needs to be expanded to get the equation into standard form
- $\frac{4}{x+1} - 1 = \frac{3x}{x+2}$ We must multiply through by the LCD and simplify into standard form

In level 2 we learnt that to solve a quadratic equation we need to follow some basic steps.



Take note!

Steps to solve a quadratic equation:

1. Get one side of the equation equal to zero (write the quadratic in standard form).
2. Factorise the quadratic expression on the one side of the equation.
3. Apply the **zero product property** to set each factor equal to zero.
4. Solve for x in each factor.
5. Check your solutions.

No matter how complicated the quadratic equation seems to be, these are always these steps to follow.

Note

Sometimes we refer to the solution of an equation as the **root of the equation**. Because quadratic equations always give two solutions, quadratic equations have at most two roots.

Solve quadratic equations by factorisation

We know that to solve a quadratic equation we need to apply the zero product property. The zero product property says that if $A \times B = 0$ then $A = 0$ or $B = 0$ or both $A = 0$ and $B = 0$. To get a quadratic equation into the form $A \times B = 0$, we need to find the factors of the quadratic expression on the left-hand side of the equation. Often, these can be found by factoristing the quadratic expression. This is always the first thing you should try.

The zero product property:

if $A \times B = 0$ then $A = 0$ or $B = 0$ or both $A = 0$ and $B = 0$



Example 1.1

Solve for x: $2x^2 + 5x + 2 = 0$.

Solution

Step 1: Get equation into standard form
The equation is already in standard form – $ax^2 + bx + c = 0$.

Step 2: Factorise the quadratic expression

$$2x^2 + 5x + 2 = 0$$
$$\therefore (2x + 1)(x + 2) = 0$$

Step 3: Apply the zero product property

$$(2x + 1)(x + 2) = 0$$
$$\therefore (2x + 1) = 0 \quad \text{or} \quad (x + 2) = 0$$

Step 4: Solve for x in each factor

$$(2x+1)=0 \quad \text{or} \quad (x+2)=0$$
$$\therefore 2x=-1 \quad \text{or} \quad x=-2$$
$$\therefore x=-\frac{1}{2} \quad \text{or} \quad x=-2$$

Step 5: Check solutions

It is always a good idea to check the solutions you get. This is not presented as part of your solution but should be done as an aside.

$x=-\frac{1}{2}$:

$$\begin{aligned}\text{LHS} &= 2\left(-\frac{1}{2}\right)^2+5\left(-\frac{1}{2}\right)+2\\ &= 2\times\frac{1}{4}-\frac{5}{2}+2\\ &= \frac{1}{2}-\frac{5}{2}+\frac{4}{2}\\ &= \frac{0}{2}\\ &= 0\\ &= \text{RHS}\end{aligned}$$

$x=-2$:

$$\begin{aligned}\text{LHS} &= 2(-2)^2+5(-2)+2\\ &= 2\times 4-10+2\\ &= 8-10+2\\ &= 0\\ &= \text{RHS}\end{aligned}$$

Both solutions are valid. Therefore, $x=-\frac{1}{2}$ or $x=-2$.

Note: It is always best to test your solutions with the original equation.



Example 1.2

Solve for a: $a(a-3)=10$

Solution

Step 1: Get equation into standard form

$$a(a-3)=10$$
$$\therefore a^2-3a=10$$
$$\therefore a^2-3a-10=0$$

Step 2: Factorise the quadratic expression

$$a^2-3a-10=0$$
$$\therefore (a-5)(a+2)=0$$

Step 3: Apply the zero product property

$(a-5)(a+2)=0$

$\therefore (a-5)=0 \quad \text{or} \quad (a+2)=0$

Step 4: Solve for a in each factor

$(a-5)=0 \quad \text{or} \quad (x+2)=0$

$\therefore a=5 \quad \text{or} \quad a=-2$

Step 5: Check solutions

$a=5$:

$$\begin{aligned}\text{LHS} &= 5(5-3)\\ &= 10\\ &= \text{RHS}\end{aligned}$$

$x=-2$:

$$\begin{aligned}\text{LHS} &= (-2)\left((-2)-3\right)\\ &= (-2)(-5)\\ &= 10\\ &= \text{RHS}\end{aligned}$$

Both solutions are valid. Therefore, $a=5 \text{ or } \quad a=-2$.

Note: Remember, it is always best to test your solutions with the original equation.



Example 1.3

Solve for x: $\dfrac{4}{x+1}-1=\dfrac{3x}{x+2}$

Solution

Step 1: Get equation into standard form

$$\frac{4}{x+1}-1=\frac{3x}{x+2} \quad \text{LCD: } (x+1)(x+2)$$

$$\therefore 4(x+2)-(x+1)(x+2)=3x(x+1)$$

$$\therefore 4x+8-(x^2+3x+2)=3x^2+3x$$

$$\therefore 4x+8-x^2-3x-2-3x^2-3x=0$$

$$\therefore -4x^2-2x+6=0$$

$$\therefore 4x^2+2x-6=0$$

$$\therefore 2(2x^2+x-3)=0$$

Because there are unknowns in the denominator, we need to place restrictions on the solutions that we can accept. We are never permitted to have zero in the denominator. Therefore:

$(x+1)\neq 0 \text{ and } (x+2)\neq 0$

$\therefore x\neq -1 \text{ and } x\neq -2$

Step 2: Factorise the quadratic expression

$2(2x^2+x-3)=0$

$\therefore 2x^2+x-3=0$

$\therefore (2x+3)(x-1)=0$

Step 3: Apply the zero product property

$(2x+3)(x-1)=0$

$\therefore (2x+3)=0 \quad \text{or} \quad (x-1)=0$

Step 4: Solve for x in each factor

$(2x+3)=0 \quad \text{or} \quad (x-1)=0$

$\therefore x=-\frac{3}{2} \quad \text{or} \quad x=1$

Step 5: Check solutions

Neither solution contradicts our restrictions.

$x=1$:

$$\begin{aligned}\text{LHS} &= \frac{4}{x+1}-1 \\ &= \frac{4}{1+1}-1 \\ &= 1 \\ \text{RHS} &= \frac{3x}{x+2} \\ &= \frac{3(1)}{1+2} \\ &= 1 \\ &= \text{LHS}\end{aligned}$$

$x=-\frac{3}{2}$:

$$\begin{aligned}\text{LHS} &= \frac{4}{\left(-\frac{3}{2}\right)+1}-1 \\ &= \frac{4}{\left(-\frac{1}{2}\right)}-1 \\ &= -8-1 \\ &= -9 \\ \text{RHS} &= \frac{3x}{x+2} \\ &= \frac{3\left(-\frac{3}{2}\right)}{\left(-\frac{3}{2}\right)+2} \\ &= \frac{-\frac{9}{2}}{\frac{1}{2}} \\ &= -\frac{9}{2}\times\frac{2}{1} \\ &= -9 \\ &= \text{LHS}\end{aligned}$$

Both solutions are valid. Therefore, $x=1 \text{ or } \quad x=-\frac{3}{2}$.

Exercise 1.1

Solve for the unknown in each case:

1. $6g^2 + 12g = 0$
2. $13y^2 - 26y = -13$
3. $a(2a + 1) = 15$
4. $13t^2 = 325$
5. $\frac{d-2}{d+1} = \frac{2d+1}{d-7}$
6. $\frac{5x}{x-2} + \frac{6}{y^2-2y} = -\frac{3}{y} - 2$

The full solutions are at the end of the unit.

Solve more complex quadratic equations

Let's now work through some examples of how to solve more complex quadratic equations.

Example 1.4

Solve for the unknown: $x^2 = 13$

Solution

Step 1: Get equation into standard form

$$x^2 = 13$$
$$\therefore x^2 - 13 = 0$$

Note: In this case the coefficient of the x term in $x^2 + bx + c$ is zero.

Step 2: Factorise the quadratic expression

We can factorise the quadratic expression using the difference of two squares by recognising that $\left(\sqrt{13}\right)^2 = 13$.

$$x^2 - 13 = 0$$
$$\therefore \left(x + \sqrt{13}\right)\left(x - \sqrt{13}\right)) = 0$$

Step 3: Apply the zero product property

$$\left(x + \sqrt{13}\right)\left(x - \sqrt{13}\right)) = 0$$
$$\therefore \left(x + \sqrt{13}\right) = 0 \quad \text{or} \quad \left(x - \sqrt{13}\right) = 0$$

Step 4: Solve for x in each factor

$$\left(x + \sqrt{13}\right) = 0 \quad \text{or} \quad \left(x - \sqrt{13}\right) = 0$$
$$\therefore x = -\sqrt{13} \quad \text{or} \quad x = \sqrt{13}$$

Step 5: Check solutions

$x = -\sqrt{13}$:

$$\begin{aligned}\text{LHS} &= \left(-\sqrt{13}\right)^2 \\ &= 13 \\ &= \text{RHS}\end{aligned}$$

$x = \sqrt{13}$:

$$\begin{aligned}\text{LHS} &= \left(\sqrt{13}\right)^2 \\ &= 13 \\ &= \text{RHS}\end{aligned}$$

Both solutions are valid. Therefore, $x = -\sqrt{13}$ or $x = \sqrt{13}$.



Example 1.5

Solve for the unknown: $n + 2 = \sqrt{7 + 2n}$.

Solution

Step 1: Get equation into standard form

We need to 'get rid' of the square root on the RHS. Therefore, we need to square both sides of the equation. However, before we do this, it is important to recognise that the value inside the square root sign must be greater than or equal to zero. In other words:

$$7 + 2n \geq 0$$

$$\therefore 2n \geq -7$$

$$\therefore n \geq -\frac{7}{2}$$

$$n + 2 = \sqrt{7 + 2n} \qquad \text{Restriction: } \geq -\frac{7}{2}$$

$$\therefore (n + 2)^2 = \left(\sqrt{7 + 2n}\right)^2 \qquad \text{Make sure you square the whole of the LHS}$$

$$\therefore n^2 + 4n + 4 = 7 + 2n$$

$$\therefore n^2 + 2n - 3 = 0$$

Step 2: Factorise the quadratic expression

$$n^2 + 2n - 3 = 0$$

$$\therefore (n + 3)(n - 1) = 0$$

Step 3: Apply the zero product property

$$(n + 3)(n - 1) = 0$$

$$\therefore (n + 3) = 0 \text{ or } (n - 1) = 0$$

Step 4: Solve for x in each factor

$$(n + 3) = 0 \text{ or } (n - 1) = 0$$

$$\therefore n = -3 \text{ or } n = 1$$

Step 5: Check solutions

You can either check your solutions by testing to see if they make the $\text{LHS} = \text{RHS}$ or you can test them against the original restriction that $n \geq -\frac{7}{2}$.

$n = -3$:

$$\begin{aligned}\text{LHS} &= n+2\\ &= -3+2\\ &= -1\\ \text{RHS} &= \sqrt{7+2n}\\ &= \sqrt{7+2(-3)}\\ &= 1 \neq \text{LHS}\end{aligned}$$

$n = -3$ conflicts with the restriction that $n \geq -\frac{7}{2}$.

$n = 1$:

$$\begin{aligned}\text{LHS} &= n+2\\ &= 1+2\\ &= 3\\ \text{RHS} &= \sqrt{7+2n}\\ &= \sqrt{7+2(1)}\\ &= 3\\ &= \text{LHS}\end{aligned}$$

$n = 1$ does not conflict with the restriction that $n \geq -\frac{7}{2}$.

Only one solution is valid. Therefore, $n = 1$.



Example 1.6

Solve for y: $3\sqrt{2y+1} - 5 = -2(2y+1)$

Solution

The first thing we need to recognise is that there are restrictions because of the square root.

$$2y+1 \geq 0$$
$$\therefore y \geq -\frac{1}{2}$$

$3\sqrt{2y+1} - 5 = -2(2y+1)$ Restriction: $y \geq -\frac{1}{2}$

While we could square both sides to 'get rid' of the square root, a better technique would be to recognise that if $A = \sqrt{2y+1}$ then $(2y+1) = A^2$. Therefore, we can start by doing a substitution to make the equation simpler.

Let $A = \sqrt{2y+1}$

$$3A - 5 = -2A^2$$
$$\therefore 2A^2 + 3A - 5 = 0$$
$$\therefore (2A+5)(A-1) = 0$$
$$\therefore A = -\frac{5}{2} \text{ or } A = 1$$

Now substitute A back for $\sqrt{2y+1}$:

$\sqrt{2y+1} = -\frac{5}{2}$ or $\sqrt{2y+1} = 1$

$\sqrt{2y-1} = -\frac{5}{2}$

$\therefore 2y - 1 = \frac{25}{4}$

$\therefore 2y = \frac{25+4}{4} = \frac{29}{4}$

$\therefore y = \frac{29}{8}$

This does not conflict with the resistriction that $y \geq -\frac{1}{2}$ but we need to check the solution.

$$\begin{aligned}
\text{LHS} &= 3\sqrt{2y+1} - 5 \\
&= 3\sqrt{2\left(\frac{29}{8}\right)+1} - 5 \\
&= 3\sqrt{\frac{29}{4}+\frac{4}{4}} - 5 \\
&= 3\sqrt{\frac{33}{4}} - 5 \\
&= 3\frac{\sqrt{33}}{2} - 5 \\
&= \frac{3\sqrt{33}-10}{2} \\
\text{RHS} &= -2(2y+1) \\
&= -2\left(2\left(\frac{29}{8}\right)+1\right) \\
&= -2\left(\frac{29}{4}+\frac{4}{4}\right) \\
&= -2\left(\frac{33}{4}\right) \\
&= -\frac{66}{4}
\end{aligned}$$

$\text{LHS} \neq \text{RHS}$ so we cannot accept the solution.

$\sqrt{2y+1} = 1$

$\therefore 2y + 1 = 1$

$\therefore y = 0$

Check the solution:

$$\begin{aligned}
\text{LHS} &= 3\sqrt{2y+1} - 5 \\
&= 3\sqrt{1} - 5 \\
&= -2 \\
\text{RHS} &= -2(2y+1) \\
&= -2 \\
&= \text{LHS}
\end{aligned}$$

Therefore, $y = 0$ only.

Exercise 1.2

Solve for the unknowns in the following:

1. $17 = a^2$
2. $4 + a = \sqrt{a + 6}$
3. $2k = \sqrt{5 - 12k}$
4. $5\sqrt{5y + 1} - 4 = 5y + 1$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve quadratic equations using factorisation.
- How to apply the zero product property.
- How to solve quadratic equations involving square roots.
- How to find and apply restrictions to quadratic equations.

Unit 1: Assessment

Suggested time to complete: 35 minutes

1. Solve for the unknown in each of the following:
   a. $21 + 5a = 6a^2$
   b. $\frac{x}{x+3} + \frac{x+9}{x^2-9} = \frac{2}{x-3}$
   c. $2x^3 = 10x$
   d. $\frac{1}{2}p = \sqrt{-p-5} - 4$
   e. $\frac{z-1}{z^2-z-6} - \frac{4}{z^2-4} = \frac{4}{z+2}$
2. Solve for y in terms of k: $y^2 - 6yk = 7k^2$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

$$6g^2 + 12g = 0$$
$$\therefore 6(g^2 + 2g) = 0$$
$$\therefore g^2 + 2g = 0$$
$$\therefore g(g + 2) = 0$$
$$\therefore g = 0 \text{ or } g = -2$$

2.
$$13y^2 - 26y = -13$$
$$\therefore 13y^2 - 26y + 13 = 0$$
$$\therefore 13(y^2 - 2y + 1) = 0$$
$$\therefore y^2 - 2y + 1 = 0$$
$$\therefore (y - 1)(y - 1) = 0$$
$$\therefore y = 1$$

Note: It may look like there is only one solution or root. However, there are still two roots. They just happen to be the same.

3.
$$2(2a + 1) = 15$$
$$\therefore 2a^2 + a - 15 = 0$$
$$\therefore (2a - 5)(a + 3) = 0$$
$$\therefore a = \frac{5}{2} \text{ or } a = -3$$

4.
$$13t^2 = 325$$
$$\therefore 13t^2 - 325 = 0$$
$$\therefore 13(t^2 - 25) = 0$$
$$\therefore t^2 - 25 = 0$$
$$\therefore (t + 5)(t - 5) = 0$$
$$\therefore t = -5 \text{ or } t = 5$$

5.
$$\frac{d - 2}{d + 1} = \frac{2d + 1}{d - 7} \qquad \text{Restriction: } d \neq -1, d \neq 7$$
$$\therefore (d - 2)(d - 7) = (2d + 1)(d + 1)$$
$$\therefore d^2 - 9d + 14 = 2d^2 + 3d + 1$$
$$\therefore -d^2 - 12d + 13 = 0$$
$$\therefore d^2 + 12d - 13 = 0$$
$$\therefore (d - 1)(d + 13) = 0$$
$$\therefore d = 1 \text{ or } d = -13$$

6.
$$\frac{5x}{x - 2} + \frac{6}{x^2 - 2x} = -\frac{3}{x} - 2$$
$$\therefore \frac{5x}{x - 2} + \frac{6}{x(x - 2)} = -\frac{3}{x} - 2 \qquad \text{Restriction: } x \neq 2, x \neq 0$$
$$\therefore 5x(x) + 6 = -3(x - 2) - 2x(x - 2)$$
$$\therefore 5x^2 + 6 = -3x + 6 - 2x^2 + 4x$$
$$\therefore 7x^2 - x = 0$$
$$\therefore x(7x - 1) = 0$$
$$\therefore \cancel{x = 0} \text{ or } x = \frac{1}{7}$$
$$\therefore x = \frac{1}{7}$$

Back to Exercise 1.1

Exercise 1.2

1.

$$17 = a^2$$
$$\therefore a^2 - 17 = 0$$
$$\therefore \left(a + \sqrt{17}\right)\left(a - \sqrt{17}\right) = 0$$
$$\therefore a = -\sqrt{17} \text{ or } a = \sqrt{17}$$

2.

$$4 + a = \sqrt{a + 6} \qquad \text{Restriction: } a \geq -6$$
$$\therefore (4 + a)^2 = a + 6$$
$$\therefore 16 + 8a + a^2 = a + 6$$
$$\therefore a^2 + 7a + 10 = 0$$
$$\therefore (a + 2)(a + 5) = 0$$
$$\therefore a = -2 \text{ or } \cancel{a = -5}$$
$$\therefore a = -2$$

3.

$$2k = \sqrt{5 - 8k} \qquad \text{Restriction: } k \leq \frac{5}{8}$$
$$\therefore 4k^2 = 5 - 8k$$
$$\therefore 4k^2 + 8k - 5 = 0$$
$$\therefore (2k - 1)(2k + 5) = 0$$
$$\therefore k = \frac{1}{2} \text{ or } \cancel{k = -\frac{5}{2}}$$
$$\therefore k = \frac{1}{2}$$

4.

$$5\sqrt{5y + 1} - 4 = 5y + 1$$
$$\text{Let } \sqrt{5y + 1} = A$$
$$\therefore 5A - 4 = A^2$$
$$\therefore A^2 - 5A + 4 = 0$$
$$\therefore (A - 4)(A - 1) = 0$$
$$\therefore A = 4 \text{ or } A = 1$$
$$\therefore \sqrt{5y + 1} = 4 \text{ or } \sqrt{5y + 1} = 1$$
$$\therefore 5y + 1 = 16 \text{ or } 5y + 1 = 1$$
$$\therefore 5y = 15 \text{ or } 5y = 0$$
$$\therefore y = 3 \text{ or } y = 0$$

Back to Exercise 1.2

Unit 1: Assessment

1.

a.

$$21 + 5a = 6a^2$$
$$\therefore 6a^2 - 5a - 21 = 0$$
$$\therefore (3a - 7)(2a + 3) = 0$$
$$\therefore a = \frac{7}{3} \text{ or } a = -\frac{3}{2}$$

b.

$$\frac{x}{x+3}+\frac{x+9}{x^2-9}=\frac{2}{x-3}$$

$$\therefore \frac{x}{x+3}+\frac{x+9}{(x+3)(x-3)}=\frac{2}{x-3} \qquad \text{Restrictions: } x\neq -3, x\neq 3$$

$$\therefore x(x-3)+(x+9)=2(x+3)$$

$$\therefore x^2-3x+x+9=2x+6$$

$$\therefore x^2-4x+3=0$$

$$\therefore (x-1)(x-3)=0$$

$$\therefore x=1 \text{ or } \cancel{x=3}$$

$$\therefore x=1$$

c.

$$2x^3=10x$$

$$\therefore 2x^3-10x=0$$

$$\therefore 2x(x^2-5)=0$$

$$\therefore 2x\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)=0$$

$$\therefore x-0 \text{ or } x=-\sqrt{5} \text{ or } x=\sqrt{5}$$

d.

$$\frac{1}{2}p=\sqrt{-p-5}-4$$

$$\therefore \frac{1}{2}p+4=\sqrt{-p-5}$$

$$\therefore \left(\frac{1}{2}p+4\right)^2=-p-5$$

$$\therefore \frac{1}{4}p^2+4p+16=-p-5$$

$$\therefore p^2+16p+64+4p+20=0$$

$$\therefore p^2+20p+84=0$$

$$\therefore (p+6)(p+14)=0$$

$$\therefore p=-6 \text{ or } \cancel{p=14}$$

e.

$$\frac{z-1}{z^2-z-6}-\frac{4}{z^2-4}=\frac{4}{z+2}$$

$$\therefore \frac{z-1}{(z-3)(z+2)}-\frac{4}{(z+2)(z-2)}=\frac{4}{(z+2)} \qquad \text{Restrictions: } z\neq 3, z\neq \pm 2$$

$$\therefore (z-1)(z-2)-4(z-3)=4(z-3)(z-2)$$

$$\therefore z^2-3z+2-4z+12=4z^2-20z+24$$

$$\therefore 3z^2-13z+10=0$$

$$\therefore (3z-10)(z-1)=0$$

$$\therefore z=\frac{10}{3} \text{ or } z=1$$

2.

$$y^2-6yk=7k^2$$

$$\therefore y^2-6yk-7k^2=0$$

$$\therefore (y-7k)(y+k)=0$$

$$\therefore y=7k \text{ or } y=-k$$

Back to Unit 1: Assessment

Unit 2: Solve quadratic equations by completing the square

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve a quadratic equation by completing the square.

What you should know

Before you start this unit, make sure you can:

- Solve simple quadratic equations by factorisation. Refer to level 2 subject outcome 2.3 unit 1 if you need help with this.
- Complete the square of a quadratic expression. Refer to level 3 subject outcome 2.2 unit 3 if you need help with this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Complete the square of the following quadratic expressions:

1. $x^2 - 4x - 4$
2. $2x^2 - 6x + 1$

Solutions

1. $$\begin{aligned} x^2 - 4x - 4 &= x^2 - 4x + 4 - 4 - 4 \\ &= (x^2 - 4x + 4) - 8 \\ &= (x + 2)^2 - 8 \end{aligned}$$
2. $$\begin{aligned} 2x^2 - 6x + 1 &= 2\left(x^2 - 3x + \frac{1}{2}\right) \\ &= 2\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + \frac{1}{2}\right) \\ &= 2\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{2} + 1 \\ &= 2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2} \end{aligned}$$

Introduction

As we saw in the previous unit, most of the quadratic equations that you will come across can be solved by factorising the quadratic expression when you have the equation in the standard form of $ax^2 + bx + c = 0$. You must always try to factorise this quadratic expression first before trying anything else.

However, there are some quadratic expressions that are hard to factorise. In these cases, you can use a technique called **completing the square**. You learnt why it is called completing the square in subject outcome 2.2 unit 3.

Solve quadratic equation by completing the square

You should know how to complete the square of a quadratic expression. However, let's look at one example before we move on.



Example 2.1

Factorise the following quadratic expression fully by first completing the square: $x^2 - 10x - 11$

Solution

$x^2 - 10x + 11$ cannot be easily factorised but we can complete the square.

$x^2 - 10x + 11 = x^2 - 10x + 25 - 25 + 11$

We add 25 to the expression because $\left(\frac{10}{2}\right)^2 = 25$.

However, we need to subtract 25 from the expression as well to keep the value of the whole expression the same.

$$\begin{aligned} x^2 - 10x + 11 &= x^2 - 10x + 25 - 25 + 11 \\ &= (x-5)(x-5) - 25 + 11 \\ &= (x-5)^2 - 14 \end{aligned}$$

We have completed the square. But now we have a difference of two squares, so we can factorise further.

$$\begin{aligned} (x-5)^2 - 14 &= \left((x-5) + \sqrt{14}\right)\left((x-5) - \sqrt{14}\right) \\ &= \left(x - 5 + \sqrt{14}\right)\left(x - 5 - \sqrt{14}\right) \end{aligned}$$

Hopefully you can see how the technique of completing the square allows us to solve quadratic equations that are not easily factorised.



Example 2.2

Solve for x: $x^2 - 4x = 3$

Solution

As usual, we need to get the equation into standard form.

$$x^2 - 4x = 3$$
$$\therefore x^2 - 4x - 3 = 0$$

Now the quadratic expression is not easy to factorise, so we continue by completing the square.

$$x^2 - 4x - 3 = 0$$
$$\therefore x^2 - 4x + 4 - 4 - 3 = 0$$
$$\therefore (x-2)^2 - 7 = 0$$

At this point, we can either continue to factorise the resulting difference of two squares or proceed as follows. The choice is yours.

$$(x-2)^2 - 7 = 0$$
$$\therefore (x-2)^2 = 7 \qquad \text{Take the square root of both sides}$$
$$\therefore (x-2) = \pm\sqrt{7} \qquad \text{Remember that the RHS is } \pm\sqrt{7}$$
$$\therefore x = 2 + \sqrt{7} \text{ or } x = 2 - \sqrt{7}$$

Make sure that you can get the same roots by factorising the difference of two squares.

Remember to check your solutions.

$x = 2 + \sqrt{7}$:

$$\begin{aligned} \text{LHS} &= x^2 - 4x \\ &= (2+\sqrt{7})^2 - 4(2+\sqrt{7}) \\ &= 4 + 4\sqrt{7} + 7 - 8 - 4\sqrt{7} \\ &= 3 \\ &= \text{RHS} \end{aligned}$$

$x = 2 - \sqrt{7}$:

$$\begin{aligned} \text{LHS} &= x^2 - 4x \\ &= (2-\sqrt{7})^2 - 4(2-\sqrt{7}) \\ &= 4 - 4\sqrt{7} + 7 - 8 + 4\sqrt{7} \\ &= 3 \\ &= \text{RHS} \end{aligned}$$

Both solutions are valid. Therefore, $x = 2 + \sqrt{7}$or $x = 2 - \sqrt{7}$.

Note: It is always best to test your solutions with the original equation.



Example 2.3

Solve for a: $3a^2 - 9a = 15$

Solution

Get the equation into standard form.

$$3a^2 - 9a = 15$$
$$\therefore 3a^2 - 9a - 15 = 0$$

We cannot factorise this quadratic expression easily, so we proceed by completing the square.

$$3a^2 - 9a - 15 = 0$$
$$\therefore 3(a^2 - 3a - 5) = 0$$
$$\therefore a^2 - 3a - 5 = 0$$
$$\therefore a^2 - 3a + \frac{9}{4} - \frac{9}{4} - 5 = 0$$
$$\therefore \left(a - \frac{3}{2}\right)^2 - \frac{29}{4} = 0$$
$$\therefore \left(a - \frac{3}{2}\right)^2 = \frac{29}{4}$$ Take the square root of both sides
$$\therefore a - \frac{3}{2} = \frac{\pm\sqrt{29}}{2}$$ Remember that the RHS is $\frac{\pm\sqrt{29}}{2}$
$$\therefore a = \frac{3}{2} + \frac{\sqrt{29}}{2} \text{ or } a = \frac{3}{2} - \frac{\sqrt{29}}{2}$$
$$\therefore a = \frac{3 + \sqrt{29}}{2} \text{ or } a = \frac{3 - \sqrt{29}}{2}$$

Check your solutions.

$a = \frac{3 + \sqrt{29}}{2}$:

$$\begin{aligned}
\text{LHS} &= 3a^2 - 9a \\
&= 3\left(\frac{3 + \sqrt{29}}{2}\right)^2 - 9\left(\frac{3 + \sqrt{29}}{2}\right) \\
&= 3\left(\frac{9 + 6\sqrt{29} + 29}{4}\right) - 9\left(\frac{3 + \sqrt{29}}{2}\right) \\
&= \frac{27 + 18\sqrt{29} + 87}{4} - \frac{54 + 18\sqrt{29}}{4} \\
&= \frac{27 + 18\sqrt{29} + 87 - 54 - 18\sqrt{29}}{4} \\
&= \frac{60}{4} \\
&= 15 \\
&= \text{RHS}
\end{aligned}$$

$x = \frac{3 - \sqrt{29}}{2}$:

$$\begin{aligned}
\text{LHS} &= 3a^2 - 9a \\
&= 3\left(\frac{3 - \sqrt{29}}{2}\right)^2 - 9\left(\frac{3 - \sqrt{29}}{2}\right) \\
&= 3\left(\frac{9 - 6\sqrt{29} + 29}{4}\right) - 9\left(\frac{3 - \sqrt{29}}{2}\right) \\
&= \frac{27 - 18\sqrt{29} + 87}{4} - \frac{54 - 18\sqrt{29}}{4} \\
&= \frac{27 - 18\sqrt{29} + 87 - 54 = 18\sqrt{29}}{4} \\
&= \frac{60}{4} \\
&= 15 \\
&= \text{RHS}
\end{aligned}$$

Both solutions are valid. Therefore, $a = \frac{3 \pm \sqrt{29}}{2}$.



Exercise 2.1

1. Solve for the unknown in each case by completing the square:
   a. $a^2 + 3a = -2$
   b. $x^2 - 10x + 2 = 0$
   c. $y^2 = 7 - 8y$
   d. $2x^2 = 13x$
2. Solve the equation in 1.d. WITHOUT completing the square.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve quadratic equations by completing the square.

Unit 2: Assessment

Suggested time to complete: 30 minutes

1. Solve for the unknown in each of the following by completing the square:
   a. $x^2 + 12x - 3 = 0$
   b. $3a^2 + a - 4 = 0$
   c. $3(6x + x^2) = -5$
   d. $\frac{4x}{(x+3)} - \frac{2x}{(x-4)} = \frac{7}{x^2 - x - 12}$
2. Solve for x in terms of a: $x^2 - 6x = 2a$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

a.

$$\begin{aligned} a^2 + 3a &= -2 \\ \therefore a^2 + 3a + 2 &= 0 \\ \therefore a^2 + 3a + \frac{9}{4} - \frac{9}{4} + 2 &= 0 \\ \therefore \left(a + \frac{3}{2}\right)^2 &= \frac{1}{4} \\ \therefore a + \frac{3}{2} &= \pm\frac{1}{2} \\ \therefore a = -\frac{3}{2} + \frac{1}{2} &\text{ or } a = -\frac{3}{2} - \frac{1}{2} \\ \therefore a = -1 &\text{ or } a = -2 \end{aligned}$$

b.

$$\begin{aligned} x^2 - 10x + 2 &= 0 \\ \therefore x^2 - 10x + 25 - 25 + 2 &= 0 \\ \therefore (x-5)^2 - 23 &= 0 \\ \therefore (x-5)^2 &= 23 \\ \therefore x &= 5 \pm \sqrt{23} \end{aligned}$$

c.

$$\begin{aligned} y^2 &= 7 - 8y \\ \therefore y^2 + 8y - 7 &= 0 \\ \therefore y^2 + 8y + 16 - 16 - 7 &= 0 \\ \therefore (y+4)^2 - 23 = 0 \therefore (y+4)^2 &= 23 \\ \therefore y &= -4 \pm \sqrt{23} \end{aligned}$$

d.

$$\begin{aligned} 2x^2 &= 13x \\ \therefore 2x^2 - 13x &= 0 \\ \therefore 2\left(x^2 - \frac{13}{2}x\right) &= 0 \\ \therefore x^2 - \frac{13}{2}x &= 0 \\ \therefore x^2 - \frac{13}{2} + \frac{169}{4} - \frac{169}{4} &= 0 \\ \therefore \left(x - \frac{13}{4}\right)^2 &= \frac{169}{4} \\ \therefore x - \frac{13}{4} &= \pm\frac{13}{4} \\ \therefore x &= \frac{13}{4} \pm \frac{13}{4} \\ \therefore x = 0 &\text{ or } x = \frac{13}{2} \end{aligned}$$

2.

$$2x^2 = 13x$$
$$\therefore 2x^2 - 13x = 0$$
$$\therefore x(2x - 13) = 0$$
$$\therefore x = 0 \text{ or } x = \frac{13}{2}$$

Back to Exercise 2.1

Unit 2: Assessment

1.

a.

$$x^2 + 12x - 3 = 0$$
$$\therefore x^2 + 12x + 36 - 36 - 3 = 0$$
$$\therefore (x + 6)^2 - 39 = 0$$
$$\therefore (x + 6)^2 = 39$$
$$\therefore x + 6 = \pm\sqrt{39}$$
$$\therefore x = -6 \pm \sqrt{39}$$

b.

$$3a^2 + a - 4 = 0$$
$$\therefore 3\left(a^2 + \frac{1}{3}a\right) = 4$$
$$\therefore a^2 + \frac{1}{3}a = \frac{4}{3}$$
$$\therefore a^2 + \frac{1}{3}a + \frac{1}{36} = \frac{4}{3} + \frac{1}{36}$$
$$\therefore \left(a + \frac{1}{6}\right)^2 = \frac{49}{36}$$
$$\therefore a + \frac{1}{6} = \pm\frac{7}{6}$$
$$\therefore a = -\frac{1}{6} + \frac{7}{6} \text{ or } a = -\frac{1}{6} - \frac{7}{6}$$
$$\therefore a = 1 \text{ or } a = -\frac{8}{6} = -\frac{4}{3}$$

c.

$$3(6x + x^2) = -5$$
$$\therefore x^2 + 6x = -\frac{5}{3}$$
$$\therefore x^2 + 6x + 9 = -\frac{5}{3} + 9$$
$$\therefore (x + 3)^2 = \frac{22}{3}$$
$$\therefore x + 3 = \pm\sqrt{\frac{22}{3}}$$
$$\therefore x = -3 \pm \sqrt{\frac{22}{3}}$$

d.

$$\frac{4x}{(x+3)} - \frac{2x}{(x-4)} = \frac{7}{x^2 - x - 12}$$
$$\therefore \frac{4x}{(x+3)} - \frac{2x}{(x-4)} = \frac{7}{(x+3)(x-4)}, x \neq -3, x \neq 4$$
$$\therefore 4x(x-4) - 2x(x+3) = 7$$
$$\therefore 4x^2 - 16x - 2x^2 - 6x - 7 = 0$$
$$\therefore 2x^2 - 22x - 7 = 0$$
$$\therefore 2(x^2 - 11x) = 7$$
$$\therefore x^2 - 11x = \frac{7}{2}$$
$$\therefore x^2 - 11x + \frac{121}{4} = \frac{7}{2} + \frac{121}{4}$$
$$\therefore \left(x - \frac{11}{2}\right)^2 = \frac{135}{4}$$
$$\therefore x - \frac{11}{2} = \frac{\pm\sqrt{135}}{2}$$
$$\therefore x = \frac{11 \pm \sqrt{135}}{2}$$

2.

$$x^2 - 6x = 2a$$
$$\therefore x^2 - 6x + 9 = 2a + 9$$
$$\therefore (x-3)^2 = 2a + 9$$
$$\therefore x - 3 = \pm\sqrt{2a+9}$$
$$\therefore x = 3 \pm \sqrt{2a+9}$$

Back to Unit 2: Assessment

Unit 3: Solve quadratic equations using the quadratic formula

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Understand when to use the quadratic formula to solve quadratic equations.
- Apply the quadratic formula when necessary.

What you should know

Before you start this unit, make sure you can:

- Solve a quadratic equation by completing the square. Refer to unit 2 in this subject outcome if you need help with this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Complete the square of the following quadratic expressions:

1. $x^2 - 4x - 4 = 0$
2. $2x^2 = 6x - 1$

Solutions

1.

$$\begin{aligned} x^2 - 4x - 4 &= 0 \\ \therefore x^2 - 4x + 4 - 4 - 4 &= 0 \\ \therefore (x-2)^2 - 8 &= 0 \\ \therefore (x-2)^2 &= 8 \\ \therefore (x-2) &= \pm\sqrt{8} \\ \therefore x &= 2 \pm \sqrt{8} \end{aligned}$$

2.

$$
\begin{aligned}
2x^2 &= 6x - 1 \\
\therefore 2x^2 - 6x &= 1 \\
\therefore 2(x^2 - 3x) &= 1 \\
\therefore x^2 - 3x &= \frac{1}{2} \\
\therefore x^2 - 3x + \frac{9}{4} &= \frac{1}{2} + \frac{9}{4} \\
\therefore \left(x - \frac{3}{2}\right)^2 &= \frac{11}{4} \\
\therefore x - \frac{3}{2} &= \frac{\pm\sqrt{11}}{2} \\
\therefore x &= \frac{3 \pm \sqrt{11}}{2}
\end{aligned}
$$

Introduction

Completing the square is a powerful technique that can be used to factorise any quadratic expression and hence solve any quadratic equation. In fact, it is so powerful that it can even be used to create a general formula with which any quadratic equation can be solved. This formula is known as the **quadratic formula**.

However, the quadratic formula is not just a useful tool for finding the roots of any quadratic equation. It can also be used to predict what kind of roots these will be. Will there be two distinct roots or will the roots be the same? Will the roots be real numbers or complex numbers?

The quadratic formula can, therefore, be used to determine where, or even if, a quadratic function cuts the x-axis.

Did you know?

Quadratic equations have been around for a long time and the quadratic formula has a long history. Its development is a wonderful example of how civilisations across time have contributed to the body of knowledge we have inherited today.

The ancient Egyptians solved quadratic equations geometrically. Diophantus (250 AD) started to solve them algebraically which was further developed by the Indian mathematician Brahmagupta (597–668 AD) who described a version of the quadratic formula in words. Incidentally, Brahmagupta was also the first known mathematician to give rules for computing with zero.

In the 9th century, the highly influential Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī (depicted in Figure 1) presented the first systematic solution to linear and quadratic equations.



Figure 1: A statue of Muḥammad ibn Mūsā al-Khwārizmī

A more modern form of the quadratic formula (which covers all possible cases) was first discovered/ invented (depending on your view) by Simon Stevin in 1594. Finally, in 1637, René Descartes published the form of the formula we use today.

The quadratic formula

The quadratic formula can be easily derived by completing the square of the standard form of the quadratic equation. Before reading on and looking at the derivation, see if you can do it yourself.

Complete the square to solve for x in terms of a, b and c in $ax^2 + bx + c = 0$.

Here is the full derivation of the quadratic formula by completing the square.

$$ax^2 + bx + c = 0$$
$$\therefore ax^2 + bx = -c$$
$$\therefore a\left(x^2 + \frac{b}{a}x\right) = -c$$
$$\therefore x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \left[\frac{1}{2} \times \frac{b}{a} = \frac{b}{2a}. \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}\right]$$
$$\therefore x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$
$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$\therefore x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: We can write both roots separately if we want to.

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is really easy to use. You simply substitute values for a, b and c into the formula. The formula will work for any quadratic equation, even those that can be easily factorised. However, solving quadratic equations by factorisation is still the simplest and quickest method. So, use factorisation whenever you can.



Example 3.1

Solve for x using the quadratic formula and check your answer by factorisation:
$x^2 - 7x + 6 = 0$

Solution

$x^2 - 7x + 6 = 0$. The equation is in standard form. Therefore, $a = 1$, $b = -7$ and $c = 6$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{7 \pm \sqrt{49 - 24}}{2}$$
$$= \frac{7 \pm \sqrt{25}}{2}$$
$$= \frac{7 \pm 5}{2}$$
$$\therefore x = 6 \text{ or } x = 1$$

We can easily solve the quadratic equation by factorising the original quadratic expression as follows:

$$x^2 - 7x + 6 = 0$$
$$\therefore (x - 6)(x - 1) = 0$$
$$\therefore x = 6 \text{ or } x = 1$$



Example 3.2

Solve for x using the quadratic formula, leaving your answer in simplest surd form:

$4x^2 = 5x + 13$

Solution

The equation is not yet in standard form.

$$4x^2 = 5x + 13$$
$$\therefore 4x^2 - 5x - 13 = 0$$
$$a = 4, b = -5, c = -13$$

Now we can solve for x by using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(4)(-13)}}{2(4)}$$
$$= \frac{5 \pm \sqrt{25 + 208}}{8}$$
$$= \frac{5 \pm \sqrt{233}}{8}$$
$$\therefore x = \frac{5 + \sqrt{233}}{8} \text{ or } x = \frac{5 - \sqrt{233}}{8}$$

We cannot simplify these roots any further. They are in simplest surd form.



Example 3.3

Solve for x using the quadratic formula, leaving your answer in simplest surd form:
$8x - 21 = 3x^2$

Solution

The equation is not in standard form.

$$8x - 21 = 3x^2$$
$$\therefore -3x^2 + 8x - 21 = 0$$

You can decide whether to proceed with the equation above where $a = -3$ or to multiply both sides by -1 to make $a > 0$. You will get the same result. Let's proceed with $a = -3$.

$a = -3, b = 8, c = -21$

Now we can solve for x by using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(8) \pm \sqrt{(8)^2 - 4(-3)(-21)}}{2(-3)}$$
$$= \frac{-8 \pm \sqrt{64 - 252}}{-6}$$
$$= \frac{8 \pm \sqrt{-188}}{6}$$

Therefore, x has no real solution.

In this case we end up with a negative number under the square root sign. Therefore, the roots to this quadratic equation are not real. They are complex.



Exercise 3.1

1. Solve for the unknown in each case using the quadratic equation, leaving your answer in simplest surd form:
   a. $x^2 - 9x + 2 = 0$
   b. $-2f^2 + 3f = -2$
   c. $y^2 = 7(3 - 2y)$
   d. $2x^2 = 12x$
   e. $6x^2 = 7$
2. Solve for x using the quadratic equation and round your solution(s) to two decimal places:
   $3x^2 + 7x = 21$

The full solutions are at the end of the unit.

The nature of roots

Please note that this section on the nature of roots is not technically part of the NC(V) level 3 curriculum.

However, it has been included here as a helpful addition to students' knowledge about the roots of quadratic equations and how these directly relate to the kinds of intercepts that quadratic functions make with the x-axis.

As we said in the introduction to this unit, the quadratic formula can be used to not only calculate the roots of a quadratic equation but also predict what kind of roots they are. This can be very useful when dealing with quadratic functions.

There is one part of the quadratic formula that does this. It is the portion under the square root sign and is called the **discriminant**. It is sometimes given the symbol Δ (delta), the Greek letter 'd'.

Therefore, $\Delta = b^2 - 4ac$ (the discriminant).

If $\Delta > 0$, then the number under the square root is **positive** and the roots of the equation are **distinct** (different), and both are **real numbers**.

$x = \dfrac{-3 \pm \sqrt{27}}{2}$

$\Delta = 27$ and the roots are $x = \dfrac{-3 + \sqrt{27}}{2}$ or $\dfrac{-3 - \sqrt{27}}{2}$ (the roots are distinct and real numbers)

If $\Delta < 0$, then the number under the square root sign is **negative** and the roots of the equation are **distinct**, and both are **complex (non-real) numbers**.

$x = \dfrac{-3 \pm \sqrt{-27}}{2}$

$\Delta = -27$ and the roots are $x = \dfrac{-3 + \sqrt{-27}}{2}$ or $\dfrac{-3 - \sqrt{-27}}{2}$ (the roots are distinct and complex (non-real) numbers)

If $\Delta = 0$, then the roots are **real** and **equal**.

$x = \dfrac{-3 \pm \sqrt{0}}{2}$

$\Delta = 0$ and the roots are $x = \dfrac{-3 + \sqrt{0}}{2} = -\dfrac{3}{2}$ or $\dfrac{-3 - \sqrt{-27}}{2} = -\dfrac{3}{2}$ (the roots are the same and real numbers)

There is a graphical interpretation of this as well. In the context of the quadratic function $f(x) = ax^2 + bx + c$, when we solve the quadratic equation $ax^2 + bx + c = 0$ we are really finding the x-intercepts of the function.

If $\Delta > 0$, then the roots are distinct and real. Therefore, the function has two x-intercepts (figure 2).

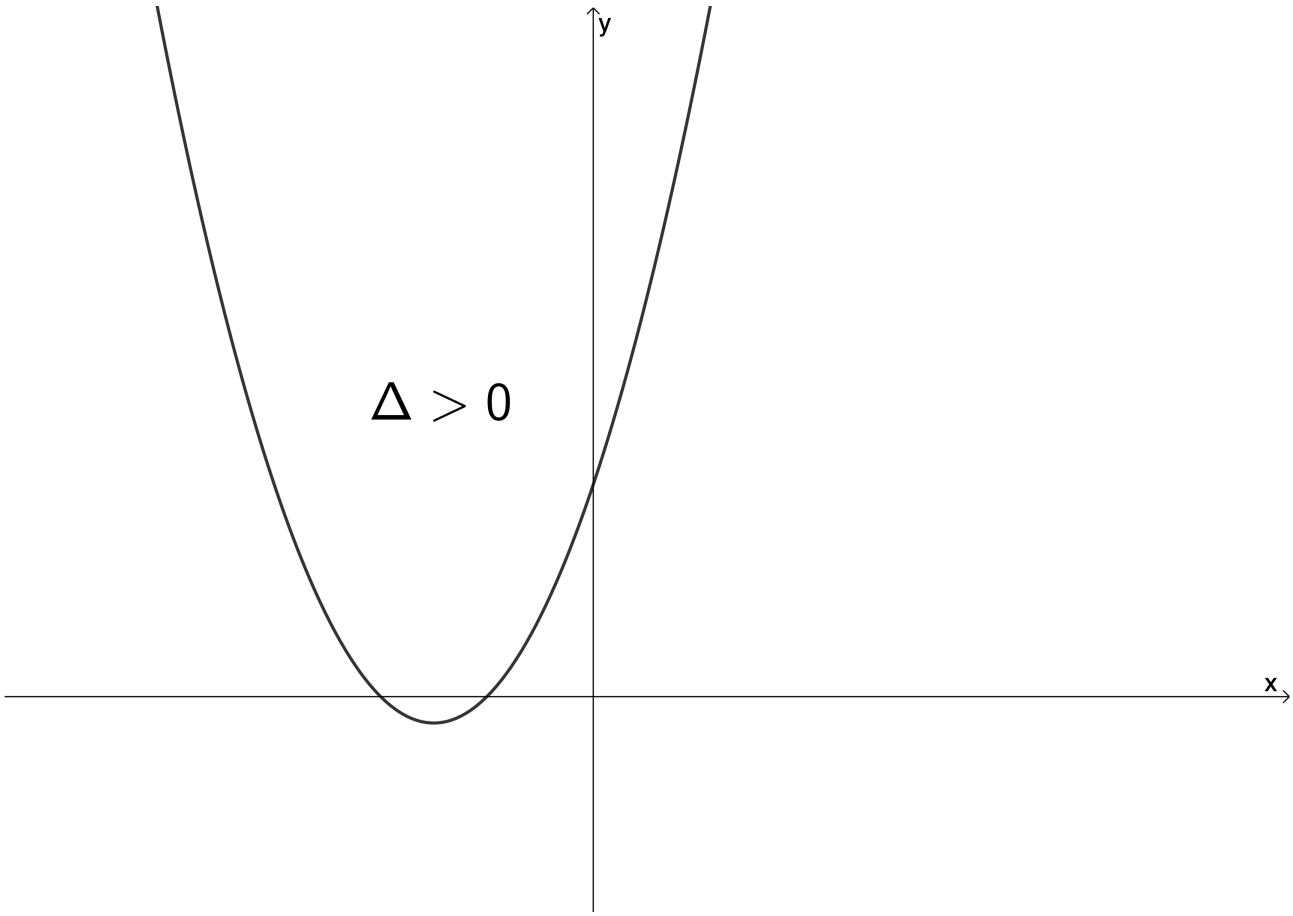


Figure 2: Two real roots

If $\Delta < 0$, then the roots are distinct and complex (non-real). Therefore, the function has no x-intercepts.

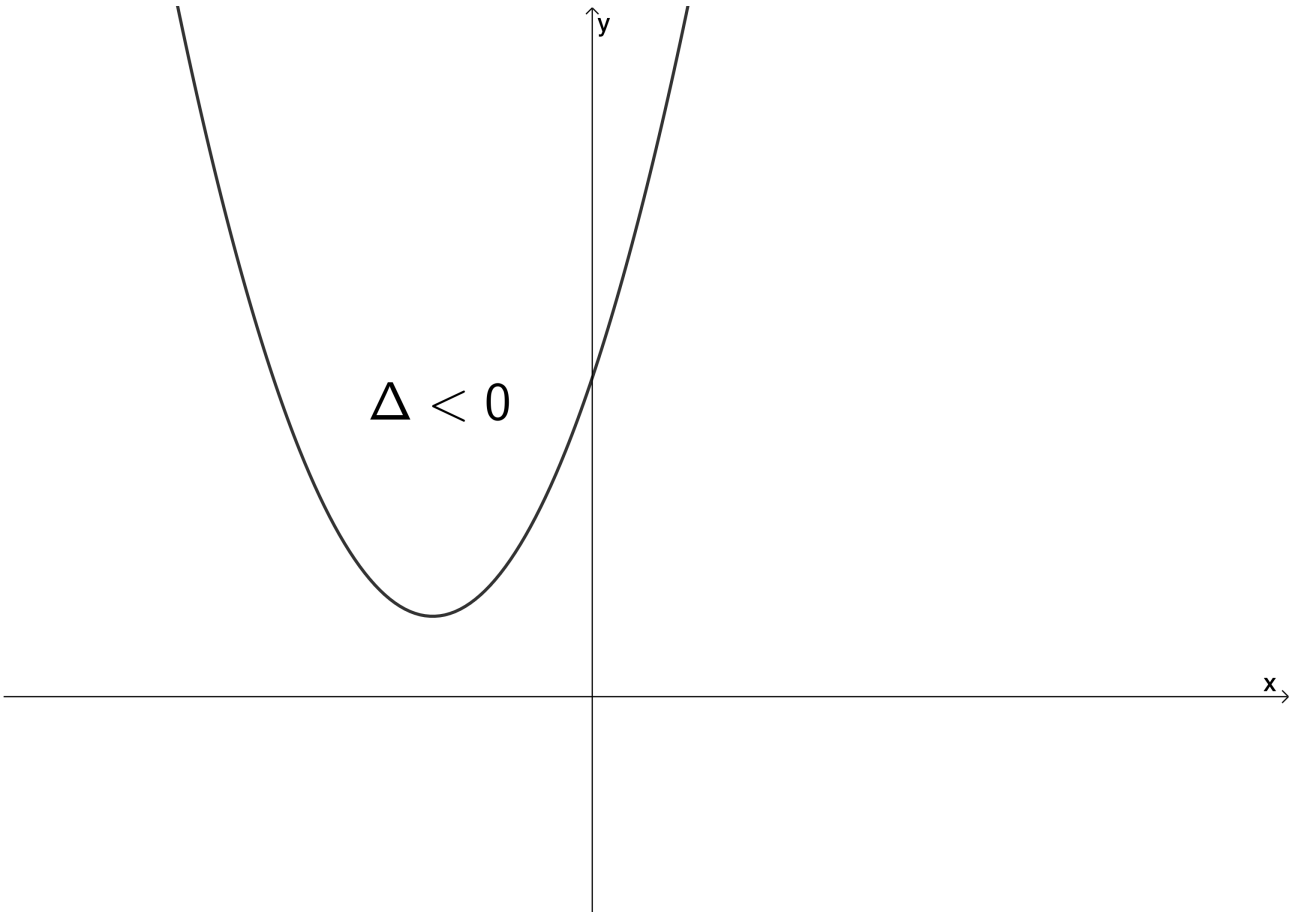
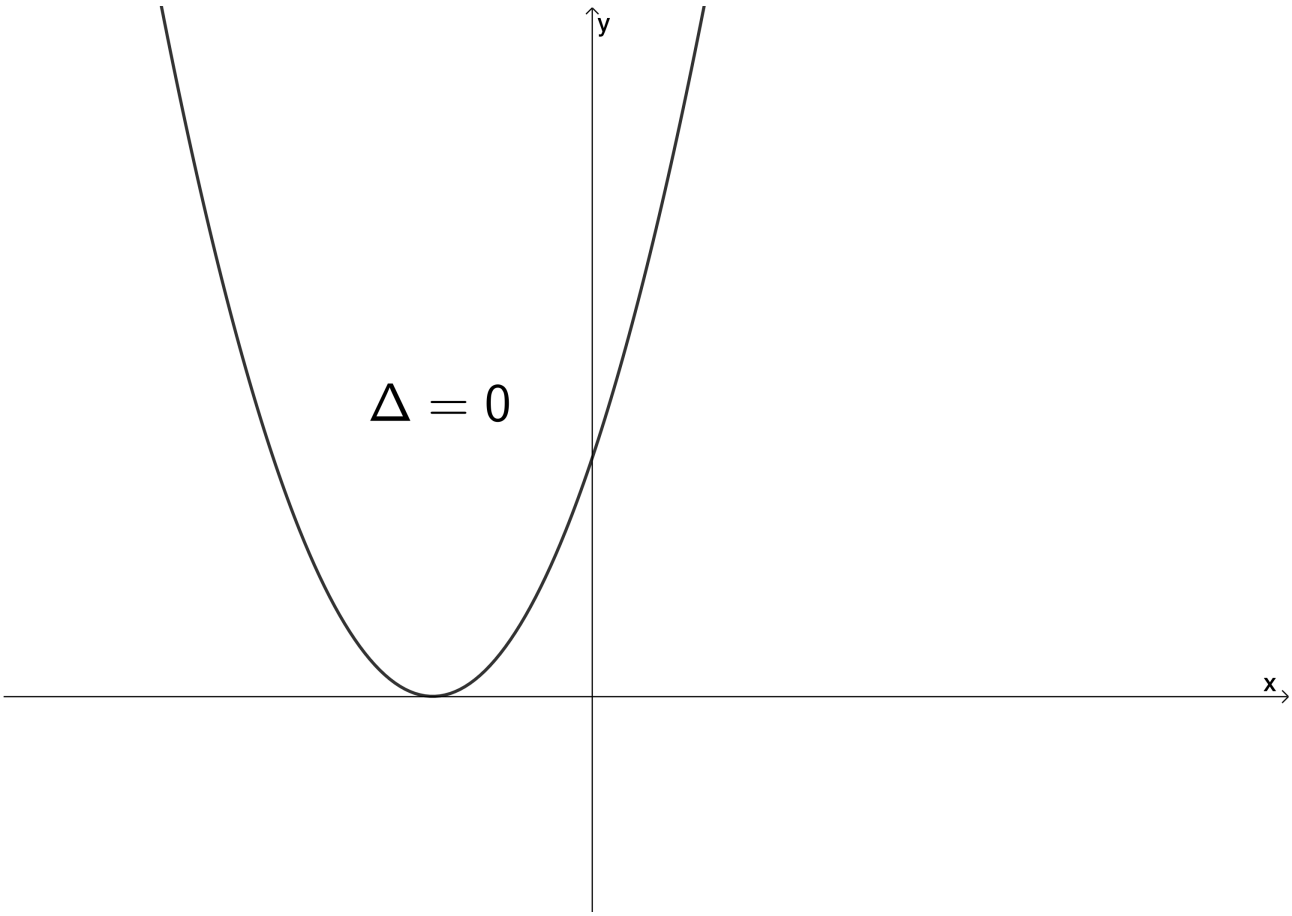


Figure 3: No real roots

If $\Delta = 0$, then the roots are real and equal. This means the function just touches the x-axis at one point.



Take note!

$\Delta = b^2 - 4ac$

If $\Delta > 0$, the roots are distinct and real.
If $\Delta < 0$, the roots are distinct and complex (non-real).
If $\Delta = 0$, the roots are the same and real.

The discriminant only tells you about the kind of roots. It **does not** tell you what these roots are.

Exercise 3.2

Determine the nature of the roots for the following quadratic equations:

1. $x^2 + 2x + 2$
2. $3x^3 = 6x + 1$
3. $-2x^2 = 0$
4. $7x - 4 = 5x^2$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve quadratic equations using the quadratic formula.
- *How to determine the nature of the roots of a quadratic equation by evaluating the discriminant (additional information).*

Unit 3: Assessment

Suggested time to complete: 50 minutes

1. Solve for the unknown in each of the following by factorisation if possible, leaving your answer in simplest surd form where necessary:
   a. $-4x^2 - 41x - 45 = 0$
   b. $3a^2 + 10a - 48 = 0$
   c. $63 - 5g^2 = 26g$
   d. $x(x - 8) + 14 = 0$
   e. $\frac{3x}{(x-3)} - \frac{2x}{(x+4)} = \frac{3}{x^2 + x - 12}$
   f. $2x\left(x - \frac{5}{2}\right) - 5 = -\frac{5}{2x^2 - 5x}$

Question 2 is adapted from *Everything Maths Grade 11 Exercise 2-11 question 22*.

2.
   a. Solve for x in $x^2 - 6 = x$.
   b. Hence solve for y in $(y - 7)^2 = y - 1$.
3. Solve for x in $2(x^2 + 3)^{\frac{1}{2}} = 9$.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
   a.

$$x^2 - 9x + 2 = 0$$
$$a = 1, b = -9, c = 2$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{9 \pm \sqrt{81 - 8}}{2}$$
$$= \frac{9 \pm \sqrt{73}}{2}$$

b.

$$-2f^2 + 3f = -2$$
$$\therefore -2f^2 + 3f + 2 = 0$$
$$\therefore 2f^2 - 3f - 2 = 0$$
$$a = 2, b = -3, c = -2$$
$$f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore f = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$
$$= \frac{3 \pm \sqrt{9 + 16}}{4}$$
$$= \frac{3 \pm \sqrt{25}}{4}$$
$$\therefore f = \frac{3 + 5}{4} = 2 \text{ or } f = \frac{3 - 5}{4} = -\frac{1}{2}$$

c.

$$y^2 = 7(3 - 2y)$$
$$\therefore y^2 = 21 - 14y$$
$$\therefore y^2 + 14y - 21 = 0$$
$$a = 1, b = 14, c = -21$$
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore y = \frac{-(14) \pm \sqrt{(14)^2 - 4(1)(-21)}}{2(1)}$$
$$= \frac{-14 \pm \sqrt{196 + 84}}{2}$$
$$= \frac{-14 \pm \sqrt{280}}{2}$$
$$= \frac{-14 \pm 2\sqrt{70}}{2} \qquad \sqrt{280} = \sqrt{4 \times 70} = 2\sqrt{70}$$
$$= -7 \pm \sqrt{70}$$

d.

$$2x^2 = 12x$$
$$\therefore 2x^2 - 12x = 0$$
$$\therefore x^2 - 6x = 0$$
$$a = 1, b = -6, c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(0)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36}}{2}$$
$$\therefore x = \frac{12}{2} = 6 \text{ or } x = \frac{0}{2} = 0$$

e.

$$6x^2 = 7$$
$$\therefore 6x^2 - 7 = 0$$
$$a = 6, b = 0, c = -7$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(0) \pm \sqrt{(0)^2 - 4(6)(-7)}}{2(4)}$$
$$= \frac{\pm\sqrt{168}}{8}$$
$$= \frac{\pm 2\sqrt{42}}{8}$$
$$= \frac{\pm\sqrt{42}}{4}$$

2.

$$3x^2 + 7x = 21$$
$$\therefore 3x^2 + 7x - 21 = 0$$
$$a = 3, b = 7, c = -21$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(7) \pm \sqrt{(7)^2 - 4(3)(-21)}}{2(3)}$$
$$= \frac{-7 \pm \sqrt{49 + 252}}{6}$$
$$= \frac{-7 \pm \sqrt{301}}{6}$$
$$\therefore x = 1.72 \text{ or } x = -4.06$$

Back to Exercise 3.1

Exercise 3.2

1.

$$x^2 + 2x + 2 = 0$$
$$a = 1, b = 2, c = 2$$
$$\Delta = b^2 - 4ac$$
$$= (2)^2 - 4(1)(2)$$
$$= 4 - 8$$
$$= -4$$
$$\therefore \Delta < 0$$

The roots are distinct and complex (non-real).

2.

$$3x^3 = 6x + 1$$
$$\therefore 3x^2 - 6x - 1 = 0$$
$$a = 3, b = -6, c = -1$$
$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-6)^2 - 4(3)(-1) \\ &= 36 + 12 \\ &= 48\end{aligned}$$
$$\therefore \Delta > 0$$

The roots are distinct and real. The function has two real x-intercepts.

3.

$$-2x^2 = 0$$
$$a = -2, b = 0, c = 0$$
$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (0)^2 - 4(-2)(0) \\ &= 0\end{aligned}$$
$$\therefore \Delta = 0$$

The roots are the same and real. The function has one real x-intercept or touches the x-axis at one point.

4.

$$7x - 4 = 5x^2$$
$$\therefore 5x^2 - 7x + 4 = 0$$
$$a = 5, b = -7, c = 4$$
$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(5)(4) \\ &= 49 - 80 \\ &= -31\end{aligned}$$
$$\therefore \Delta < 0$$

The roots are distinct and complex (non-real). The function has no x-intercepts.

Back to Exercise 3.2

Unit 3: Assessment

1.

a.

$$-4x^2 - 41x - 45 = 0$$
$$\therefore 4x^2 + 41x + 45 = 0$$
$$\therefore (4x + 5)(x + 9) = 0$$
$$\therefore x = -\frac{5}{4} \text{ or } x = -9$$

b.

$$3a^2 + 10a - 48 = 0$$
$$\therefore (3a - 8)(a + 6) = 0$$
$$\therefore a = \frac{8}{3} \text{ or } a = -6$$

c.

$$63 - 5g^2 = 26g$$
$$\therefore 5g^2 + 26g - 63 = 0$$
$$\therefore (5g - 9)(g + 7) = 0$$
$$\therefore x = \frac{9}{5} \text{ or } x = -7$$

d.

$$x(x - 8) + 14 = 0$$
$$\therefore x^2 - 8x + 14 = 0$$
$$a = 1, b = -8, c = 14$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(14)}}{2(1)}$$
$$= \frac{8 \pm \sqrt{64 - 56}}{2}$$
$$= \frac{8 \pm \sqrt{8}}{2}$$
$$= \frac{8 \pm 2\sqrt{2}}{2}$$
$$= 4 \pm \sqrt{2}$$

e.

$$\frac{3x}{(x - 3)} - \frac{2x}{(x + 4)} = \frac{3}{x^2 + x - 12}$$
$$\therefore \frac{3x}{(x - 3)} - \frac{2x}{(x + 4)} = \frac{3}{(x - 3)(x + 4)}, x \neq 3, x \neq -4$$
$$\therefore 3x(x + 4) - 2x(x - 3) = 3$$
$$\therefore 3x^2 + 12x - 2x^2 + 6x - 3 = 0$$
$$\therefore x^2 + 18x - 3 = 0$$
$$a = 1, b = 18, c = -3$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(18) \pm \sqrt{(-18)^2 - 4(1)(-3)}}{2(1)}$$
$$= \frac{-18 \pm \sqrt{324 + 12}}{2}$$
$$= \frac{-18 \pm \sqrt{336}}{2}$$
$$= \frac{-18 \pm 4\sqrt{21}}{2} \qquad \left[\sqrt{336} = \sqrt{16 \times 21} = 4\sqrt{21}\right]$$
$$= -9 \pm 2\sqrt{21}$$

f.

$$2x\left(x-\frac{5}{2}\right)-6=-\frac{5}{2x^2-5x}$$
$$\therefore (2x^2-5x)-6=-\frac{5}{(2x^2-5x)}$$
$$\text{Let } 2x^2-5x=A$$
$$\therefore A-6=-\frac{5}{A}$$
$$\therefore A^2-6A=-5$$
$$\therefore A^2-6A+5=0$$
$$\therefore (A-5)(A-1)=0$$
$$\therefore A=5 \text{ or } A=1$$
$$\therefore 2x^2-5x=5 \text{ or } 2x^2-5x=1$$
$$\therefore 2x^2-5x-5=0 \text{ or } 2x^2-5x-1=0$$

$$2x^2-5x-5=0$$
$$a=2, b=-5, c=-5$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$\therefore x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-5)}}{2(2)}$$
$$=\frac{5\pm\sqrt{25+40}}{4}$$
$$=\frac{5\pm\sqrt{65}}{4}$$

$$2x^2-5x-1=0$$
$$a=2, b=-5, c=-1$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$\therefore x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-1)}}{2(2)}$$
$$=\frac{5\pm\sqrt{25+8}}{4}$$
$$=\frac{5\pm\sqrt{33}}{4}$$

2.

a.

$$x^2-6=x$$
$$\therefore x^2-x-6=0$$
$$\therefore (x+2)(x-3)=0$$
$$\therefore x=-2 \text{ or } x=3$$

b.

$$(y-7)^2=y-1$$
$$\therefore (y-7)^2=(y-7)+6$$
$$\therefore (y-7)^2-6=(y-7)$$
$$\therefore (y-7)=-2 \text{ or } (y-7)=3$$
$$\therefore y=5 \text{ or } y=10$$

3.

$$2(x^2+3)^{\frac{1}{2}}=9 \quad \text{Square both sides}$$
$$\therefore 4(x^2+3)=81$$
$$\therefore 4x^2+12-81=0$$
$$\therefore 4x^2=69$$
$$\therefore x^2=\frac{69}{4}$$
$$\therefore x=\frac{\pm\sqrt{69}}{2}$$

Back to Unit 3: Assessment

Media Attributions

- figure1 © M. Tomczak is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 4: Solve simultaneous equations

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve simultaneous equations algebraically where the one equation is linear and the other equation is quadratic.
- Solve simultaneous equations graphically where the one equation is linear and the other equation is quadratic.

What you should know

Before you start this unit, make sure you can:

- Solve simultaneous equations where both equations are linear. Refer to level 2 subject outcome 2.3 unit 4 if you need help with this.
- Solve quadratic equations by factorising and using the quadratic formula. Refer to unit 1, unit 2 and unit 3 in this subject outcome if you need help with this.
- Sketch linear functions. Refer to level 2 subject outcome 2.1 unit 1 if you need help with this.
- Sketch quadratic functions. Refer to level 2 subject outcomes 2.1 unit 2 and level 3 subject outcome 2.1 unit 1 if you need help with this.

Here is a short self-assessment to make sure you have the simultaneous equation skills you need to proceed with this unit.

Solve the following equations simultaneously:

1. $5a + 3b + 11 = 0$ and $4b = -2a - 10$
2. $\dfrac{x+1}{y} = 7$ and $\dfrac{x}{y+1} = 6$

Solutions

1. $5a + 3b + 11 = 0 \quad (1)$
   $4b = -2a - 10 \quad (2)$
   From (2):
   $4b = -2a - 10$
   $\therefore 2a = -4b - 10$
   $\therefore a = -2b - 5 \quad (3)$
   Substitute (3) into (1):
   $5(-2b - 5) + 3b + 11 = 0$
   $\therefore -10b - 25 + 3b + 11 = 0$
   $\therefore -7b = 14$
   $\therefore b = -2$
   Substitute $b = -2$ into (3):

$$a = -2(-2) - 5$$
$$\therefore a = -1$$
$$a = -1$$
$$b = -2$$

2. $$\frac{x+1}{y} = 7 \qquad (1)$$
$$\frac{x}{y+1} = 6 \qquad (2)$$
From (1):
$$x + 1 = 7y$$
$$\therefore x = 7y - 1 \qquad (3)$$
From (2):
$$x = 6y + 6 \qquad (4)$$
Substitute (3) into (4):
$$7y - 1 = 6y + 6$$
$$\therefore y = 7$$
Substitute $y = 7$ into (3):
$$x = 7(7) - 1$$
$$\therefore x = 48$$
$$x = 48$$
$$y = 7$$

Introduction

We learnt how to solve simultaneous equations in level 2 subject outcome 2.3 unit 4. In that unit, we only solved two equations simultaneously at a time and both equations were linear equations.

In all cases in that unit, we got two answers or one answer pair (x, y) and we made the point that to solve for two unknowns, you need two equations. To solve for three unknowns, you would need three equations and so on.

We also learnt that we can solve equations simultaneously in a few different ways:

- Algebraically:
  - The substitution method requires us to express one of the unknowns in terms of the other from one of the equations and then to substitute this into the other equation.
  - The elimination method requires us to multiply through each equation by whatever constants will allow us to eliminate one of the unknowns by adding or subtracting the resulting equations.
- Graphically: The solution to simultaneous equations is the point where the graphs of the two functions intersect.

We will use all of these methods to solve simultaneous equations in this unit. The only difference is that one of the equations will be linear and the other will be quadratic. You may choose the method you are most comfortable with to solve simultaneously unless the instruction says otherwise.

Think for a moment how many answer pairs you can expect to get when you solve a linear and quadratic equation simultaneously.

Solve simultaneous equations algebraically: the substitution method

The best way to learn how to solve a linear and quadratic equation simultaneously by substitution is to work through an example.

Example 4.1

Solve for x and y in the following equations using substitution:

$x^2 - y = -3x - 2$
$12y - 18x = 24$

Solution

We need to express one of the variables in terms of the other. When one of the equations is quadratic and the other linear, it is often best to express the variable that is not squared in terms of the other in the linear equation.

$x^2 - y = -3x - 2 \qquad (1)$
$12y - 18x = 24 \qquad (2)$

From (2):

$$12y - 18x = 24$$
$$\therefore 12y = 18x + 24$$
$$\therefore y = \frac{18}{12}x + 2$$
$$\therefore y = \frac{3}{2}x + 2 \qquad (3)$$

Substitute (3) into (1):

$$x^2 - \left(\frac{3}{2}x + 2\right) = -3x - 2$$
$$\therefore x^2 - \frac{3}{2}x - 2 + 3x + 2 = 0$$
$$\therefore x^2 + \frac{3}{2}x = 0$$
$$\therefore x\left(x + \frac{3}{2}\right) = 0$$
$$\therefore x = 0 \text{ or } x = -\frac{3}{2}$$

Now we have two solutions for x. We need to substitute these values for x into any of the other equations to find the corresponding solutions for y.

Substitute into (3):

$x = 0$:

$$y = \frac{3}{2}(0) + 2$$
$$= 2$$

$x = -\frac{3}{2}$:

$$y = \frac{3}{2}\left(-\frac{3}{2}\right) + 2$$
$$= -\frac{9}{4} + \frac{8}{4}$$
$$= -\frac{1}{4}$$

Therefore $x = 0$ and $y = 2$ or $x = -\frac{3}{2}$ and $y = -\frac{1}{4}$. Another way of writing the answer would be as the ordered pairs $(0, 2)$ and $(-\frac{3}{2}, -\frac{1}{4})$.

We can see that because one of the equations is a quadratic, the system of simultaneous equations generates two solution pairs.



Example 4.2

Solve for x and y in the following equations using substitution:

$6x + y = x^2 + 5$
$y = -2x + 2$

Solution

$6x + y = x^2 + 5 \qquad (1)$
$y = -2x + 2 \qquad (2)$

In equation (2), y is already the subject of the formula so we can go straight to the substitution.

Substitute (2) into (1):

$$6x + (-2x + 2) = x^2 + 5$$
$$\therefore 6x - 2x + 2 - x^2 - 5 = 0$$
$$\therefore -x^2 + 4x - 3 = 0$$
$$\therefore x^2 - 4x + 3 = 0$$
$$\therefore (x - 3)(x - 1) = 0$$
$$\therefore x = 3 \text{ or } x = 1$$

Substitute into (2):

$x = 3$:
$y = -2(3) + 2$
$= -4$

$x = 1$:
$y = -2(1) + 2$
$= 0$

Therefore, $x = 3$ and $y = -4$ or $x = 1$ and $y = 0$.



Exercise 4.1

Solve the following simultaneous equations using the substitution method:

1. $y = x^2 + 2x - 4$ and $y = 2x$
2. $y = 2x^2 + 7x - 4$ and $y = 2x - 1$

3. $5x^2 + 8y = 104$ and $4x + y = 19$

The full solutions are at the end of the unit.

Solve simultaneous equations algebraically: the elimination method

The next example illustrates how to solve a system of simultaneous equations using the elimination method.



Example 4.3

Solve for x and y in the following equations using elimination:

$y = x^2 + 5x - 24$
$y - x + 19 = 0$

Solution

$y = x^2 + 5x - 24 \qquad (1)$
$y - x + 19 = 0 \qquad (2)$

From 2:
$y - x + 19 = 0$
$\therefore y = x - 19 \qquad (3)$

We do not need to multiply through either of the equations in order to create an unknown to eliminate. Both equations contain y. This is the unknown that we will eliminate.
First we subtract equation 3 from equation 1:
(1) – (3):
Formula does not parse

$x^2 + 4x - 5 = 0$
$\therefore (x - 1)(x + 5) = 0$
$\therefore x = 1 \text{ or } x = -5$

Now we can substitute both our vales of x into any of the other equations to find the corresponding values for y.

Substitute into (3):
$x = 1$:
$y = 1 - 19$
$= -18$
$x = -5$:
$y = -5 - 19$
$= -24$
Therefore $x = 1$ and $y = -18$ or $x = -5$ and $y = -24$

Most of the time solving simultaneous equations algebraically will be easier using substitution but you are free to use the elimination method if you like.

Exercise 4.2

Solve the following systems of simultaneous equations algebraically, leaving your answer in simplest surd form if necessary:

1. $y = -x + 5$ and $y = x^2 - 3x + 5$
2. $y - x^2 = 6 - 5x$ and $y + 1 = x$
3. $4y = 2x + 2$ and $y + 5 = 2x^2 - 3x$
4. $a - 2b = -1$ and $a - 12b = 2(b^2 - 2)$
5. $3x^2 - 4x = y - 1$ and $3x = 4y + 2$

The full solutions are at the end of the unit.

Solve simultaneous equations graphically

We said in the introduction that the graphical solution to simultaneous equations is the point where the functions intersect. Remember an equation like $y = x^2 + 2x + 1$ is really a function. If you think about a parabola and a straight line, it makes sense why we usually get two solution pairs. Most of the time, the straight line will intersect the parabola in two places.

Example 4.4

Solve for x and y in the following equations graphically:

$6x + y = x^2 + 5$
$y = -2x + 2$

Solution

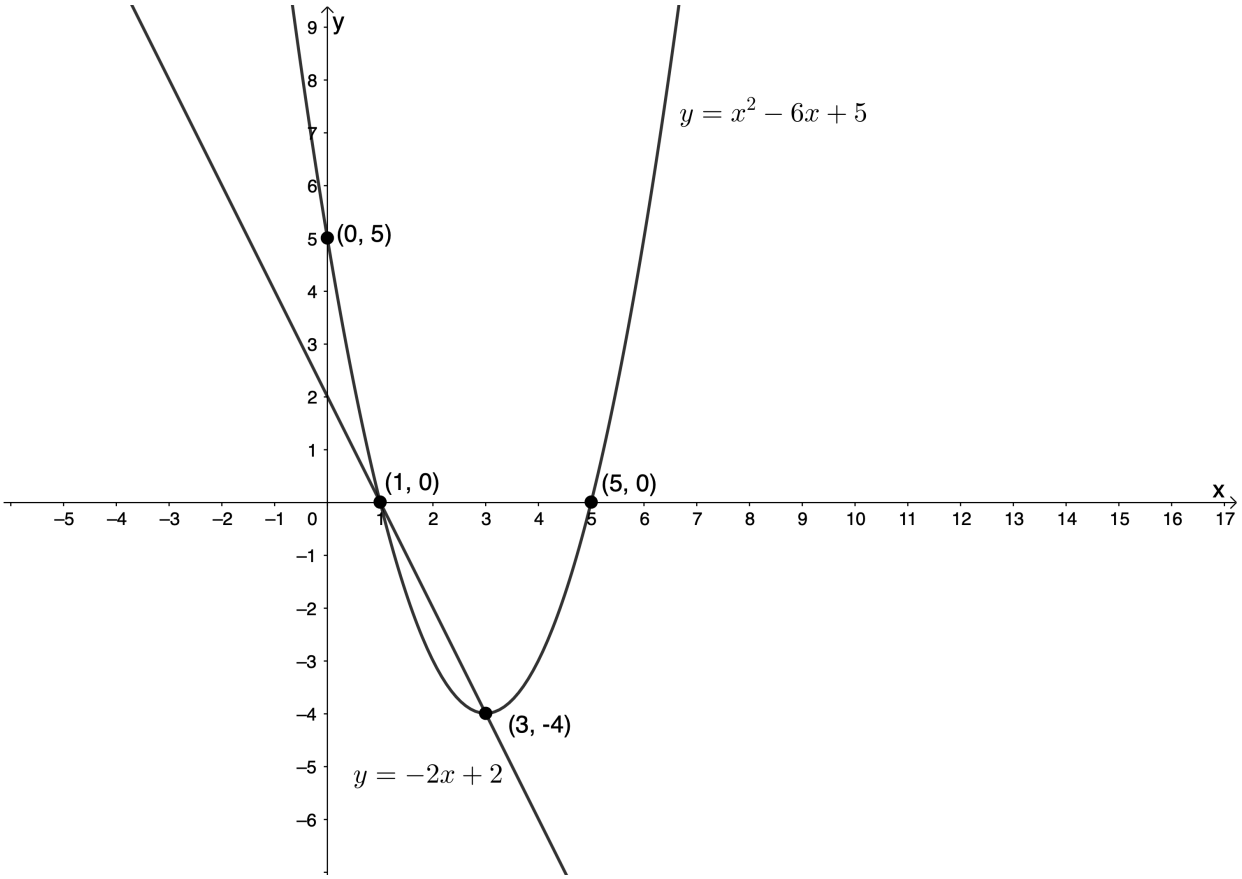
This is the same system of simultaneous equations we solved in Example 4.2 so we know that the solution is $x = 3$ and $y = -4$ or $x = 1$ and $y = 0$. We can represent the solution as the coordinates of two points on the Cartesian plane – $(3, -4)$ and $(1, 0)$. Let's see how we can arrive at the same solution graphically.

Step 1: Get the functions in standard form
To begin with, we need to express both equations in standard form ($y = mx + c / y = ax + q$ and $y = ax^2 + bx + c / y = a(x + p)^2 + q$) so that we can sketch the functions. The linear equation is already in standard form.

Formula does not parse
$y = -2x + 2 \qquad (2)$

From (1):
$$6x + y = x^2 + 5$$
$$\therefore y = x^2 - 6x + 5 \qquad (3)$$

Step 2: Sketch the functions
Next, we need to sketch both functions on the same set of axes. We are not going to go through all the steps to sketching functions here. Refer to the section on prior learning above if you need help with sketching linear and quadratic functions.



Step 3: Read off the solutions (the points of intersection)
The solution is $(1, 0)$ or $(3, -4)$. Notice that this is exactly the same result we got when solving algebraically.

Exercise 4.3

Questions 2. and 3. are taken from Everything Maths Grade 11 Exercise 2 – 9.

Solve the following systems of equations graphically. Check your solutions algebraically.

1. $-2x^2 = y$
   $y - x = -1$
2. $x^2 - 1 - y = 0$
   $y + x - 5 = 0$
3. $x + y - 10 = 0$
   $x^2 - 2 - y = 0$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve a system of two simultaneous equations algebraically where one of the equations is linear and the other is quadratic.
- How to solve a system of two simultaneous equations graphically where one of the equations is linear and the other is quadratic.

Unit 4: Assessment

Suggested time to complete: 50 minutes

1. Solve the following systems of simultaneous equations algebraically:
   a. $y = 3x + 2$
      $y = 3x^2 + 6x - 4$
   b. $2x - y = 1$
      $y - x^2 = -3x + 3$
   c. $y = (x - 2)^2 - 9$
      $y = 2x + 4$
2. Show the solution to question 1.b. above graphically.
3. Solve the following system of simultaneous equations graphically:
   $x^2 - 4x = y - 4$
   $y + 3x = 6$

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. $y = x^2 + 2x - 3 \qquad (1)$
   $y = 2x \qquad (2)$
   Substitute (2) into (1):
   $2x = x^2 + 2x - 4$
   $\therefore x^2 = 4$
   $\therefore x = \pm 2$
   Substitute into (2):
   $x = 2$: $x = -2$:
   $y = 4$ $y = -4$
   $x = 2$ and $y = 4$ or $x = -2$ and $y = -4$
2. $y = 2x^2 + 7x - 4 \qquad (1)$
   $y = 2x - 1 \qquad (2)$
   Substitute (2) into (1):

$$2x - 1 = 2x^2 + 7x - 4$$
$$\therefore 2x^2 + 5x - 3 = 0$$
$$\therefore (2x - 1)(x + 3) = 0$$
$$\therefore x = \frac{1}{2} \text{ or } x = -3$$

Substitute into (2):

$x = \frac{1}{2}$: $x = -3$:

$y = 2\left(\frac{1}{2}\right) - 1 = 0$ $y = 2(-3) - 1 = -7$

$x = \frac{1}{2}$ and $y = 0$ or $x = -3$ and $y = -7$

3. $5x^2 + 8y = 104 \quad (1)$

$4x + y = 19 \quad (2)$

From (2):

$$4x + y = 19$$
$$\therefore y = 19 - 4x \quad (3)$$

Substitute (3) into (1):

$$5x^2 + 8(19 - 4x) = 104$$
$$\therefore 5x^2 + 152 - 32x = 104$$
$$\therefore 5x^2 - 32x + 48 = 0$$
$$\therefore (5x - 12)(x - 4) = 0$$
$$\therefore x = \frac{12}{5} \text{ or } x = 4$$

Substitute into (3):

$x = \frac{12}{5}$: $x = 4$:

$y = 19 - 4\left(\frac{12}{5}\right)$

$= 19 - \frac{48}{5}$ $y = 19 - 4(4)$

$= \frac{95 - 48}{5}$ $= 19 - 16$

$= \frac{47}{5}$ $= 3$

$x = \frac{12}{5}$ and $y = \frac{47}{5}$ or $x = 4$ and $y = 3$

Back to Exercise 4.1

Exercise 4.2

1. $y = -x + 5 \quad (1)$

$y = x^2 - 3x + 5 \quad (2)$

Substitute (1) into (2):

$$-x + 5 = x^2 - 3x + 5$$
$$\therefore x^2 - 2x = 0$$
$$\therefore x(x - 2) = 0$$
$$\therefore x = 0 \text{ or } x = 2$$

Substitute into (1):

$x = 0$: $x = 2$:

$y = 0 + 5$ $y = -2 + 5$

$= 5$ $= 3$

$x = 0$ and $y = 5$ or $x = 2$ and $y = 3$

2. $y - x^2 = 6 - 5x \qquad (1)$

$y + 1 = x \qquad (2)$

From (2):

$y + 1 = x$

$\therefore y = x - 1 \qquad (3)$

Substitute (3) into (1):

$(x - 1) - x^2 = 6 - 5x$

$\therefore x^2 - 6x + 7 = 0$

$a = 1, b = -6, c = 7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 28}}{2}$$

$$= \frac{6 \pm \sqrt{8}}{2}$$

$$= \frac{6 \pm 2\sqrt{2}}{2}$$

$$= 3 \pm \sqrt{2}$$

Substitute into (3):

$x = 3 + \sqrt{2}$: $x = 3 - \sqrt{2}$:

$y = 3 + \sqrt{2} - 1$ $y = 3 - \sqrt{2} - 1$

$x = 3 + \sqrt{2}$ and $y = 3 + \sqrt{2} - 1$ or $x = 3 - \sqrt{2}$ and $y = 3 - \sqrt{2} - 1$

3. $4y = 2x + 2 \qquad (1)$

$y + 5 = 2x^2 - 3x \qquad (2)$

From (1):

$4y = 2x + 2$

$$\therefore y = \frac{2x + 2}{4}$$

$$= \frac{2(x + 1)}{4}$$

$$= \frac{x + 1}{2} \qquad (3)$$

Substitute (3) into (2):

$$\left(\frac{x + 1}{2}\right) + 5 = 2x^2 - 3x$$

$\therefore x + 1 + 10 = 4x^2 - 6x$

$\therefore 4x^2 - 7x - 11 = 0$

$\therefore (4x - 11)(x + 1) = 0$

$\therefore x = \frac{11}{4}$ or $x = -1$

Substitute into (3):

$x = \frac{11}{4}$:

$$y = \frac{\frac{11}{4} + 1}{2}$$
$$= \frac{\frac{15}{4}}{2}$$
$$= \frac{15}{4} \times \frac{1}{2}$$
$$= \frac{15}{8}$$

$x = -1$:

$$y = \frac{-1 + 1}{2}$$
$$= 0$$

$x = \frac{11}{4}$ and $y = \frac{15}{8}$ or $x = -1$ and $y = 0$

4. $a - 2b = -1 \qquad (1)$

$a - 12b = 2(b^2 - 2) \qquad (2)$

From (1):

$a - 2b = -1$

$\therefore a = 2b - 1 \qquad (3)$

Substitute (3) into (2):

$(2b - 1) - 12b = 2(b^2 - 2)$

$\therefore 2b - 1 - 12b = 2b^2 - 4$

$\therefore 2b^2 + 10b - 3 = 0$

$a = 2, b = 10, c = -3$

$b = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Don't confuse the unknown b for the coefficient b

$$\therefore b = \frac{-(10) \pm \sqrt{(10)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{10 \pm \sqrt{100 + 24}}{4}$$
$$= \frac{10 \pm \sqrt{124}}{4}$$
$$= \frac{10 \pm 2\sqrt{31}}{4}$$
$$= \frac{5 \pm \sqrt{31}}{2}$$

Substitute into (3):

$b = \frac{5 + \sqrt{31}}{2}$:

$$a = 2\left(\frac{5 + \sqrt{31}}{2}\right) - 1$$
$$= 5 + \sqrt{31} - 1$$
$$= 4 + \sqrt{31}$$

$b = \frac{5 - \sqrt{31}}{2}$:

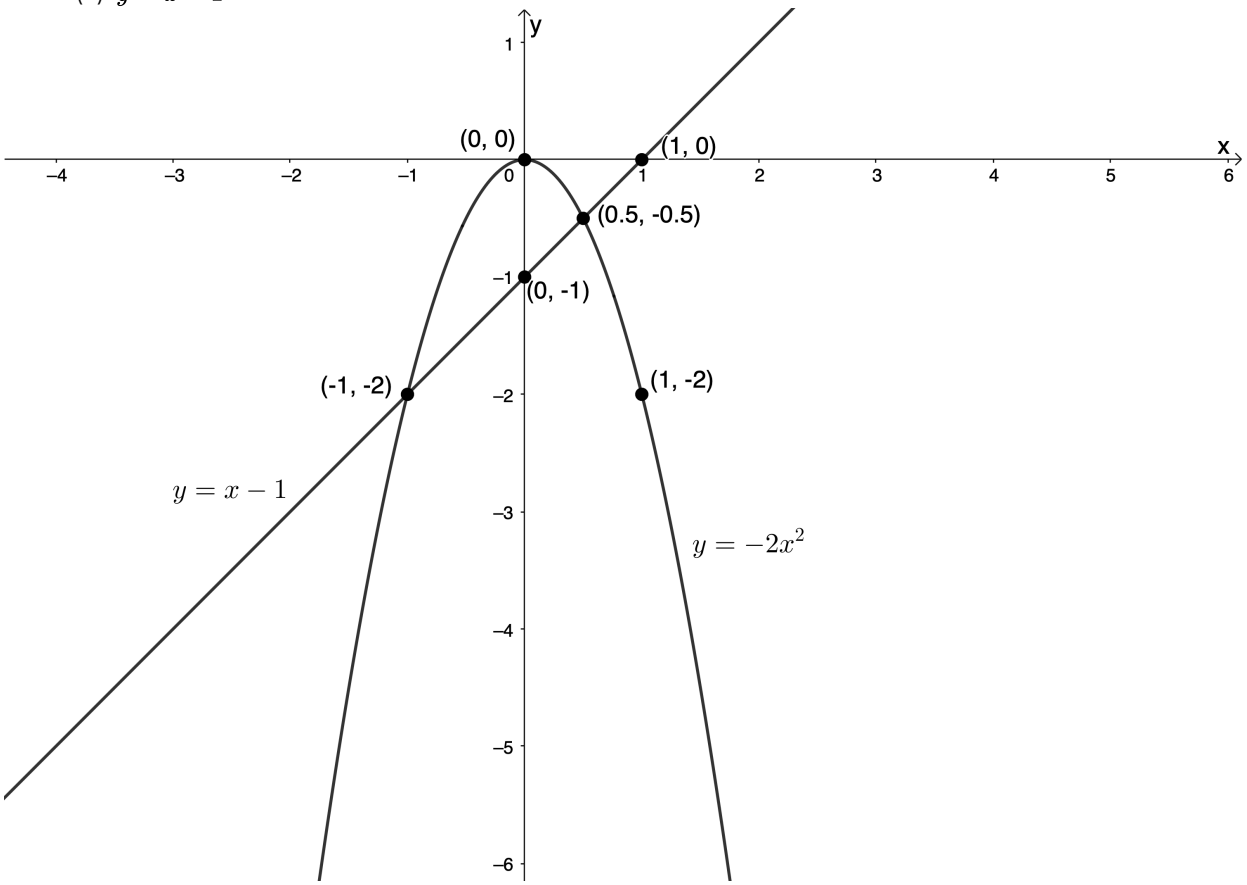
$$a = 2\left(\frac{5 - \sqrt{31}}{2}\right) - 1$$
$$= 5 - \sqrt{31} - 1$$
$$= 4 - \sqrt{31}$$

$b = \frac{5 + \sqrt{31}}{2}$ and $a = 4 + \sqrt{31}$ or $b = \frac{5 - \sqrt{31}}{2}$ and $a = 4 - \sqrt{31}$

Back to Exercise 4.2

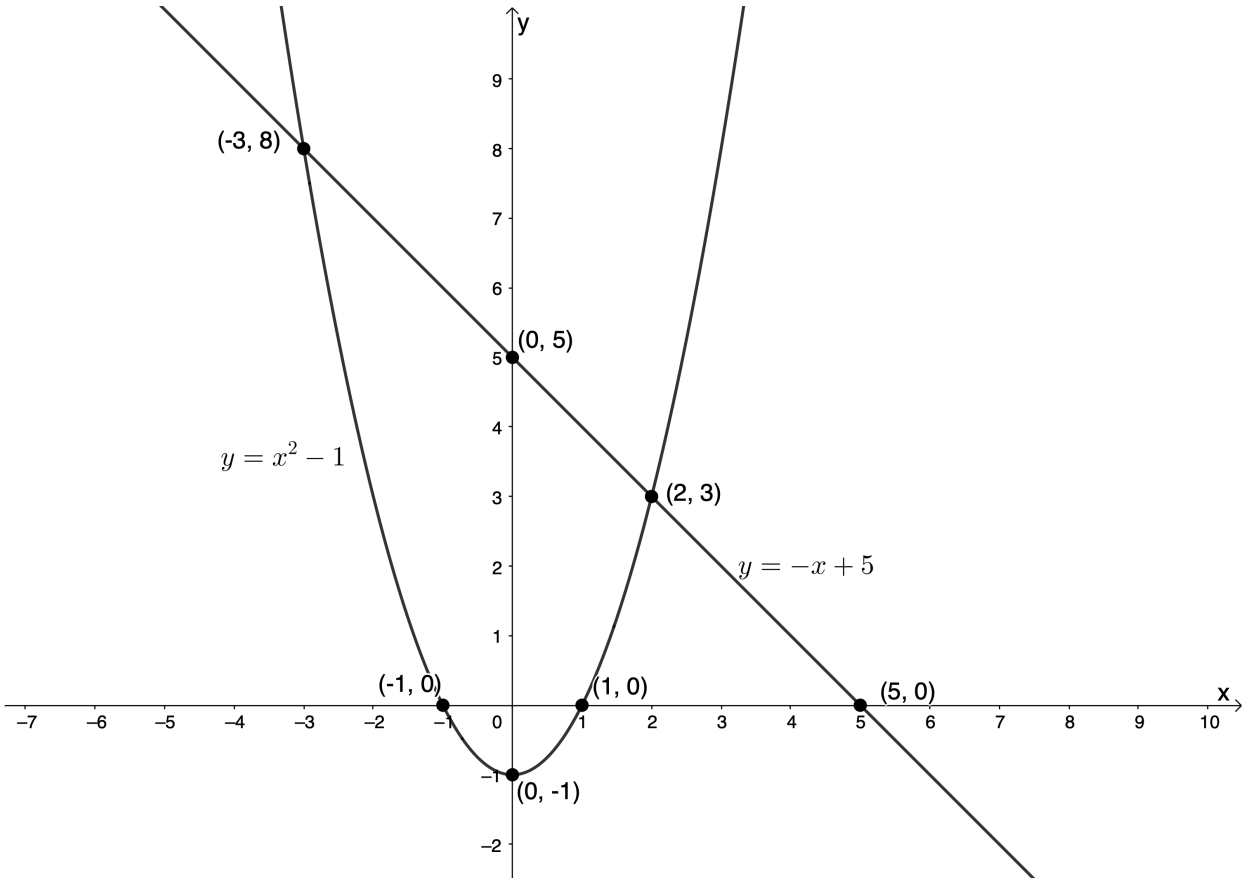
Exercise 4.3

1. $-2x^2 = y \qquad (1)$
   $y - x = -1 \qquad (2)$
   From (2): $y = x - 1$



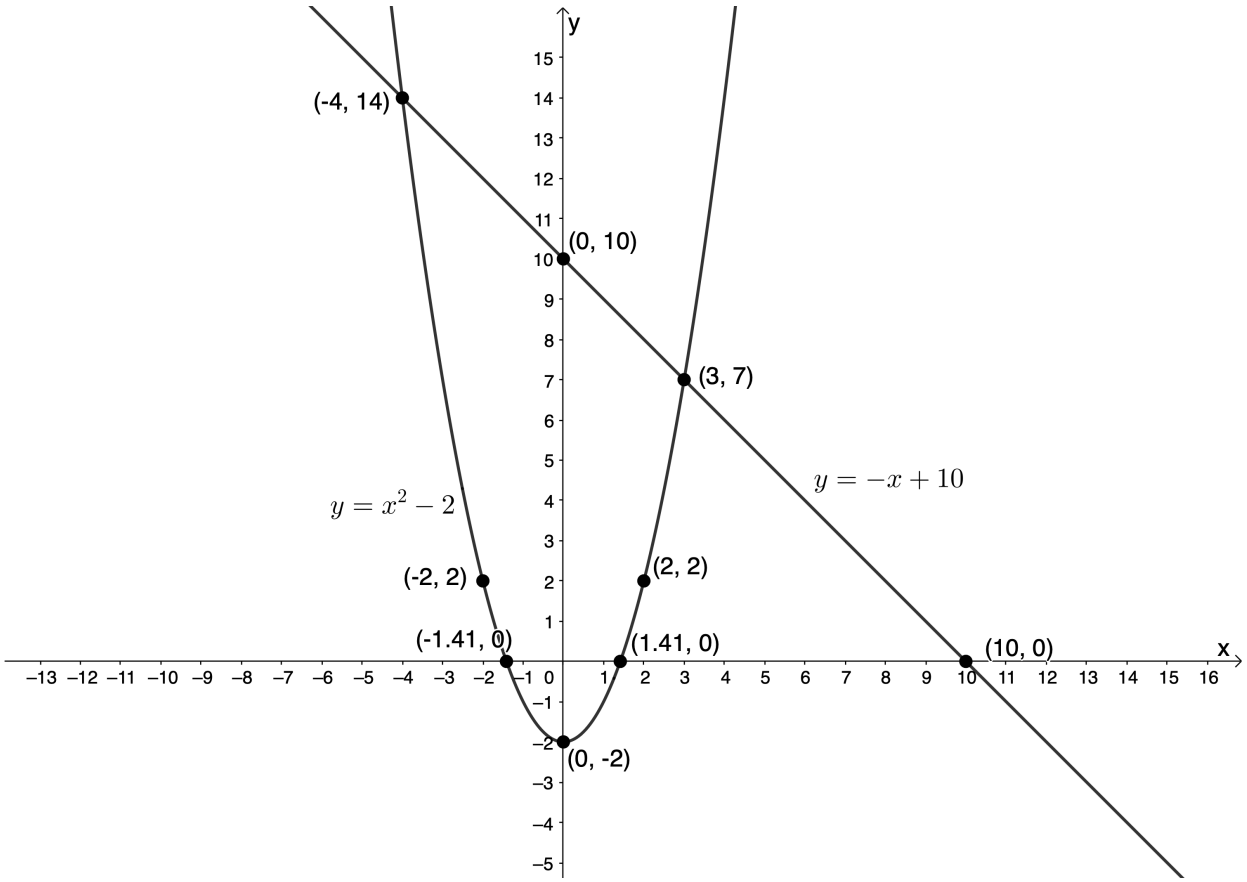
   $(-1, -2)$ or $(\frac{1}{2}, -\frac{1}{2})$

2. $x^2 - 1 - y = 0 \qquad (1)$
   $y + x - 5 = 0 \qquad (2)$
   From (1):
   $x^2 - 1 - y = 0$
   $\therefore y = x^2 - 1$
   From (2):
   $y = -x + 5$



$(2, 3)$ or $(-3, 8)$

3. $x + y - 10 = 0 \qquad (1)$
   $x^2 - 2 - y = 0 \qquad (2)$
   From (1):
   $x + y - 10 = 0$
   $\therefore y = -x + 10$
   From (2):
   $x^2 - 2 - y = 0$
   $\therefore y = x^2 - 2$



$(3, 7)$ or $(-4, 14)$

Back to Exercise 4.3

Unit 4: Assessment

1.

a. $y = 3x + 2 \quad (1)$
$y = 3x^2 + 6x - 4 \quad (2)$
Substitute (1) into (2):
$3x + 2 = 3x^2 + 6x - 4$
$\therefore 3x^2 + 3x - 6 = 0$
$\therefore (3x - 3)(x + 2) = 0$
$\therefore x = 1$ or $x = -2$
Substitute into (1):
$x = 1$: $y = 3(1) + 2 = 5$
$x = -2$: $y = 3(-2) + 2 = -4$
$x = 1$ and $y = 5$ or $x = -2$ and $y = -4$

b. $2x - y = 1 \quad (1)$
$y - x^2 = -3x + 3 \quad (2)$
From (1):
$2x - y = 1$
$\therefore y = 2x - 1 \quad (3)$
Substitute (3) into (2):

$$(2x-1)-x^2=-3x+3$$
$$\therefore x^2-5x+4=0$$
$$\therefore (x-1)(x-4)=0$$
$$\therefore x=1 \text{ or } x=4$$

Substitute into (3):

$x=1$: $y=2(1)-1$ $=1$

$x=4$: $y=2(4)-1$ $=7$

$x=1$ and $y=1$ or $x=4$ and $y=7$

c. $y=(x-2)^2-9 \quad (1)$

$y=2x+4 \quad (2)$

Substitute (2) into (1):

$$2x+4=(x-2)^2-9$$
$$\therefore 2x+4=x^2-4x+4-9$$
$$\therefore x^2-6x-9=0$$
$$a=1, b=-6, c=-9$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$\therefore x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-9)}}{2(1)}$$
$$=\frac{6\pm\sqrt{36+36}}{2}$$
$$=\frac{6\pm\sqrt{72}}{2}$$
$$=\frac{6\pm 6\sqrt{2}}{2}$$
$$=3\pm 3\sqrt{2}$$

Substitute into (2):

$x=3+3\sqrt{2}$: $y=2\left(3+3\sqrt{2}\right)+4$ $=6+6\sqrt{2}+4$ $=10+6\sqrt{2}$

$x=3-3\sqrt{2}$: $y=2\left(3-3\sqrt{2}\right)+4$ $=6-6\sqrt{2}+4$ $=10-6\sqrt{2}$

$x=3+3\sqrt{2}$ and $y=10+6\sqrt{2}$ or $x=3-3\sqrt{2}$ and $y=10-6\sqrt{2}$

2. $2x-y=1 \quad (1)$

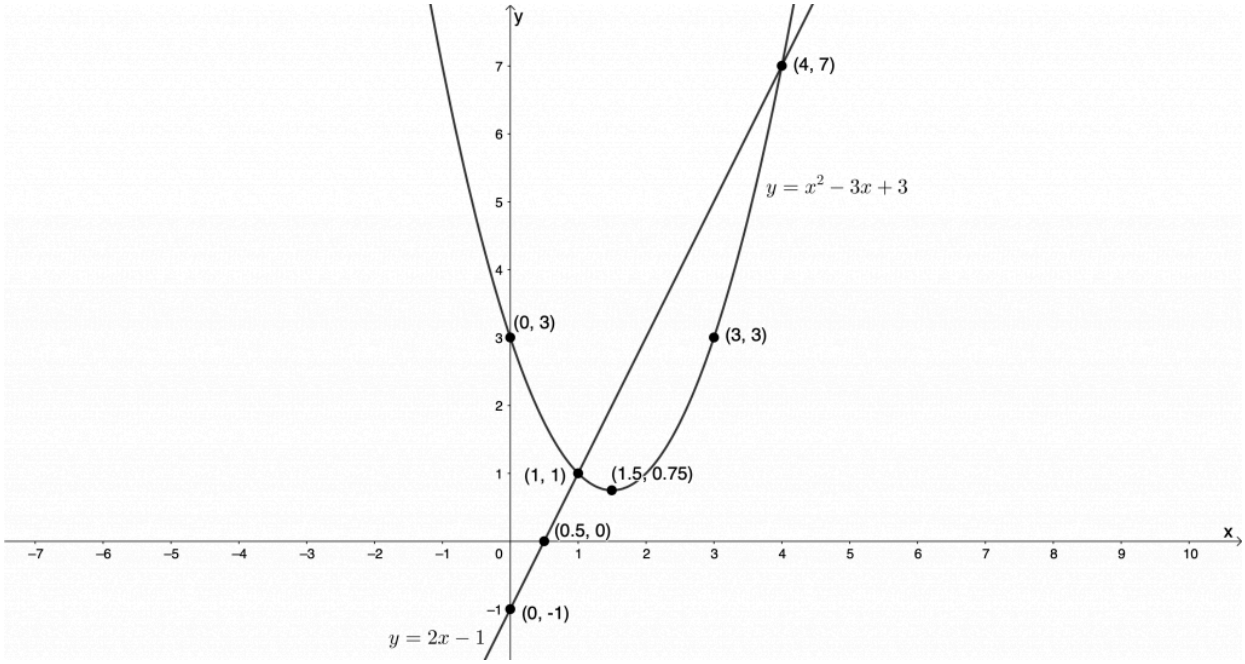
$y-x^2=-3x+3 \quad (2)$

From (1):

$$2x-y=1$$
$$\therefore y=2x-1$$

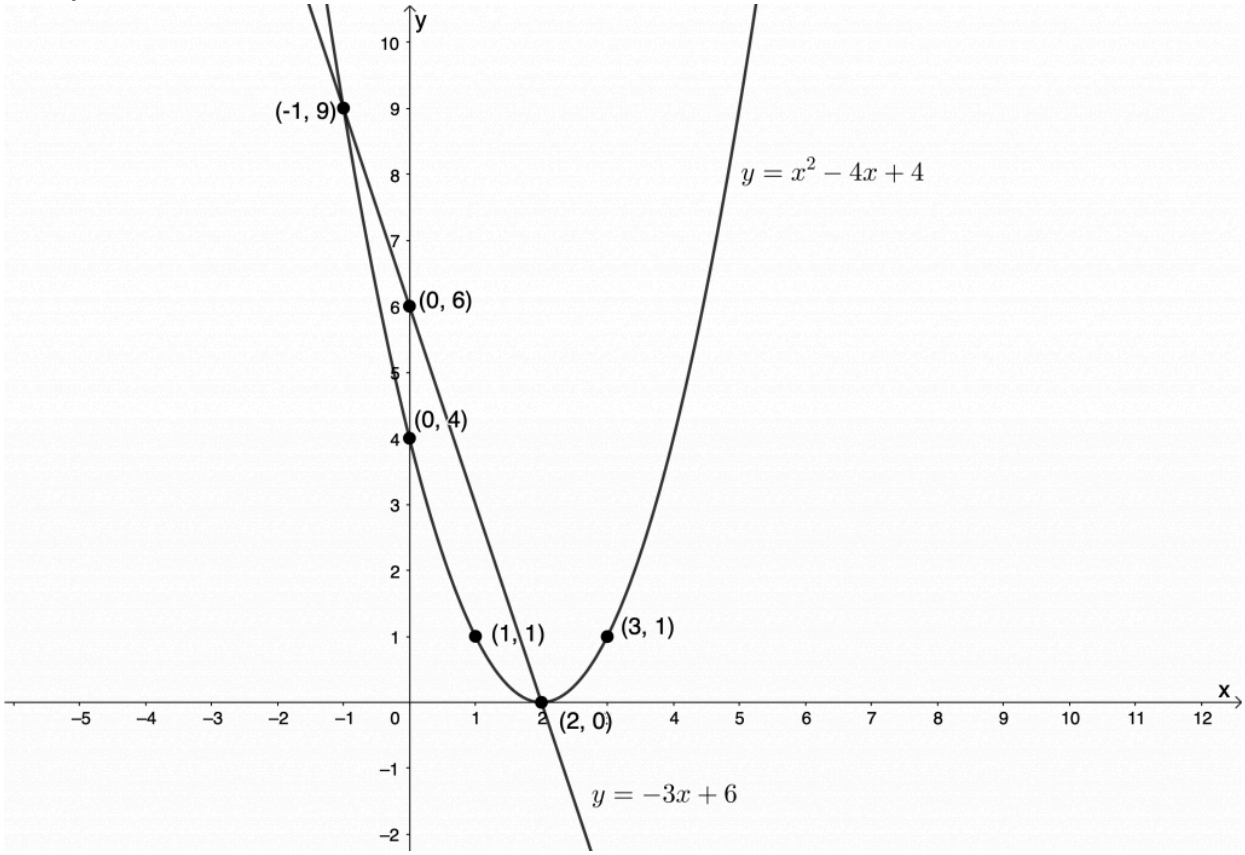
From (2):

$$y-x^2=-3x+3$$
$$\therefore y=x^2-3x+3$$



$(4, 7)$ or $(1, 1)$

3. $x^2 - 4x = y - 4 \qquad (1)$
   $y + 3x = 6 \qquad (2)$
   From (1):
   $x^2 - 4x = y - 4$
   $\therefore y = x^2 - 4x + 4$
   From (2):
   $y + 3x = 6$
   $\therefore y = -3x + 6$



$(-1, 9)$ or $(2, 0)$

Back to Unit 4: Assessment

Media Attributions

- example4.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise4.3A1 © Geogebra is licensed under a CC BY (Attribution) license
- exercise4.3A2 © Geogebra is licensed under a CC BY (Attribution) license
- exercise4.3A3 © Geogebra is licensed under a CC BY (Attribution) license
- assessmentA2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA3png © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 5: Solve quadratic inequalities

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the critical values.
- Solve quadratic inequalities using the table and graphical methods.
- Represent the solution using set builder notation, interval notation and on a number line.

What you should know

Before you start this unit, make sure you can:

- Solve linear inequalities. Refer to level 2 subject outcome 2.3 unit 3 if you need help with this.
- Represent inequalities using set interval notation, set builder notation and on a number line. Refer to level 2 subject outcome 2.3 unit 3 if you need help with this.

Here is a short self-assessment to make sure you have the basic skills you need to proceed with this unit.

Solve for x and show your answer as indicated:

1. $2-3x \leq 3-5x$ (show your answer on a number line)
2. $\dfrac{7x+3}{4} < 4(2x-7)$ (show your answer using interval and set builder notation)

Solutions

1.

$$2-3x \leq 3-5x$$
$$\therefore -3x \leq 1-5x \qquad \text{Subtract 2 from both sides of the inequality}$$
$$\therefore 2x \leq 1 \qquad \text{Add } 5x \text{ to both sides of the inequality}$$
$$\therefore x \leq \frac{1}{2}$$

2.

$$\frac{7x+3}{4} < 4(2x-7)$$
$$\therefore 7x+3 < 16(2x-7)$$
$$\therefore 7x+3 < 32x-112$$
$$\therefore -25x < -115 \qquad \text{Remember, we need to change the direction of the inequality sign when multipling by a negative number}$$
$$\therefore x > \frac{115}{25}$$
$$\therefore x > \frac{23}{5}$$

Interval

notation: $x \in (\frac{23}{5}, \infty)$

Set builder notation: $\{x | x \in \mathbb{R}, x > \frac{23}{5}\}$

Introduction

We learnt how to solve linear inequalities in level 2 subject outcome 2.3 unit 3. We saw that solving for the unknown in an inequality is very similar to solving for an unknown in an equation.

In this unit, we are going to learn how to solve quadratic inequalities. Again, the basic techniques for solving inequalities and equations are very similar. The biggest difference is still the fact that we have to flip the inequality sign around if we multiply both sides of an inequality by a negative number.

Solve simple quadratic inequalities

If $ax^2 + bx + c = 0$ is the standard form of a quadratic equation, then it stands to reason that the following are the standard forms of a quadratic inequality:

- $ax^2 + bx + c \geq 0$
- $ax^2 + bx + c > 0$
- $ax^2 + bx + c \leq 0$
- $ax^2 + bx + c < 0$

We know that quadratic equations give us two solutions or roots. But how does this translate into the solutions to quadratic inequalities? Are there always two ranges?

A useful way to think about the solutions to a quadratic inequality is to think about where the graph of the quadratic equation lies in relation to the x-axis. Think about the quadratic function $f(x) = x^2 + 3x - 4$ (see figure 1).

We know that the solution to $x^2 + 3x - 4 = 0$ gives us the x-intercepts (see Figure 1). This is where the function $f(x) = 0$. The x-values of the x-intercepts are the roots. They are the specific values that make the function value zero.

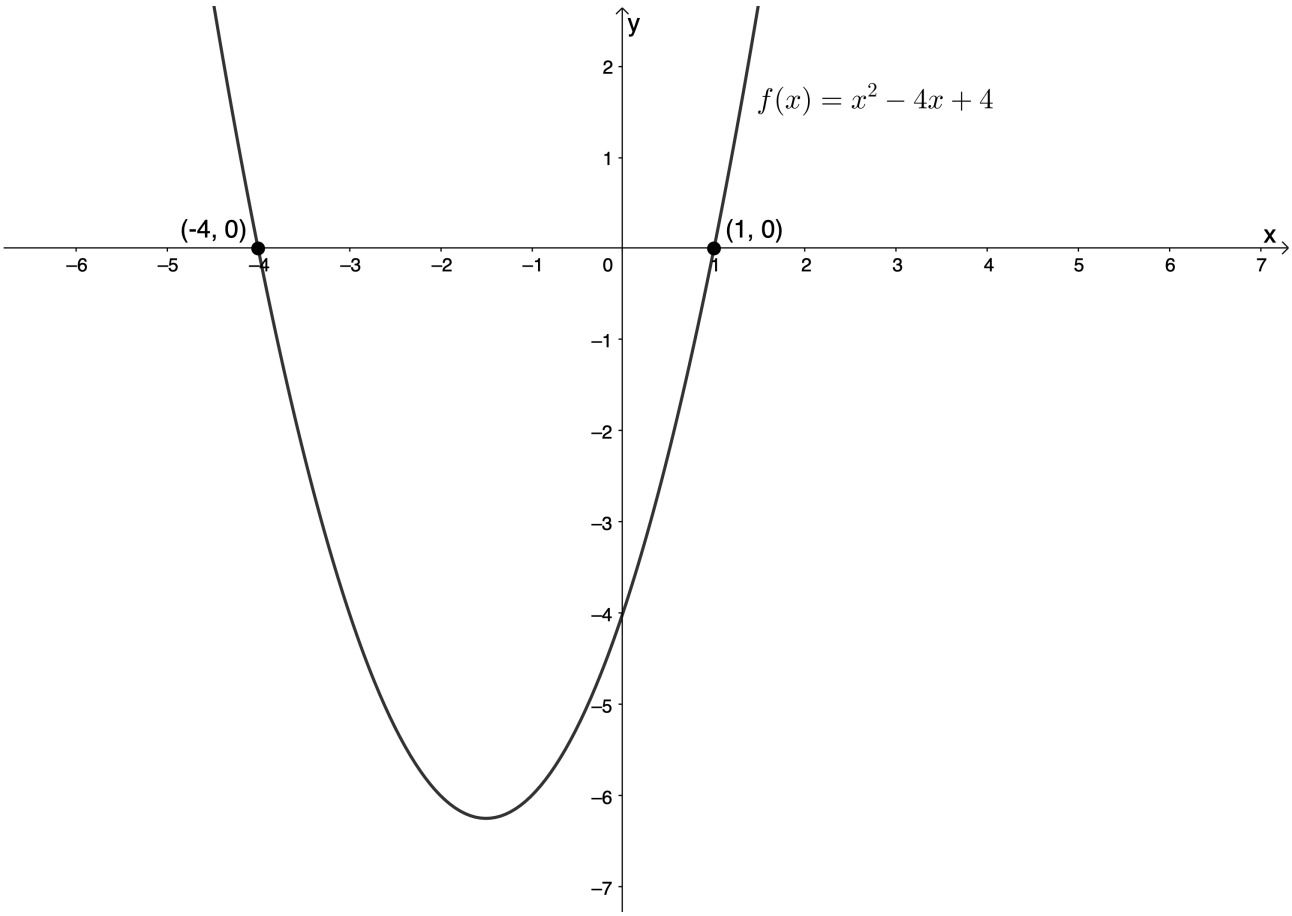


Figure 1: Graph of the function $f(x) = x^2 + 3x - 4$ showing the roots or x-intercepts

Now, if we solve the quadratic inequality of $x^2 + 3x - 4 \leq 0$, what x-values do you think will satisfy this inequality? What x-values will make $f(x) \leq 0$?

You should be able to see that any value for x that lies **between** the x-intercepts will result in $f(x) \leq 0$ (that part of the graph of the function **below** the x-axis) (see Figure 2).

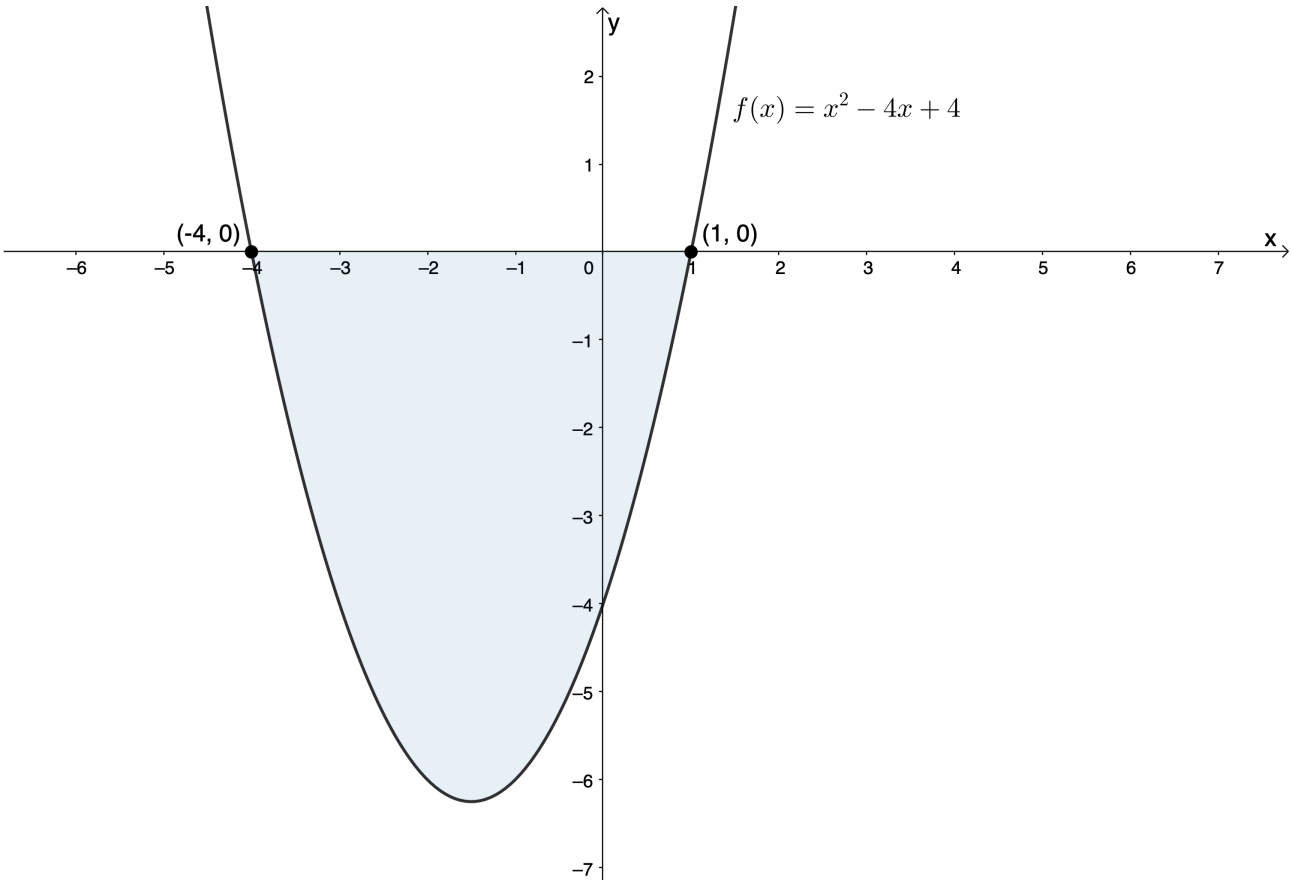


Figure 2: Region for which $x^2 + 3x - 4 \leq 0$

We can present the solution to the quadratic inequality $x^2 + 3x - 4 \leq 0$ on a number line (see Figure 3).

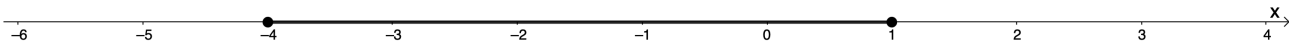


Figure 3: Solution to $x^2 + 3x - 4 \leq 0$

We can also write the solution as $x \in [-4, 1]$ (interval notation) or $\{x | x \in \mathbb{R}, -4 \leq x \leq 1\}$ (set builder notation). Note that because we are also interested in where the function is **equal** to zero, we include the **roots** in our solution.

Now, what do you think the solution to the quadratic inequality $x^2 + 3x - 4 > 0$ will be? Note that we are interested in where $f(x) > 0$ (in other words where the graph of the function is **above** the x-axis) and that we need to exclude the values where $f(x) = 0$ (in other words the roots).

Figure 4 shows the solution to $x^2 + 3x - 4 > 0$ graphically. We can see that we need x-values to the **left** of -4 and to the **right** of 1.

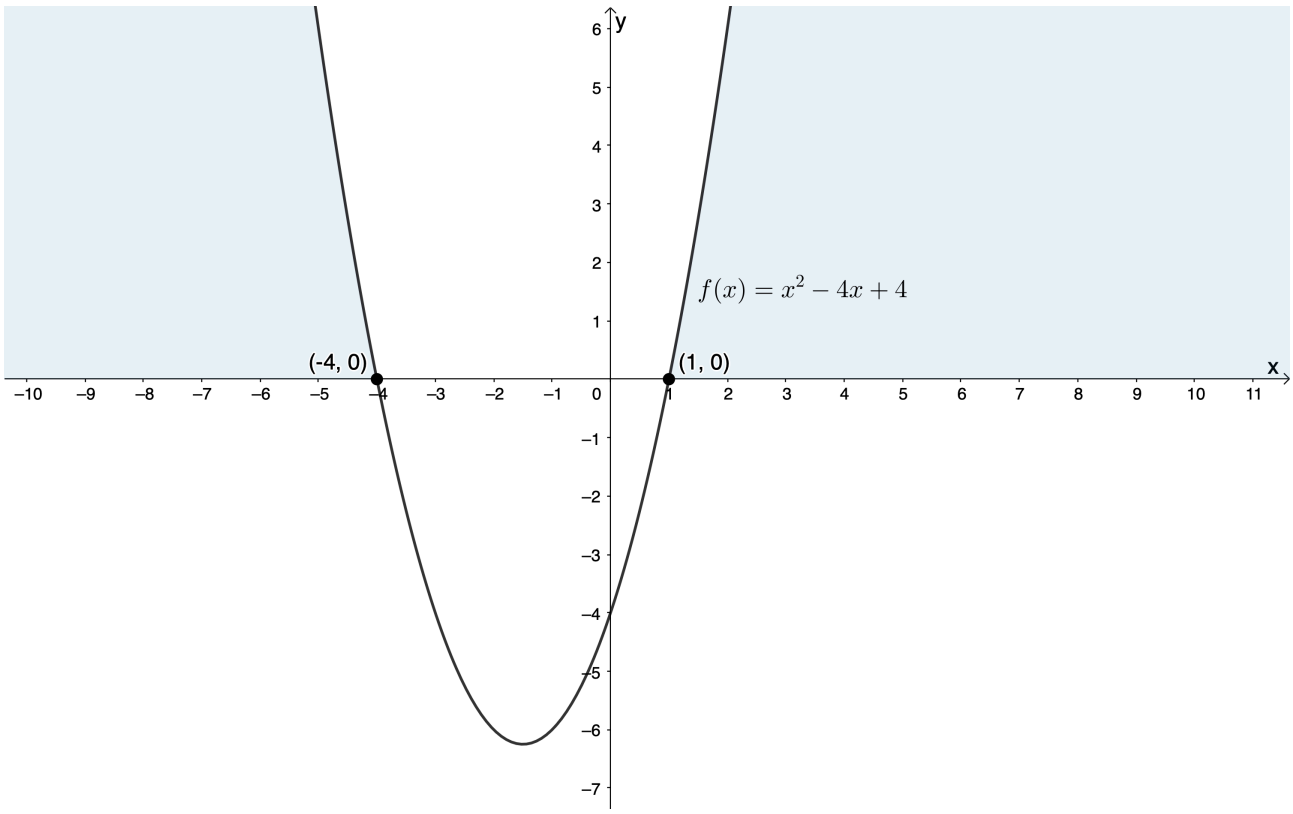


Figure 4: Region for which $x^2 + 3x - 4 > 0$

We can represent this on a number line (see Figure 5), using interval notation $x \in (-\infty, -4) \bigcup (1, \infty)$ or set builder notation $\{x | x \in \mathbb{R}, x < -4, x > 1\}$.

When representing the solution on a number line, we use a closed dot to represent values that are **included** (i.e. $\leq$ and $\geq$ or $[$ and $]$). We use an open dot to represent values that are **not included** (i.e. or $($ and $)$).

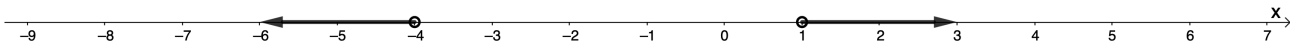


Figure 5: Solution to $x^2 + 3x - 4 > 0$

Take note!

The $\bigcup$ symbol in $x \in (-\infty, -4) \bigcup (1, \infty)$ is called **Union** and it lets us combine different intervals of values together into a single solution set.

Sketching the graph of the quadratic function each time we want to solve a quadratic inequality would be effective but not very efficient. Instead, we use a simplified representation of a table of values to figure out

where the quadratic function is positive or negative and then build our solution from here. Take a look at Example 5.1 to see how this works.

Example 5.1

Solve for x in $x^2 + 3x - 4 \geq 0$.

Solution

Notice that we are dealing with the same quadratic as above, so feel free to have a look at the figures above as you work through this solution.

Step 1: Get the inequality into standard form
The first step is always to get the quadratic inequality into standard form. Remember to flip the direction of the inequality sign if you ever multiply through by a negative number. Our inequality is already in standard form.
$x^2 + 3x - 4 \geq 0$

Step 2: Determine the critical values
The critical values are the roots of the corresponding quadratic equation. Remember these are the x-values where the function is zero as it goes from being positive to negative or negative to positive.

$$x^2 + 3x - 4 \geq 0$$
$$\therefore (x + 4)(x - 1) \geq 0$$

Our critical values are, therefore, $x = -4$ and $x = 1$.

Note

Almost always, the quadratic inequalities that you will be expected to solve will factorise easily. It would be a very nasty question that expects you to find the critical values by using the quadratic formula. If you find that your quadratic expression does not factorise, first go back to check your work before using the quadratic formula.

Step 3: Complete a table of signs
Create a table like this:

	$x < -4$	$x = -4$	$-4 < x < 1$	$x = 1$	$x > 1$
$(x + 4)(x - 1)$					

We know that each factor will be zero at its respective critical value and therefore, the whole quadratic expression will be zero at these values. So, fill this into your table.

	$x < -4$	$x = -4$	$-4 < x < 1$	$x = 1$	$x > 1$
$(x + 4)(x - 1)$		0		0	

Next, you need to determine whether the quadratic expression is positive or negative on either side of each critical value. Start by picking any value less than -4, say -5.

$(x+4) = -5+4 = -1 < 0$ and $(x-1) = -5-1 = -6 < 0$. Therefore, the overall product will be positive. Fill this into your table.

	$x < -4$	$x = -4$	$-4 < x < 1$	$x = 1$	$x > 1$
$(x+4)(x-1)$	$+$	0		0	

Complete the rest of the table by choosing the appropriate values and testing to see whether the product of the factors is positive or negative. The final table will look like this:

	$x < -4$	$x = -4$	$-4 < x < 1$	$x = 1$	$x > 1$
$(x+4)(x-1)$	$+$	0	$-$	0	$+$

Step 4: Read off the solution.
Now we can read off the solution to our inequality $x^2 + 3x - 4 \geq 0$. We need all those ranges of values that are greater than or equal to zero. These are:



$x \in (-\infty, -4] \bigcup [1, \infty)$
$\{x | x \in \mathbb{R}, x \leq -4, x \geq 1\}$

Example 5.2

Solve for x in $-x^2 + 6x \geq -7$.

Solution

Step 1: Get the inequality into standard form

$$-x^2 + 6x \geq -7$$
$$\therefore -x^2 + 6x + 7 \geq 0 \quad \text{Leave the inequality like this or multiply through by } -1$$
$$\therefore x^2 - 6x - 7 \leq 0 \quad \text{If you do multiply through by } -1 \text{ remember to flip the inequality}$$

Step 2: Determine the critical values

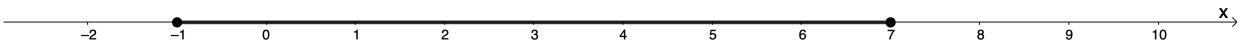
$$x^2 - 6x - 7 \leq 0$$
$$\therefore (x-7)(x+1) \geq 0$$

Our critical values are, therefore, $x = -1$ and $x = 7$.

Step 3: Complete a table of signs

	$x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$x > 7$
$(x-7)(x+1)$	$+$	0	$-$	0	$+$

Step 4: Read off the solution
$x^2 - 6x - 7 \leq 0$. Therefore:



$x \in [-1, 7]$
$\{x | x \in \mathbb{R}, -1 \leq x \leq 7\}$

Exercise 5.1

Solve for x in the following inequalities, representing your answer as indicated:

1. $x^2 + 2x - 15 < 0$ (represent your answer using set builder notation)
2. $10x^2 \leq 31x - 15$ (represent your answer using interval notation)
3. $9x(x+1) > 10$ (represent your answer using interval notation)
4. $5(2x^2 - 7) < 11x$ (represent your answer using set builder notation)
5. $\dfrac{40x - 21}{x^2} \geq 16$ (represent your answer on a number line)

The full solutions are at the end of the unit.

Solve simple quadratic inequalities graphically

Let's do a few more examples to show how to solve for x graphically.

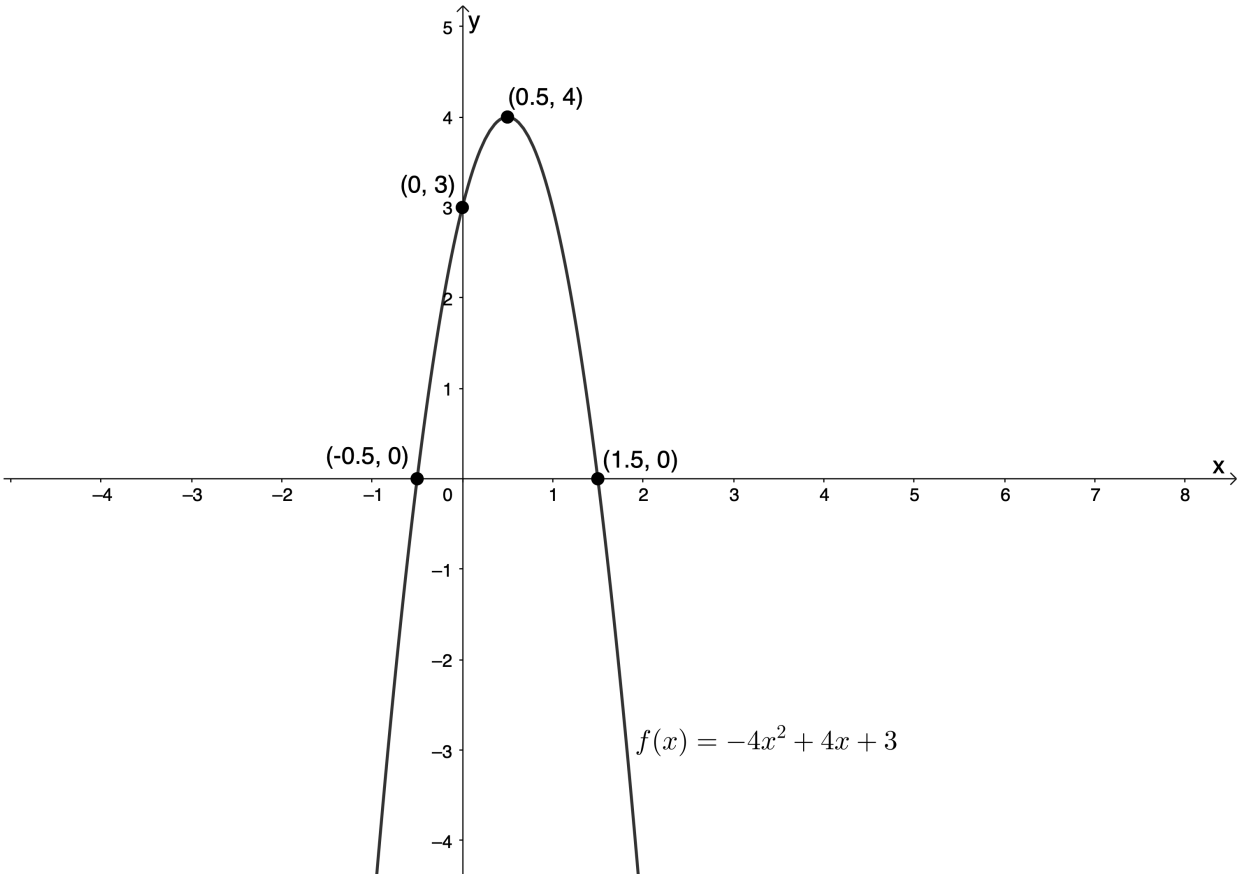
Example 5.3

Determine graphically for which values of x $f(x) \geq 0$ if $f(x) = -4x^2 + 4x + 3$.

Solution

The first thing to notice is that we need to solve for x graphically, which means we have to draw a sketch of the function and interpret this sketch to come to the solution.

Step 1: Sketch the function
If you need help sketching quadratic functions like this refer to level 3 subject outcome 2.1 unit 1.



Step 2: Read off the solution
We were asked to find where $f(x) \geq 0$, in other words, where the function f is above the x-axis. From the graph, we can see that $f(x) \geq 0$ between $-\frac{1}{2}$ and $\frac{3}{2}$. Therefore, $x \in [-\frac{1}{2}, \frac{3}{2}]$ or $\{x | x \in \mathbb{R}, -\frac{1}{2} \leq x \leq \frac{3}{2}\}$

There is something important we need to recognise about solving quadratic inequalities from Example 5.3. In this case, $f(x) = -4x^2 + 4x + 3$ and we were asked where $f(x) \geq 0$ or above the x-axis. We can see that the graph has a maximum turning point because $a < 0$ and so the solution is as shown in Figure 6.

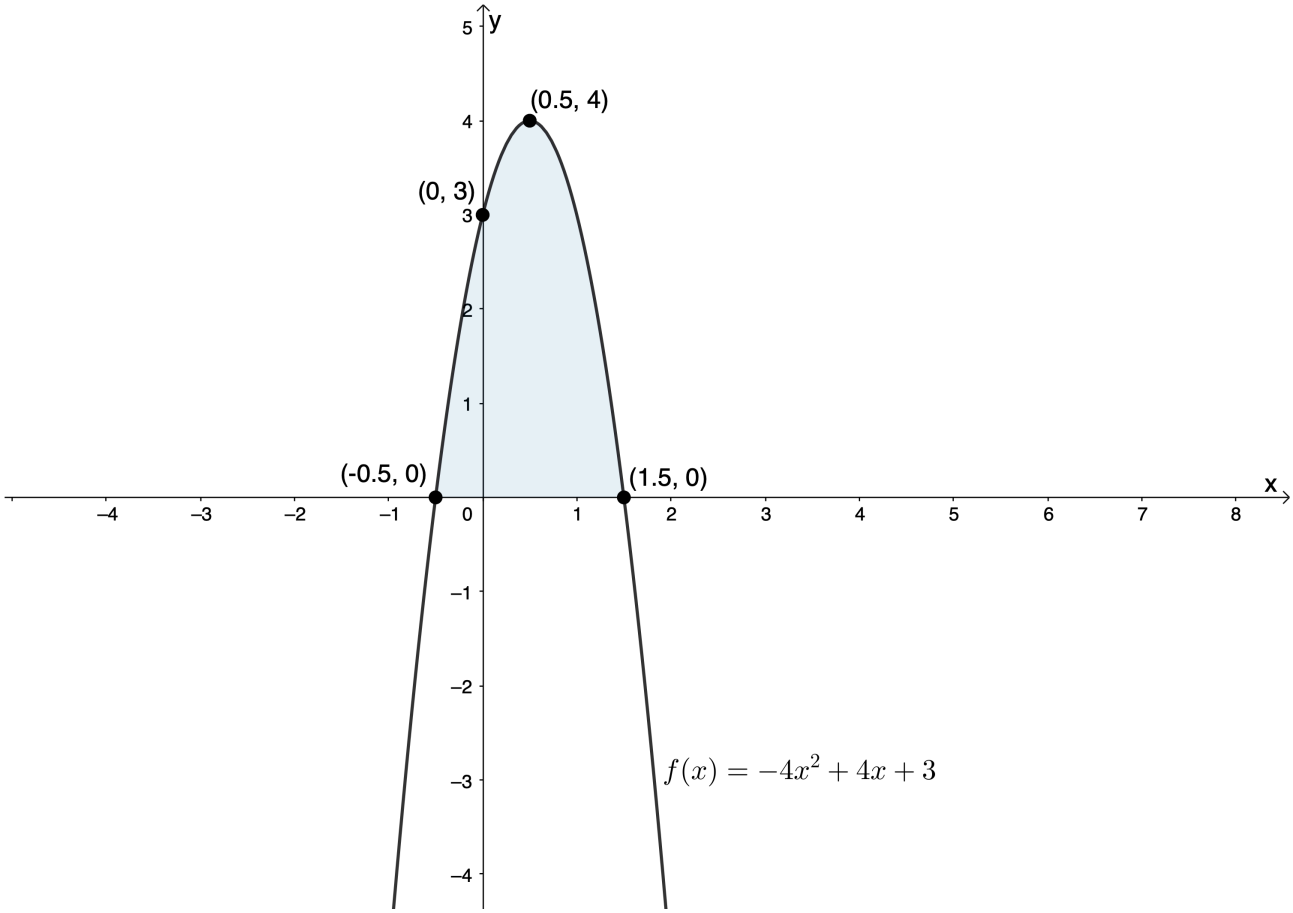


Figure 6: Solution to $f(x) \geq 0$ *if* $f(x) = -4x^2 + 4x + 3$

However, if we were solving this inequality algebraically, we would probably proceed as follows:

$-4x^2 + 4x + 3 \geq 0 \quad$ Multiply through by -1 and reverse the inequality

$\therefore 4x^2 - 4x + 3 \leq 0$

Now you might be wondering how changing the function from $f(x) = -4x^2 + 4x + 3$ to another function, for example $g(x) = 4x^2 - 4x - 3$, will not result in a completely different solution. The answer lies in the importance of reversing the inequality sign whenever we multiply both sides of an inequality by a negative number. If we look at a sketch of $g(x) = 4x^2 - 4x - 3$ we can see why (see Figure7).

Because we reversed the inequality sign, we are now interested in where $4x^2 - 4x - 3 \leq 0$, where $g(x) \leq 0$ or where $g(x)$ is below the x-axis.

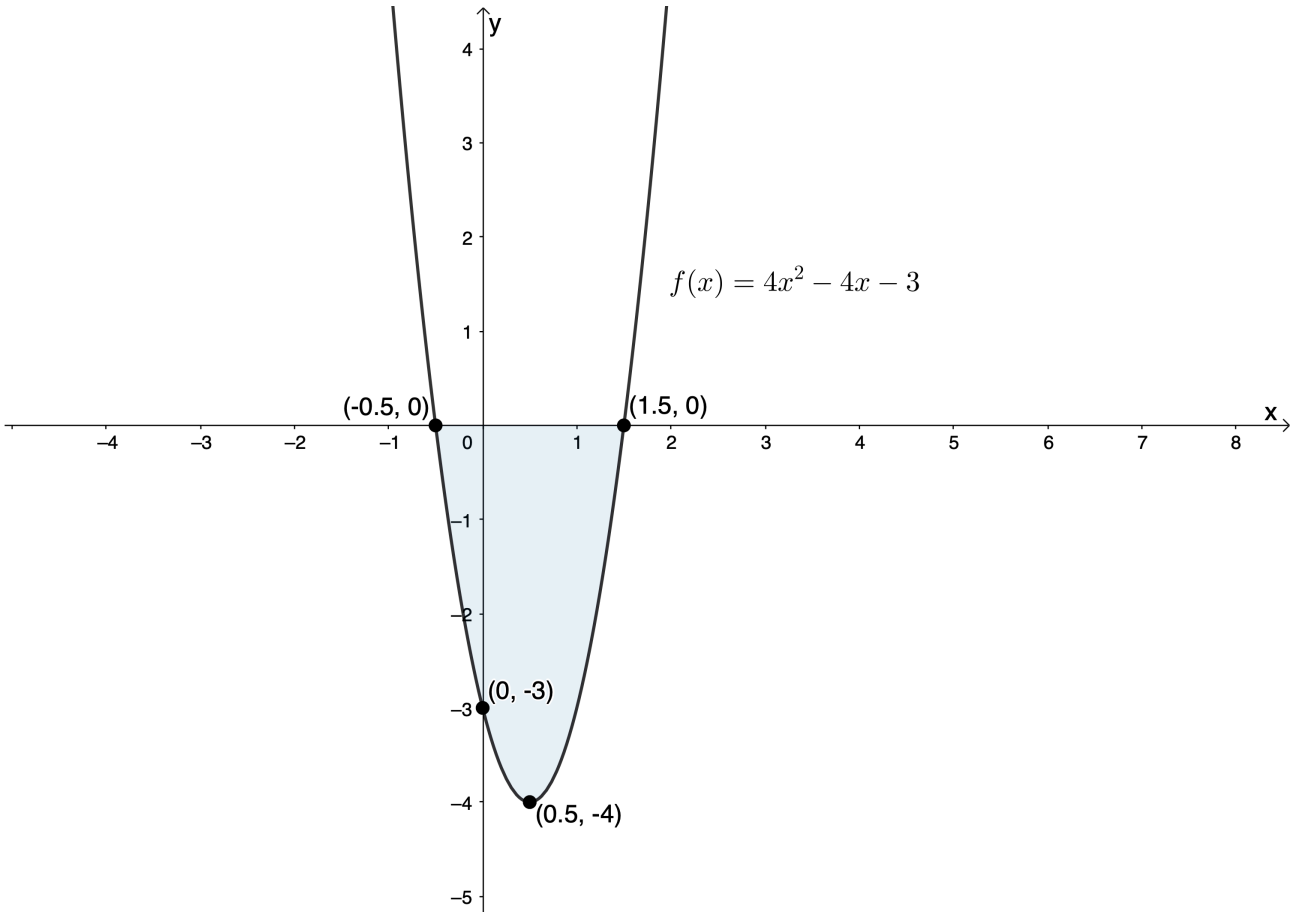


Figure 7: Solution to $g(x) \leq 0$ *if* $g(x) = 4x^2 - 4x - 3$

We can see that we get exactly the same solution: $x \in [-\frac{1}{2}, \frac{3}{2}]$ or $\{x | x \in \mathbb{R}, -\frac{1}{2} \leq x \leq \frac{3}{2}\}$.

Solve inequalities with unknowns in the denominator

The process of solving quadratic inequalities when there are fractions involved and where the denominators contain variables is slightly different.

If we were asked to solve for x in $\frac{4}{x-3} = \frac{5}{x+4}$, we would multiply both sides by the LCD of $(x-3)(x+4)$ to get:

$$\frac{4}{x-3} = \frac{5}{x+4}, x \neq 3, x \neq -4$$
$$\therefore 4(x+4) = 5(x-3)$$
$$\therefore 4x + 16 = 5x - 15$$
$$\therefore x = 31$$

We can do this because we don't need to worry about whether $(x-3) < 0$ or whether $(x+4) < 0$. But if we have an inequality, we know we have to flip the inequality around if we multiply through by a negative number. Because we don't know if $(x-3) < 0$ or if $(x+4) < 0$ we don't know whether to flip the inequality around or not. So, we need a different method.

Example 5.4

Solve for x in $\frac{4}{x-3} \leq \frac{5}{x+4}$.

Solution

Step 1: Get one side of the inequality equal to zero
To make sure that we don't have to multiply through by the LCD, we rearrange the inequality to get one side equal to zero.

$$\frac{4}{x-3} \leq \frac{5}{x+4} \quad \text{Subtract } \frac{5}{x+4} \text{ from both sides}$$

$$\therefore \frac{4}{x-3} - \frac{5}{x+4} \leq 0$$

Step 2: Add the fractions
Next, we need to subtract the fractions on the other side of the inequality.

$$\frac{4}{x-3} - \frac{5}{x+4} \leq 0$$

$$\therefore \frac{4(x+4) - 5(x-3)}{(x-3)(x+4)} \leq 0$$

$$\therefore \frac{4x + 16 - 5x + 15}{(x-3)(x+4)} \leq 0$$

$$\therefore \frac{-x + 31}{(x-3)(x+4)} \leq 0 \quad \text{We can leave the LHS as it is, or multiply through by } -1$$

$$\therefore \frac{-(x-31)}{(x-3)(x+4)} \leq 0$$

$$\therefore \frac{(x-31)}{(x-3)(x+4)} \geq 0 \quad \text{Remember to flip the inequality sign}$$

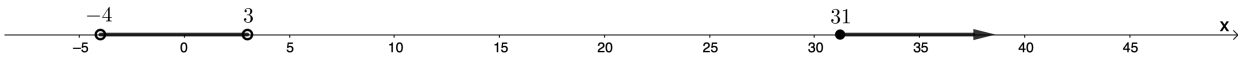
Step 3: Determine the critical values
The critical values now are not just the roots (those values that make the expression on the LHS equal to zero) but also the restrictions (those values that make the expression on the LHS undefined).

Our critical values are $x = 31$, $x = 3$ and $x = -4$.

Step 4: Complete a table of signs
Notice how we need to add additional columns to accommodate the extra critical value. As before, choose values for x to complete the table.

	$x < -4$	$x = -4$	$-4 < x < 3$	$x = 3$	$3 < x < 31$	$x = 31$	$x > 31$
$\frac{(x-31)}{(x-3)(x+4)}$	$-$	Undefined	$+$	Undefined	$-$	0	$+$

Step 5: Read off the solution
We are solving for $\frac{(x-31)}{(x-3)(x+4)} \geq 0$. Which ranges result in a positive expression? These are:



$x \in (-4, 3) \bigcup [31, \infty)$

$\{x | x \in \mathbb{R}, -4 < x < 3, x \geq 31\}$

Note: Even though we are solving for a greater than or equal to inequality, we cannot include -4 or 3 in our solutions because these values make the expression undefined.



Exercise 5.2

Solve for x in the following inequalities and represent your answers using interval and set builder notation:

1. $\dfrac{4}{x-3} \geq 1$
2. $\dfrac{15-x}{x} \geq 2x$
3. $\dfrac{-4}{(x-3)(x-1)} < 3$
4. $\dfrac{x^2+3}{3x-2} \leq 0$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the critical values.
- How to solve quadratic inequalities using the table method.
- How to solve quadratic inequalities using the graphical method.
- How to represent the solution to a quadratic inequality on a number line.
- How to solve inequalities with fractions where there are unknowns in the denominator.

Unit 5: Assessment

Suggested time to complete: 60 minutes

1. Solve for x in the following inequalities and represent your answers as indicated:

a. $2 + x - x^2 \leq 0$ (represent your answer on a number line)

b. $(x - 1)(x - 2) < 6$ (represent your answer using set builder notation)

c. $-10x < -x^2$ (represent your answer using set builder notation)

d. $\frac{x^2 + 4}{x - 7} \leq 0, x \neq 7$ (represent your answer using interval notation)

e. $x^2 + 4x < -4$ (represent your answer using set builder notation)

f. $\frac{x + 2}{x} > 1, x \neq 0$ (represent your answer using interval notation)

2. Solve for x in $-6 \leq x^2 - 5x \leq 6$ and represent your answer on a number line.
Hint: Split the expression into two separate inequalities first.

The full solutions are at the end of the unit.

Unit 5: Solutions

Exercise 5.1

1. $x^2 + 2x - 15 < 0$
$\therefore (x - 3)(x + 5) < 0$
Critical values: $x = -5$ and $x = 3$

	$x < -5$	$x = -5$	$-5 < x < 3$	$x = 3$	$x > 3$
$(x - 3)(x + 5)$	$+$	0	$-$	0	$+$

$\{x | x \in \mathbb{R}, -5 < x < 3\}$

2. $10x^2 \leq 31x - 15$
$\therefore 10x^2 - 31x + 15 \leq 0$
$\therefore (2x - 5)(5x - 3) \leq 0$
Critical values: $x = \frac{5}{2}$ and $x = \frac{3}{5}$

	$x < \frac{3}{5}$	$x = \frac{3}{5}$	$\frac{3}{5} < x < \frac{5}{2}$	$x = \frac{5}{2}$	$x > \frac{5}{2}$
$(2x - 5)(5x - 3)$	$+$	0	$-$	0	$+$

$x \in [\frac{3}{5}, \frac{5}{2}]$

3. $9x(x + 1) > 10$
$\therefore 9x^2 + 9x - 10 > 0$
$\therefore (3x - 2)(3x + 5) > 0$
Critical values: $x = \frac{2}{3}$ and $x = -\frac{5}{3}$

	$x < -\frac{5}{3}$	$x = -\frac{5}{3}$	$-\frac{5}{3} < x < \frac{2}{3}$	$x = \frac{2}{3}$	$x > \frac{2}{3}$
$(3x-2)(3x+5)$	$+$	0	$-$	0	$+$

$x \in (-\infty, -\frac{5}{3}) \bigcup (\frac{2}{3}, \infty)$

4.

$$5(2x^2 - 7) < 11x$$
$$\therefore 10x^2 - 11x - 35 < 0$$
$$\therefore (5x+7)(2x-5) < 0$$

Critical values: $x = -\frac{7}{5}$ and $x = \frac{5}{2}$

	$x < -\frac{7}{5}$	$x = -\frac{7}{5}$	$-\frac{7}{5} < x < \frac{5}{2}$	$x = \frac{5}{2}$	$x > \frac{5}{2}$
$(5x+7)(2x-5)$	$+$	0	$-$	0	$+$

$\{x | x \in \mathbb{R}, -\frac{7}{5} < x < \frac{5}{2}\}$

5.

$$\frac{40x - 21}{x^2} \geq 16, x^2 \neq 0 \therefore x \neq 0$$

You can multiply both sides by x^2 because we know that $x^2 > 0$

$$\therefore 40x - 21 \geq 16x^2$$
$$\therefore 16x^2 - 40x + 21 \leq 0 \quad \text{Be careful of the direction of the inequality}$$
$$\therefore (4x-3)(4x-7) \leq 0$$

Critical values: $x = \frac{3}{4}$ and $x = \frac{7}{4}$

	$x < \frac{3}{4}$	$x = \frac{3}{4}$	$\frac{3}{4} < x < \frac{7}{4}$	$x = \frac{7}{4}$	$x > \frac{7}{4}$
$(4x-3)(4x-7)$	$+$	0	$-$	0	$+$



Back to Exercise 5.1

Exercise 5.2

1.

$$\frac{4}{x-3} \geq 1, x \neq 3$$

$$\therefore \frac{4}{x-3} - 1 \geq 0$$

$$\therefore \frac{4-(x-3)}{(x-3)} \geq 0$$

$$\therefore \frac{-x+7}{(x-3)} \geq 0 \quad \text{Multiply through by } -1 \text{ and reverse the inequality sign}$$

$$\therefore \frac{x-7}{(x-3)} \leq 0$$

Critical values: $x = 7$ and $x = 3$

	$x < 3$	$x = 3$	$3 < x < 7$	$x = 7$	$x > 7$
$\frac{(x-7)}{(x-3)}$	$+$	Undefined	$-$	0	$+$

$x \in (3, 7]$

$\{x | x \in \mathbb{R}, 3 < x \leq 7\}$ Remember that we cannot include 3 in our solution.

2.

$$\frac{15-x}{x} \geq 2x, x \neq 0$$

$$\therefore \frac{15-x}{x} - 2x \geq 0$$

$$\therefore \frac{15-x-2x^2}{x} \geq 0$$

$$\therefore \frac{(3-x)(5+2x)}{x} \geq 0$$

Critical values: $x = 3$, $x = -\frac{5}{2}$ and $x = 0$

	$x < -\frac{5}{2}$	$x = -\frac{5}{2}$	$-\frac{5}{2} < x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$\frac{(3-x)(5+2x)}{x}$	$+$	0	$-$	Undef	$+$	0	$-$

$x \in (-\infty, -\frac{5}{2}] \bigcup (0, 3]$

$\{x | x \in \mathbb{R}, x \leq -\frac{5}{2}, 0 < x \leq 3\}$

3.

$$\frac{-3}{(x-3)(x-1)} < 3, x \neq 3, x \neq 1$$

$$\therefore \frac{-3}{(x-3)(x-1)} - 3 < 0$$

$$\therefore \frac{-3-3(x-3)(x-1)}{(x-3)(x-1)} < 0$$

$$\therefore \frac{-3-3(x^2-4x+3)}{(x-3)(x-1)} < 0$$

$$\therefore \frac{-3-3x^2+12x-9}{(x-3)(x-1)} < 0$$

$$\therefore \frac{-3x^2+12x-12}{(x-3)(x-1)} < 0 \quad \text{Multiply through by } -1 \text{ and reverse the inequality}$$

$$\therefore \frac{3x^2-12x+12}{(x-3)(x-1)} > 0 \quad \text{You can take out a common factor of 3 or factorise as is}$$

$$\therefore \frac{(3x-6)(x-2)}{(x-3)(x-1)} > 0$$

$$\therefore \frac{3(x-2)(x-2)}{(x-3)(x-1)} > 0$$

Critical values: $x = 1$, $x = 2$ and $x = 3$

	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$x > 3$
$\frac{3(x-2)(x-2)}{(x-3)(x-1)}$	$+$	Undef	$-$	0	$-$	Undef	$+$

$x \in (-\infty, 1) \bigcup (3, \infty)$

$\{x | x \in \mathbb{R}, x < 1, x > 3\}$

You could also recognise that the numerator $3(x-2)^2 \geq 0$. Therefore for the fraction to be positive, the denominator must be positive. Hence, you could consider a condensed table of critical values.

	$x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$x > 3$
$\frac{3(x-2)(x-2)}{(x-3)(x-1)}$	$+$	Undef	$-$	Undef	$+$

4.

$\frac{x^2+3}{3x-2} \leq 0, x \neq \frac{2}{3}$

$x^2 \geq 0 \therefore x^2 + 3 \geq 3$. Therefore, the numerator is always positive.

Critical values: $x = \frac{2}{3}$

	$x < \frac{2}{3}$	$x = \frac{2}{3}$	$x > \frac{2}{3}$
$\frac{x^2+3}{3x-2}$	$-$	Undef	$+$

$x \in (-\infty, \frac{2}{3})$

$\{x | x \in \mathbb{R}, x < \frac{2}{3}\}$

Back to Exercise 5.2

Unit 5: Assessment

1.

a.

$$2 + x - x^2 \leq 0$$
$$\therefore x^2 - x - 2 \geq 0$$
$$\therefore (x+1)(x-2) \geq 0$$

Critical values: $x = -1$ and $x = 2$

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$(x+1)(x-2)$	$+$	0	$-$	0	$+$



b.

$$(x-1)(x-2) < 6$$
$$\therefore x^2 - 3x + 2 - 6 < 0$$
$$\therefore x^2 - 3x - 4 < 0$$
$$\therefore (x+1)(x-4) < 0$$

Critical values: $x = -1$ and $x = 4$

	$x < -1$	$x = -1$	$-1 < x < 4$	$x = 4$	$x > 4$
$(x+1)(x-4)$	$+$	0	$-$	0	$+$

$\{x | x \in \mathbb{R}, -1 < x < 4\}$

c.

$$-10x < -x^2$$
$$\therefore x^2 - 10x < 0$$
$$\therefore x(x-10) < 0$$

Critical values: $x = 0$ and $x = 10$

	$x < 0$	$x = 0$	$0 < x < 10$	$x = 10$	$x > 10$
$x(x-10)$	$+$	0	$-$	0	$+$

$\{x | x \in \mathbb{R}, 0 < x < 10\}$

d.

$\frac{x^2+4}{x-7} \leq 0, x \neq 7$

$x^2 \geq 0 \therefore x^2 + 4 \geq 4$. The numerator is always positive.

Critical value: $x = 7$

	$x < 7$	$x = 7$	$x > 7$
$\frac{x^2+4}{x-7}$	$-$	Undefined	$+$

$x \in (-\infty, 7)$

e.

$$x^2 + 4x < -4$$
$$\therefore x^2 + 4x + 4 < 0$$
$$\therefore (x+2)(x+2) < 0$$

Critical value: $x = -2$

	$x < -2$	$x = -2$	$x > -2$
$(x+2)(x+2)$	$+$	0	$+$

No solution

Alternative approach:

$$x^2 + 4x < -4$$
$$\therefore x^2 + 4x + 4 < 0$$
$$\therefore (x+2)(x+2) < 0$$
$$\therefore (x+2)^2 < 0$$

But $(x+2)^2$ is always positive. Therefore, there is no solution.

f.

$$\frac{x+2}{x} > 1, x \neq 0$$
$$\therefore \frac{x+2}{x} - 1 > 0$$
$$\therefore \frac{x+2-x}{x} > 0$$
$$\therefore \frac{2}{x} > 0$$

Critical value: $x = 0$

	$x < 0$	$x = 0$	$x > 0$
$\frac{2}{x}$	$-$	Undefined	$+$

$x \in (0, \infty)$

2.

$-6 \leq x^2 - 5x \leq 6$
$-6 \leq x^2 - 5x$ (1)
$x^2 - 5x \leq 6$ (2)

From (1):

$$-6 \leq x^2 - 5x$$
$$\therefore x^2 - 5x + 6 \geq 0$$
$$\therefore (x-2)(x-3) \geq 0$$

Critical values: $x = 2$ and $x = 3$

	$x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$x > 3$
$(x-2)(x-3)$	$+$	0	$-$	0	$+$

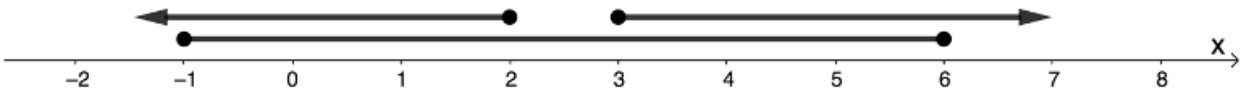
From (2):

$$x^2 - 5x \leq 6$$
$$\therefore x^2 - 5x - 6 \leq 0$$
$$\therefore (x+1)(x-6) \leq 0$$

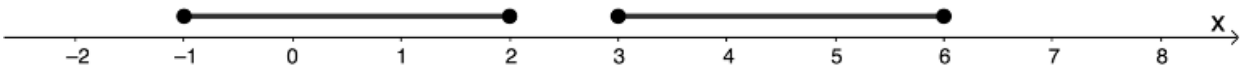
Critical values: $x = -1$ and $x = 6$

	$x < -1$	$x = -1$	$-1 < x < 6$	$x = 6$	$x > 6$
$(x-1)(x-6)$	$+$	0	$-$	0	$+$

If we plot the results of the two sign tables above on the same number line, we get the following. We can see that the regions overlap (make $-6 \leq x^2 - 5x$ AND $x^2 - 5x \leq 6$ at the same time) when $-1 \leq x \leq 2$ or when $3 \leq x \leq 6$.



Therefore, the final solution is $x \in [-1, 2] \bigcup [3, 6]$.



Back to Unit 5: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure6 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure7 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise5.1A4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2part1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2part2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

SUBJECT OUTCOME V

FUNCTIONS AND ALGEBRA: USE A VARIETY OF TECHNIQUES TO SKETCH AND INTERPRET INFORMATION FROM GRAPHS OF FUNCTIONS

Subject outcome

Subject outcome 2.1: Use a variety of techniques to sketch and interpret information from graphs of functions

Learning outcomes

- Use a variety of techniques to sketch and interpret information from graphs of functions. (Sketching of graphs using point by point plotting is an option.)
  - $y = a(x+p)^2 + q$
  - $y = ax^2 + bx + c$
  - $y = \frac{a}{x+p} + q$
  - $y = a.b^{x+p} + q, b > 0$
  - $y = a\sin kx$
  - $y = a\cos kx$
  - $y = a\tan kx$
  - $y = a\sin(x+p)$
  - $y = a\cos(x+p)$
  - $y = a\tan(x+p)$

    Note: Cubic functions will only be done in differential Calculus in level 4.
- Investigate and generalize the impact of k, p, a, b, c and q on the functions listed above.
- Identify the following characteristics of functions:
  - Domain and range
  - Intercepts with axes
  - Turning points, minima and maxima
  - Asymptotes
  - Shape and symmetry
  - Periodicity and amplitude
  - Functions or non-functions
  - Continuous or discontinuous
  - Intervals in which a function increases/decreases.
- Find the equation of the graphs by calculations or using the method of inspection (investigating the transformation of the graph).

Unit 1 outcomes

By the end of this unit you will be able to:

- Sketch quadratic functions of the form $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$.
- Find the turning point of $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$.

Unit 2 outcomes

By the end of this unit you will be able to:

- Find the equation of a quadratic function when given the x-intercepts and another point on the graph.
- Find the equation of a quadratic function when given the turning point and another point on the graph.

Unit 3 outcomes

By the end of this unit you will be able to:

- Determine the domain and range of hyperbolic functions of the form $y = \frac{a}{(x + p)} + q$.
- Find the asymptotes of hyperbolic functions of the form $y = \frac{a}{(x + p)} + q$.
- Find the axes of symmetry of hyperbolic functions of the form $y = \frac{a}{(x + p)} + q$.
- Find the intercepts with the axes of hyperbolic functions of the form $y = \frac{a}{(x + p)} + q$.
- Sketch hyperbolic functions of the form $y = \frac{a}{(x + p)} + q$.

Unit 4 outcomes

By the end of this unit you will be able to:

- Find the equation of a hyperbola in the form $y = \frac{a}{x + p} + q$.

Unit 5 outcomes

By the end of this unit you will be able to:

- Determine the domain and range of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- Find the asymptote of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- Find the intercepts with the axes of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- Sketch exponential functions of the form $y = a.b^{x+p} + q, b > 0$.

Unit 6 outcomes

By the end of this unit you will be able to:

- Find the equation of exponential graphs in the form $y = a.b^{x+p} + q,\ \ b > 0$.
- Analyse exponential functions.

Unit 7 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = a \sin k\theta$.
- Determine the effects of a and k on the sine graph of the form $y = a \sin k\theta$.
- Find the values of a and k from a given sine graph of the form $y = a \sin k\theta$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \sin x$ and $y = \sin \theta$ are the same function.

Unit 8 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = \sin(\theta + p)$.
- Determine the effects of positive and negative values of p on the sine graph $y = \sin(\theta + p)$.
- Find the value of p from a given sine graph of the form $y = \sin(\theta + p)$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \sin x$ and $y = \sin \theta$ are the same function.

Unit 9 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = a \cos k\theta$.
- Determine the effects of a and k on the cosine graph of the form $y = a \cos k\theta$.
- Find the values of a and k from a given cosine graph of the form $y = a \cos k\theta$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \cos x$ and $y = \cos \theta$ are the same function.

Unit 10 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = \cos(\theta + p)$.
- Determine the effects of positive and negative values of p on the cosine graph $y = \cos(\theta + p)$.
- Find the value of p from a given cosine graph of the form $y = \cos(\theta + p)$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \cos x$ and $y = \cos \theta$ are the same function.

Unit 11 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = a \tan k\theta$.
- Determine the effects of a and k on the tangent graph of the form $y = a \tan k\theta$.
- Find the values of a and k from a given tangent graph of the form $y = a \tan k\theta$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \tan x$ and $y = \tan \theta$ are the same function.

Unit 12 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = \tan(\theta + p)$.
- Determine the effects of positive and negative values of p on the tangent graph $y = \tan(\theta + p)$.
- Find the value of p from a given tangent graph of the form $y = \tan(\theta + p)$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \tan x$ and $y = \tan \theta$ are the same function.

Unit 1: Sketch quadratic functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Sketch quadratic functions of the form $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$.
- Find the turning point of $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$.

What you should know

Before you start this unit, make sure you can:

- Solve quadratic equations by:
  - Factorisation
  - Completing the square
  - Using the quadratic formula.
- Plot quadratic functions of the form $y = ax^2 + q$.
- Determine the x- and y-intercepts of a quadratic function.
- Determine the domain and range of a quadratic function.
- Determine the turning point (TP) of a quadratic function of the form $y = ax^2 + q$.

Review unit 1 of subject outcome 2.3 for level 2 and unit 1, unit 2 and unit 3 of subject outcome 2.3 for level 3 for help with solving quadratic functions. Review unit 2 of subject outcome 2.1 for level 2 for help with plotting quadratic functions of the form $y = ax^2 + q$.

Introduction

In level 2 we were introduced to the quadratic function of the form $y = ax^2 + q$. We learnt that quadratic functions were useful in real life for modelling the flights of objects through the air, designing bridges and describing the shapes of objects such as satellite dishes.

Before we go on with this unit, let's briefly revise what we know about quadratic functions of the form $y = ax^2 + q$.

Take note!

- The function of the form $y = ax^2 + q$ is a parabola with an axis of symmetry on the y-axis (or the line $x = 0$) and a y-intercept of $(0, q)$.

- Because the axis of symmetry is the line $x = 0$, the turning point (TP) and the y-intercept are the same point, therefore, the TP is the point $(0, q)$.
- If $a > 0$ the graph curves upwards and has a minimum TP.
- If $a < 0$ the graph curves downwards and has a maximum TP.
- If $a > 0$ and $q > 0$ or if $a < 0$ and $q < 0$ the graph has no x-intercepts and the equation has no real roots.
- If $q = 0$ the graph touches the origin – the point where the x-axis and y-axis intersect, and the equation has one real root.

This is all illustrated in Figure 1.

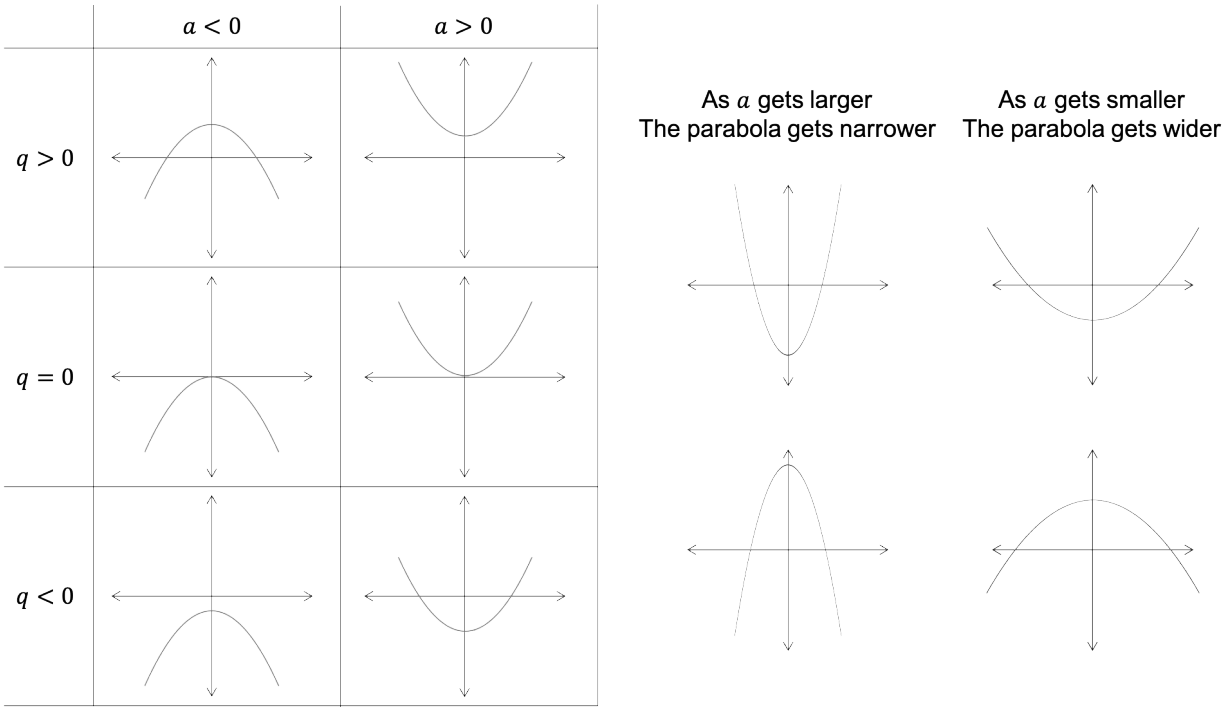


Figure 1: Summary of the effects of a and q on the function $y = ax^2 + q$

The turning point form

We know that a affects the shape of the parabola and whether it is **concave up** (a 'smile') or **concave down** (a 'frown') and that q shifts the parabola **vertically up or down**. But in all cases, the graph is symmetrical about the y-axis (or the line $x = 0$).
Now let's look at how to shift the graph horizontally left or right.

Activity 1.1: Find the turning points of the function $y = a(x + p)^2 + q$

Time required: 15 minutes

What you need:

- paper
- pen or pencil

What to do:

1. Make a copy of the following table on a blank piece of paper and then complete it by finding the y-values for each of the x-values for each function. Each function is of the form $y = a(x + p)^2 + q$.

x-values	-3	-2	-1	0	1	2	3
$f(x) = (x - 2)^2$							
$g(x) = \frac{1}{2}(x + 1)^2$							
$h(x) = -(x + 2)^2 + 9$							

2. Plot each of these functions on the same set of axes.
3. What is the axis of symmetry of each function and how does this compare to the value of p in each case?
4. What is the turning point of each function and how does this compare to the values of p and q in each function?
5. Does the value of a affect the shape of the function of the form $y = a(x + p)^2 + q$ in the same way as the function of the form $y = ax^2 + q$?
6. How does the value of a affect whether the function has a minimum or maximum turning point?
7. Based on what you have discovered, what will the turning point of the following functions be, and will it be a minimum or maximum turning point?
   a. $b(x) = -(x + 3)^2 + 1$
   b. $c(x) = 2(x - 1.5)^2 - \frac{17}{4}$
   c. $d(x) = -\frac{1}{3}(x + r)^2 + s$

What did you find?

1. Here is the completed table of values.

x-values	-3	-2	-1	0	1	2	3
$f(x) = (x - 2)^2$	25	16	9	4	1	0	1
$g(x) = \frac{1}{2}(x + 1)^2$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	4.5	8
$h(x) = -(x + 2)^2 + 9$	8	9	8	5	0	-7	-16

2. Figure 2 shows all three functions plotted on the same system of axes.

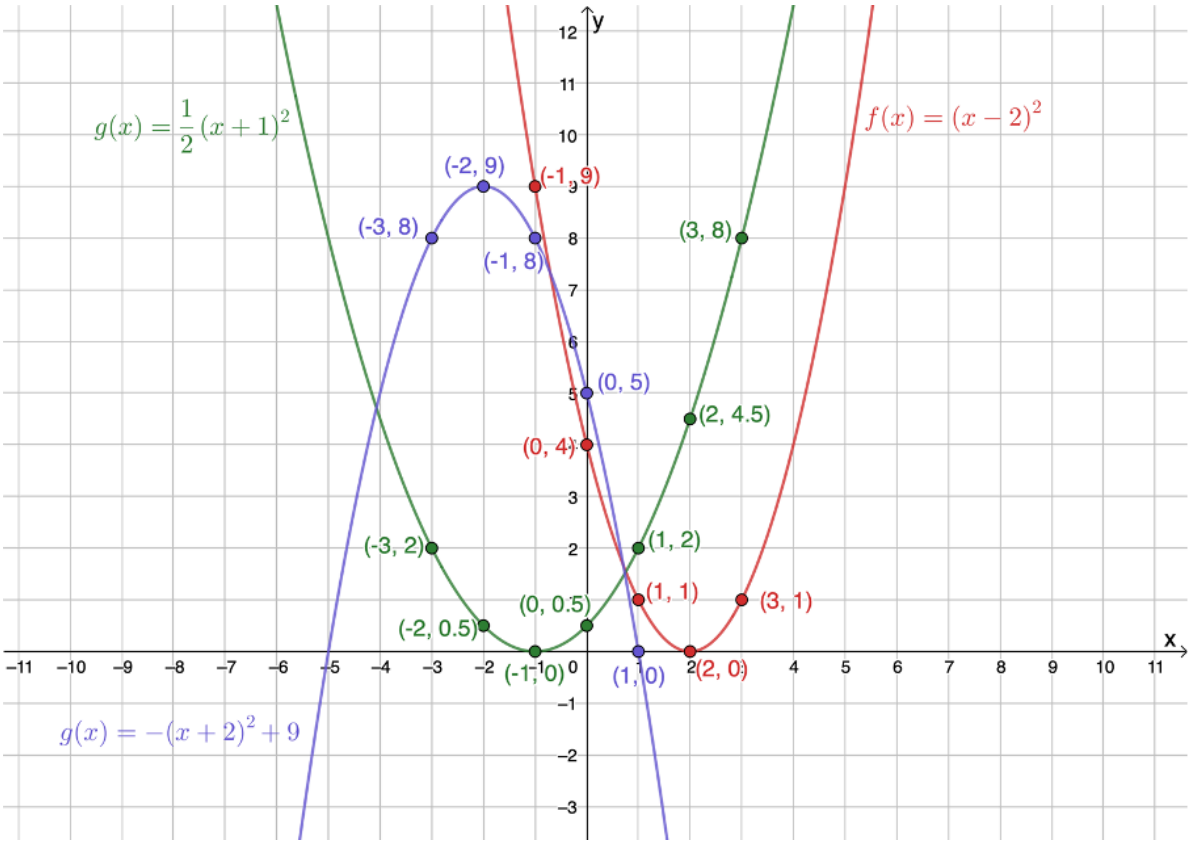


Figure 2: Plots of $f(x)$, $g(x)$ and $h(x)$

3. The axis of symmetry of $f(x)$, $g(x)$ and $h(x)$ are $x = 2$, $x = -1$ and $x = -2$ respectively. In each case, the axis of symmetry is the value of p multiplied by -1, in other words $-p$.
4. The turning points of $f(x)$, $g(x)$ and $h(x)$ are $(2, 0)$, $(-1, 0)$ and $(-2, 9)$ respectively. In each case, the coordinates of the turning point are given by $(-p, q)$.
5. The value of a has the same effect in $y = a(x+p)^2 + q$ as it does in $y = ax^2 + q$. $a > 0$ results in a concave up ('smiley') graph while $a < 0$ results in a concave down graph. The greater the value of a the narrower the graph. The smaller the value of a the wider the graph.
6. If $a > 0$, the graph has a minimum turning point. If $a < 0$, the graph has a maximum turning point.
7. We have discovered that a quadratic function of the form $y = (x+p)^2 + q$ has a turning point of $(-p, q)$. Therefore, the turning point of
   a. $b(x) = -(x+3)^2 + 1$ is $(-3, 1)$ and it will be a maximum.
   b. $c(x) = 3(x-1.5)^2 - \frac{17}{4}$ is $\left(1.5, -\frac{17}{4}\right)$ and it will be a minimum.
   c. $d(x) = -\frac{1}{3}(x+r)^2 + s$ is $(-r, s)$ and it will be a maximum.

In Activity 1.1 we discovered that the turning point (TP) of the quadratic function of the form $y = a(x+p)^2 + q$ is the point $(-p, q)$. We call this form of the quadratic function the **turning point form** because we can read the TP directly from the function. Because the x-coordinate of the TP is always the same as the axis of symmetry (A/S), we know that the A/S is the line $x = -p$.

$y = a(x+p)^2 + q$ is the turning point form of the quadratic function and has a TP of $(-p, q)$ and an axis of symmetry of $x = -p$.

If $a > 0$, the TP will be a minimum. If $a < 0$, the TP will be a maximum.



Example 1.1

$j(x) = (x-4)^2 - 1$.

1. Determine the:
   a. x-intercepts
   b. y-intercept
   c. TP
   d. A/S
   e. Domain
   f. Range
2. Make a neat sketch of $j(x)$.

Solutions

1.

a. x-intercepts: Let $j(x) = 0$

$$0 = (x-4)^2 - 1$$
$$\therefore x^2 - 8x + 16 - 1 = 0$$
$$\therefore x^2 - 8x + 15 = 0$$
$$\therefore (x-5)(x-3) = 0$$
$$\therefore x = 5 \text{ or } x = 3$$

b. y-intercept: Let $x = 0$

$$j(0) = (0-4)^2 - 1$$
$$\therefore j(0) = 15$$

c. The TP is the point $(-p, q)$. Therefore, the TP is $(-(-4), -1) = (4, -1)$.

d. The A/S is the line $x = -p$. Therefore, the A/S is $x = 4$.

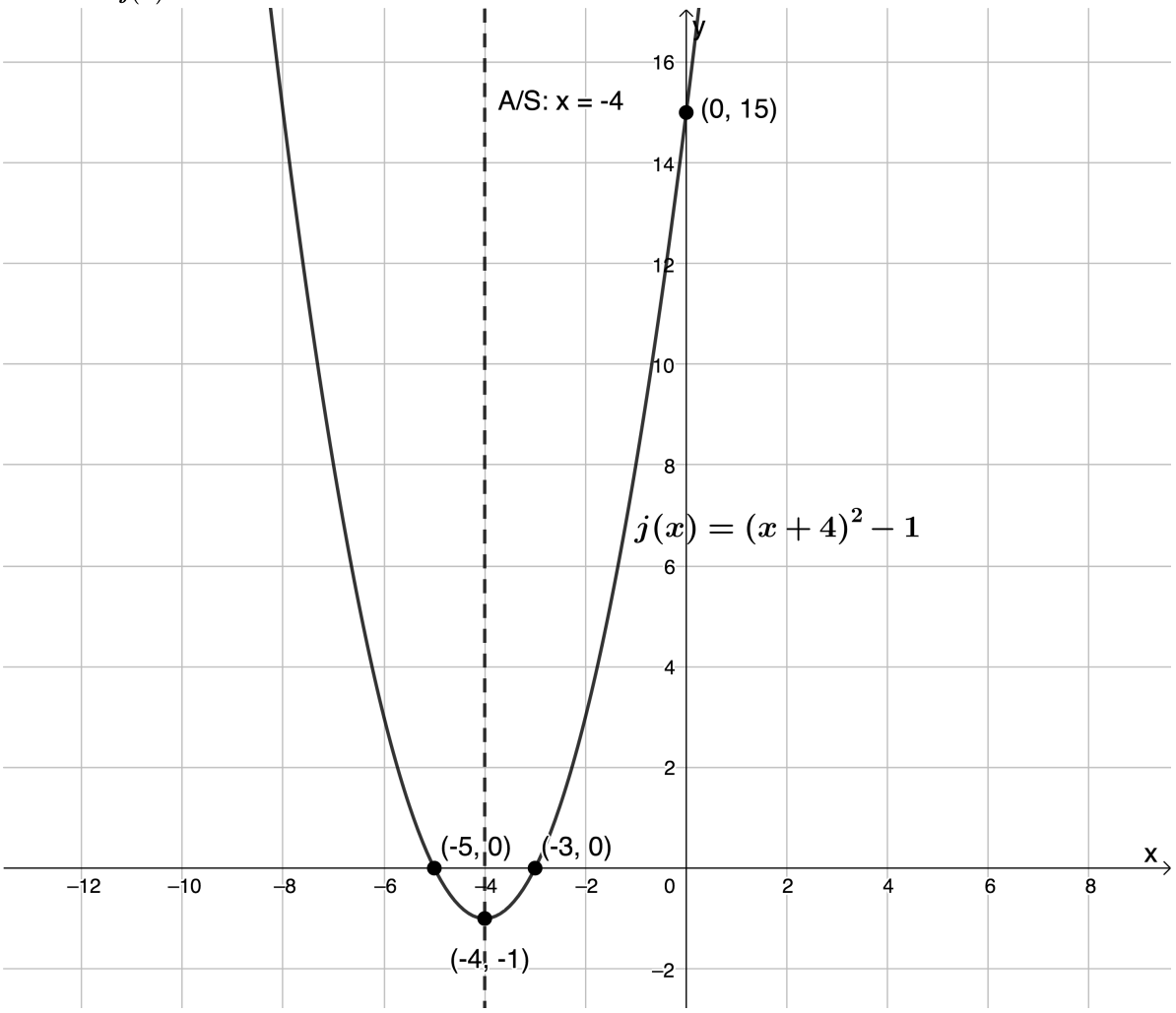
e. The domain is $\{x : x \in \mathbb{R}\}$ because there is no value of x for which $j(x)$ is undefined.

f. We know already that, because $a > 0$, the function has a minimum TP. We also know that the TP is the point $(4, -1)$. Therefore, we know that the range is $\{j(x) : j(x) \geq -1\}$ or $[-1, \infty)$ because the TP represents the minimum function value.

We can also determine the range by setting up and solving an inequality.

$(x-4)^2 \geq 0$	Any term squared must be greater than zero
$\therefore (x-4)^2 - 1 \geq -1$	We add -1 to both sides of the inequality to create the function $j(x)$ on one side
$\therefore j(x) \geq -1$	We can replace the LHS with $j(x)$ and read the range

2. Sketch of $j(x)$.



Take note!

In order to sketch functions of the form $y = a(x+p)^2 + q$, we need to determine five key characteristics of the graph, as follows:

1. the sign of a: this will help us determine if the TP is a maximum or a minimum
2. the turning point: remember, we can read the TP directly from this form of the function as $(-p, q)$
3. the axis of symmetry: remember this is the vertical line $x = -p$
4. the y-intercept
5. the x-intercepts (if they exist).

Exercise 1.1

1. $q(x) = 4(x-3)^2 - 1$
2. $r(x) = -(x-5)^2 + 4$.
   Sketch both functions on the same set of axes by first determining the following for each:
   a. x-intercepts
   b. y-intercept
   c. TP
   d. A/S
   e. domain
   f. range.
   Mark the intercepts, TP and A/S on your sketch.

The full solutions are at the end of the unit.

Completing the square

From our work on solving quadratic equations, we know that the standard form of the quadratic function is $y = ax^2 + bx + c$. In fact, in the section above on the turning point form, we expanded the turning point form to get the equation into the standard form in order to find the x-intercepts by factorising.

If we can convert the standard form into the turning point form, we would have a useful way of finding the TP and A/S of a quadratic function. We can do this by using a process called **completing the square**. You learnt how to complete the square in unit 2 of subject outcome 2.3 in level 3. If you need more help with completing the square, you should review that unit.

We can expand $y = 2(x+3)^2 - 1$ into standard form to get $y = 2x^2 + 12x + 17$. Check that you can do the expansion and get the same result.

Now we need to find a way to get back to the original equation.

Step 1: We know that we need the coefficient of x inside the square brackets to be 1. So, take out a common factor of 2 from each term.

$$y = 2\left(x^2 + 6x + \frac{17}{2}\right)$$

Step 2: We know that the $6x$ term was the result of $3x + 3x$. So, divide the coefficient of the x term by 2 and then square it to get 9. To keep the expression balanced, we need to add 9 to and subtract 9 from the right-hand side.

$$y = 2\left(x^2 + 6x + 9 - 9 + \frac{17}{2}\right)$$

Step 3: Now the $x^2 + 6x + 9$ part of the right-hand side can be factorised to $(x+3)^2$ which is what we want. But remember, the whole right-hand side is being multiplied by 2. When we take the -9 and the $\frac{17}{2}$ out of the bracket we have to multiply them by 2.

$$y = 2(x+3)^2 - 18 + 17$$

Step 4: Finally, we can simplify to get back to our original equation.

$y = 2(x+3)^2 - 1$

We can use this process of completing the square to convert any quadratic function in standard form into turning point form.



Example 1.2

Determine the TP and A/S of $t(x) = 2 - 10x + 5x^2$ by completing the square to get the function into the turning point form.

Solution

The first thing we need to do is to write $t(x)$ in standard form.
$t(x) = 5x^2 - 10x + 2$

Now we need to take out a common factor of 5 so that the cooefficient of $x^2$is 1.

$$t(x) = 5\left(x^2 - 2x + \frac{2}{5}\right)$$

Next, we have to halve the coefficient of x and then square it. Add and subtract the result from the right-hand side.

$$t(x) = 5\left(x^2 - 2x + 1 - 1 + \frac{2}{5}\right)$$

The $x^2 - 2x + 1$ can be factorised into a perfect square. Remember that the other terms must be multiplied by 5.

$t(x) = 5(x-1)^2 - 5 + 2$

Now simplify.

$t(x) = 5(x-1)^2 - 3$

It is always a good idea to check yourself by expanding to see if you get back to the original expression.

$$\begin{aligned} t(x) &= 5(x-1)^2 - 3 \\ &= 5(x^2 - 2x + 1) - 3 \\ &= 5x^2 - 10x + 5 - 3 \\ &= 5x^2 - 10x + 2 \end{aligned}$$

We get back to the original expression, so we know that we have completed the square correctly.

Now if $t(x) = 5(x-1)^2 - 3$, then the TP is $(1, -3)$ and the A/S is the line $x = 1$.



Exercise 1.2

Change $y = -2x^2 + 3x - 4$ into the turning point form by completing the square.

The full solutions are at the end of the unit.

Activity 1.2: Derive a general result for the TP from the standard form of a quadratic equation

Time required: 15 minutes

- What you need:
- a piece of paper
- a pen or pencil

What to do:

1. Get $y = ax^2 + bx + c$ into the turning point form by completing the square.
2. What is the x-coordinate of the TP?
3. What is the axis of symmetry?
4. What is the y-coordinate of the TP?

What did you find?

1. The first thing we have to do is take out the common factor a.$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

   Now we need to halve the x coefficient and square it and then add it and subtract it.

   $$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right)$$
   $$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

   Now $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ can be factorised as a perfect square.

   $$y = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2}{4a^2}\right) + a\left(\frac{c}{a}\right)$$

   Lastly, we need to simplify.

   $$y = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2}{4a^2}\right) + a\left(\frac{c}{a}\right)$$
   $$= a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + \frac{ac}{a}$$
   $$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$
   $$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 + 4ac}{4a}$$

2. From the TP form, which we found by completing the square, we can see that the x-coordinate of the TP is given by $-\frac{b}{2a}$.

3. This means that the A/S is the line $x = -\frac{b}{2a}$.
4. The y-coordinate of the TP is given by $-\frac{b^2 - 4ac}{4a}$.

The TP of $y = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$.

The A/S of $y = ax^2 + bx + c$ is the line $x = -\frac{b}{2a}$.



Example 1.3

Determine the TP of $f(x) = x^2 - 6x + 8$ and $g(x) = 18 + 6x - 3x^2$ without changing the functions into the turning point form.

Solution

For $f(x) = x^2 - 6x + 8$, $a = 1$, $b = -6$ and $c = 8$.

The x-coordinate of the TP is given by $-\frac{b}{2a} = -\frac{(-6)}{2(1)} = 3$.

The y-coordinate of the TP is given by $-\frac{b^2 - 4ac}{4a} = -\frac{(-6)^2 - 4(1)(8)}{4(1)} = -\frac{36 - 32}{4} = -\frac{4}{4} = -1$.

We can also find the y-coordinate of the TP by finding the function value of the x-coordinate.

$f(3) = 3^2 - 6(3) + 8 = 9 - 18 + 8 = -1$.

For $g(x) = 18 + 6x - 3x^2$, $a = -3$, $b = 6$ and $c = 18$.

The x-coordinate of the TP is given by $-\frac{b}{2a} = -\frac{6}{2(-3)} = 1$.

The y-coordinate of the TP is given by $-\frac{b^2 - 4ac}{4a} = -\frac{6^2 - 4(-3)(18)}{4(-3)} = -\frac{36 + 216}{-12} = -\frac{252}{-12} = 21$.

We can also find the y-coordinate of the TP by finding the function value of the x-coordinate.

$g(1) = 18 + 6(1) - 3(1)^2 = 18 + 6 - 3 = 21$.

Sketching quadratic functions in standard form

We can now sketch the graphs of quadratic functions of the form $y = ax^2 + bx + c$ using a number of different methods.

We find the x-intercepts (if they exist) by setting the function equal to zero. We find the y-intercept by setting x equal to zero. However, this means that we can read the y-intercept directly from the standard form. In $y = ax^2 + bx + c$, the y-intercept is c because if we substitute $x = 0$ both the ax^2 and bx terms will equal zero.

We can find the TP either by changing the function into the TP form by completing the square or by using the fact that the TP is the point $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$. Remember you can also find the y-coordinate of the TP by finding the function value of the x-coordinate of the TP, in other words, by finding $f\left(-\frac{b}{2a}\right)$.



Example 1.4

Sketch $f(x) = \frac{1}{2}x^2 - 4x + \frac{7}{2}$.

Solution

Remember that to sketch a quadratic function we need to determine the intercepts, the TP and the A/S. We also need to determine whether the TP is a maximum or minimum.

x-intercepts: Let $f(x) = 0$

$$\frac{1}{2}x^2 - 4x + \frac{7}{2} = 0$$
$$\therefore x^2 - 8x + 7 = 0$$
$$\therefore (x - 1)(x - 7) = 0$$
$$\therefore x = 1 \text{ or } x = 7$$

y-intercept: Let $x = 0$

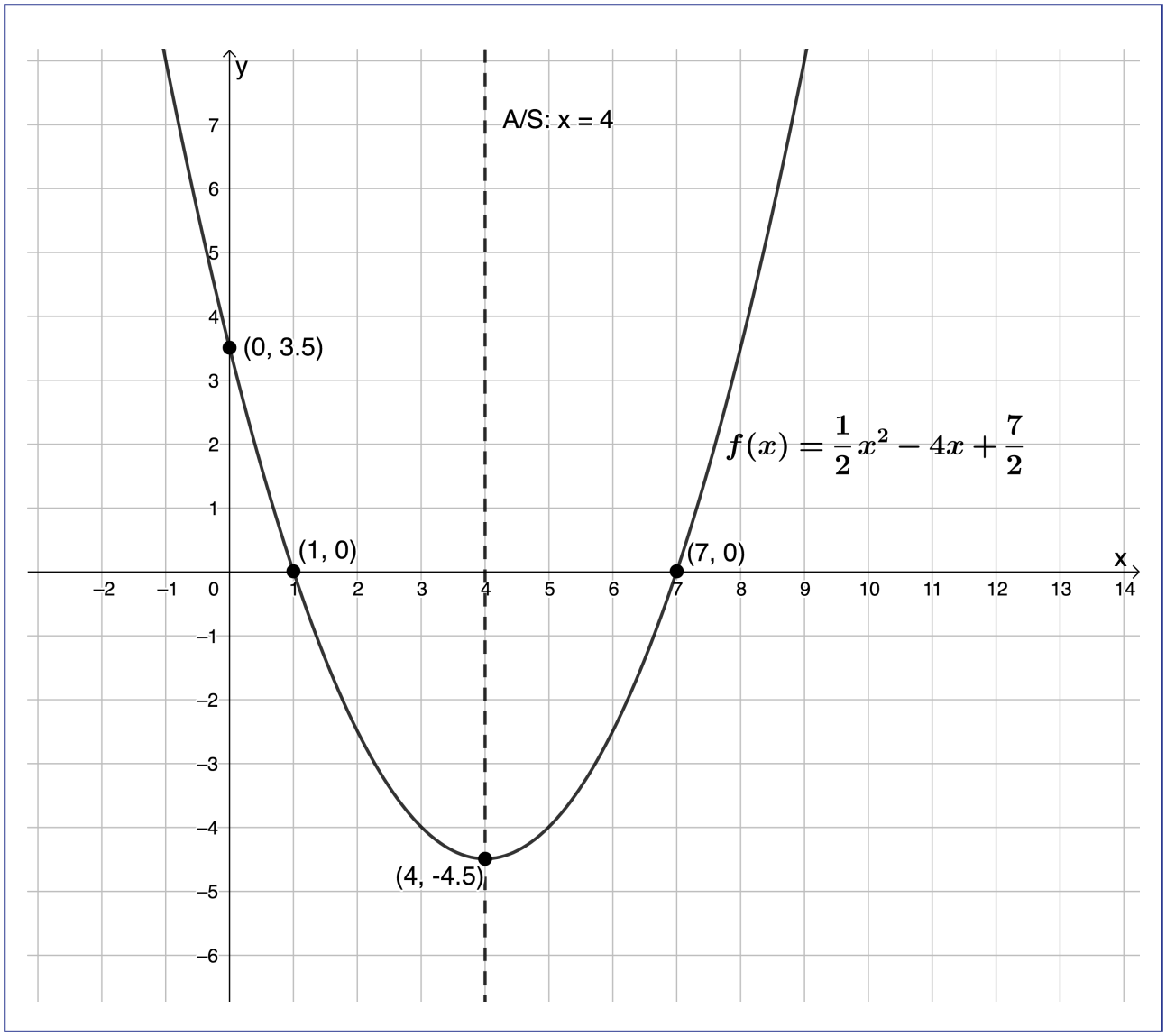
$$f(0) = \frac{1}{2}(0)^2 - 4(0) + \frac{7}{2}$$
$$= \frac{7}{2}$$

The x-coordinate of the TP is $-\frac{b}{2a} = -\frac{(-4)}{2\left(\frac{1}{2}\right)} = 4$. Therefore, the A/S is the line $x = 4$.

The y-coordinate of the TP can be found by finding $f(4)$

$$f(4) = \frac{1}{2}(4)^2 - 4(4) + \frac{7}{2}$$
$$= 8 - 16 + \frac{7}{2}$$
$$= -4.5$$

$a > 0$, so the TP is a minimum.



Exercise 1.3

Question adapted from NC(V) Mathematics First Paper November 2014

Given $y = -x^2 + 2x + 3$:

1. Make a neat sketch of the graph showing all intercepts with the axes, the TP and the A/S.
2. Write down the range of $y = -x^2 + 2x + 3$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- That the TP of a quadratic function in the turning point form $y = a(x + p)^2 + q$ is $(-p, q)$.

- How to sketch the graph of a quadratic function that is in the turning point form.
- How to turn a quadratic function from standard form into turning point form by completing the square.
- That the TP of a quadratic function in standard form $y = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$.
- That the y-intercept of a quadratic function in standard form $y = ax^2 + bx + c$ is $(0, c)$.
- How to sketch the graph of a quadratic function that is in standard form.

Unit 1: Assessment

Suggested time to complete: 30 minutes

1. Given $f(x) = -x^2 - 2x + 3$:
   a. Make a neat sketch of the graph showing all intercepts with the axes, the TP and the A/S.
   b. Write down the axis of symmetry of the graph.
   c. Write down the domain of the function.
   d. Write down the range of the function.
   (Question adapted from NC(V) Mathematics First Paper Supplementary 2014)
2. Make a neat sketch of $y = (x - 2)^2 - 7$ showing all intercepts with the axes, the TP and the A/S.
3. $y = 4\left(x + \frac{7}{2}\right)^2 - 25$. Make a neat sketch of the function showing all intercepts with the axes, the TP and the A/S.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $q(x) = 4(x - 3)^2 - 1$
   a. x-intercepts: Let $y = 0$
   $$\begin{aligned} 4(x - 3)^2 - 1 &= 0 \\ \therefore 4(x^2 - 6x + 9) - 1 &= 0 \\ \therefore 4x^2 - 24x + 36 - 1 &= 0 \\ \therefore 4x^2 - 24x + 35 &= 0 \\ \therefore (2x - 5)(2x - 7) &= 0 \\ \therefore x = \frac{5}{2} \text{ or } x &= \frac{7}{2} \end{aligned}$$
   b. y-intercept: Let $x = 0$
   $$\begin{aligned} y &= 4(0 - 3)^2 - 1 \\ \therefore y &= 4 \times 9 - 1 \\ \therefore y &= 35 \end{aligned}$$
   c. The function is in the turning point form $y = a(x + p)^2 + q$. Therefore, the TP is the point

$(-p, q) = (3, -1)$.

d. The A/S is the line $x = -p$. Therefore, the A/S is $x = 3$.

e. There are no values of x for which $q(x)$ is undefined. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

f. $a > 0$. Therefore, the TP is a minimum and the range is $[-1, \infty)$.

2. $r(x) = -(x-5)^2 + 4$.

a. x-intercepts: Let $y = 0$

$$-(x-5)^2 + 4 = 0$$
$$\therefore -(x^2 - 10x + 25) + 4 = 0$$
$$\therefore -x^2 + 10x - 25 + 4 = 0$$
$$\therefore -x^2 + 10x - 21 = 0$$
$$\therefore x^2 - 10x + 21 = 0$$
$$\therefore (x-3)(x-7) = 0$$
$$\therefore x = 3 \text{ or } x = 7$$

b. y-intercept: Let $x = 0$

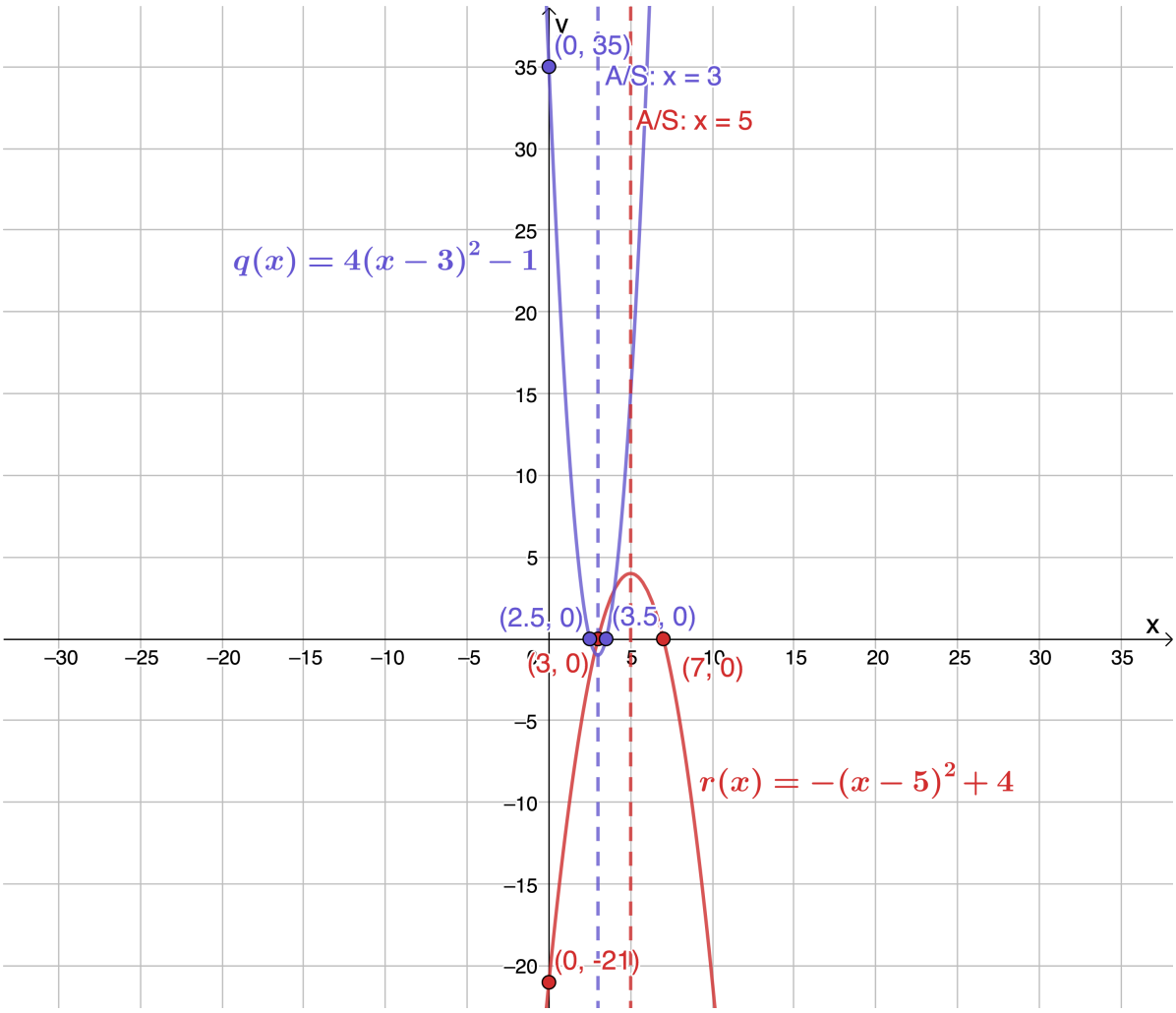
$$y = -(0-5)^2 + 4$$
$$\therefore y = -(25) + 4$$
$$\therefore y = -21$$

c. The function is in the turning point form $y = a(x+p)^2 + q$. Therefore, the TP is the point $(-p, q) = (5, 4)$.

d. The A/S is the line $x = -p$. Therefore, the A/S is $x = 5$.

e. There are no values of x for which $r(x)$ is undefined. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

f. $a < 0$. Therefore, the TP is a maximum and the range is $(\infty, 4]$.



Back to Exercise 1.1

Exercise 1.2

$$y = -2x^2 + 3x - 4$$

$$= -2\left(x^2 - \frac{3}{2}x + 2\right)$$ Take out the common factor so the leading coefficient is 1

$$= -2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} + 2\right)$$ Halve and then square the coefficient of x and add and subtract it.

$$= -2\left(x - \frac{3}{4}\right)^2 + \frac{9}{8} - 4$$ Factorise $x^2 - \frac{3}{2}x + \frac{9}{16}$ into $\left(x - \frac{3}{4}\right)^2$

$$= -2\left(x - \frac{3}{4}\right)^2 - \frac{23}{8}$$ Simplify

Back to Exercise 1.2

Exercise 1.3

1. $y = -x^2 + 2x + 3$
   x-intercepts: Let $y = 0$

$$-x^2 + 2x + 3 = 0$$
$$\therefore x^2 - 2x - 3 = 0$$
$$\therefore (x-3)(x+1) = 0$$
$$\therefore x = 3 \text{ or } x = -1$$

y-intercept: Let $x = 0$

$$y = -(0)^2 + 2(0) + 3$$
$$= 3$$

TP:

x-coordinate of the TP: $-\frac{b}{2a} = -\frac{2}{2(-1)} = 1$

y-coordinate of the TP:

$$y = -(1)^2 + 2(1) + 3$$
$$= -1 + 2 + 3$$
$$= 4$$

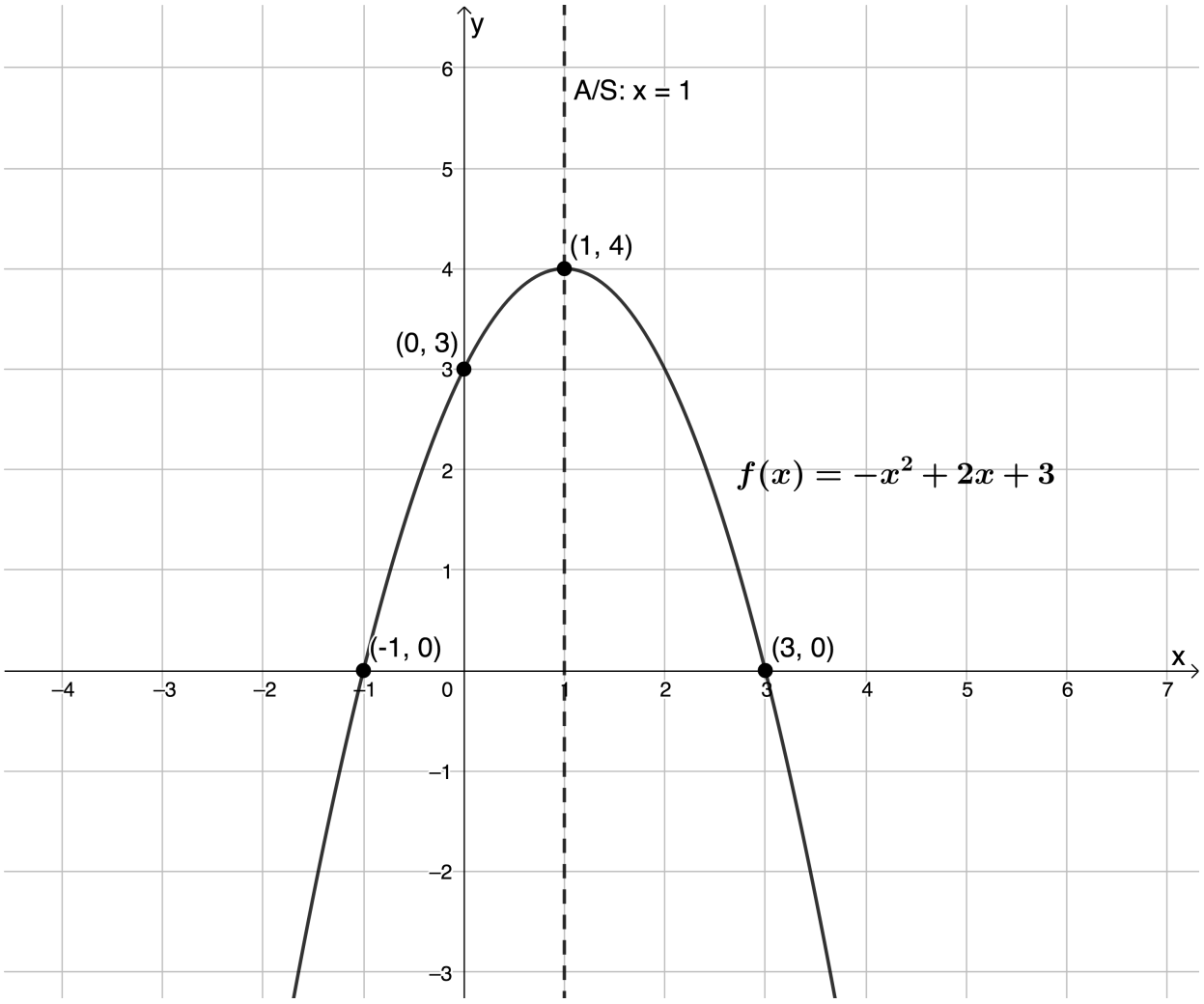
or

$$-\frac{b^2 - 4ac}{4a} = -\frac{2^2 - 4(-1)(3)}{4(-1)}$$
$$= -\frac{4 + 12}{(-4)} = 4$$

A/S:

$$x = -\frac{b}{2a} = 1$$

$a < 0$. Therefore, the TP is a maximum.



2. The range of $f(x)$ is $(-\infty, 4)$.

Back to Exercise 1.3

Unit 1: Assessment

1. $f(x) = -x^2 - 2x + 3$

   a. x-intercepts: Let $f(x) = 0$

   $$-x^2 - 2x + 3 = 0$$
   $$\therefore x^2 + 2x - 3 = 0$$
   $$\therefore (x + 3)(x - 1) = 0$$
   $$\therefore x = -3 \text{ or } x = 1$$

   y-intercept: $x = 0$

   $$f(0) = -(0)^2 - 2(0) + 3$$
   $$= 3$$

   TP:

   x-coordinate of the TP: $-\frac{b}{2a} = -\frac{(-2)}{2(-1)} = -1$

   y-coordinate of the TP:

   $$y = -(-1)^2 - 2(-1) + 3$$
   $$= -1 + 2 + 3$$
   $$= 4$$

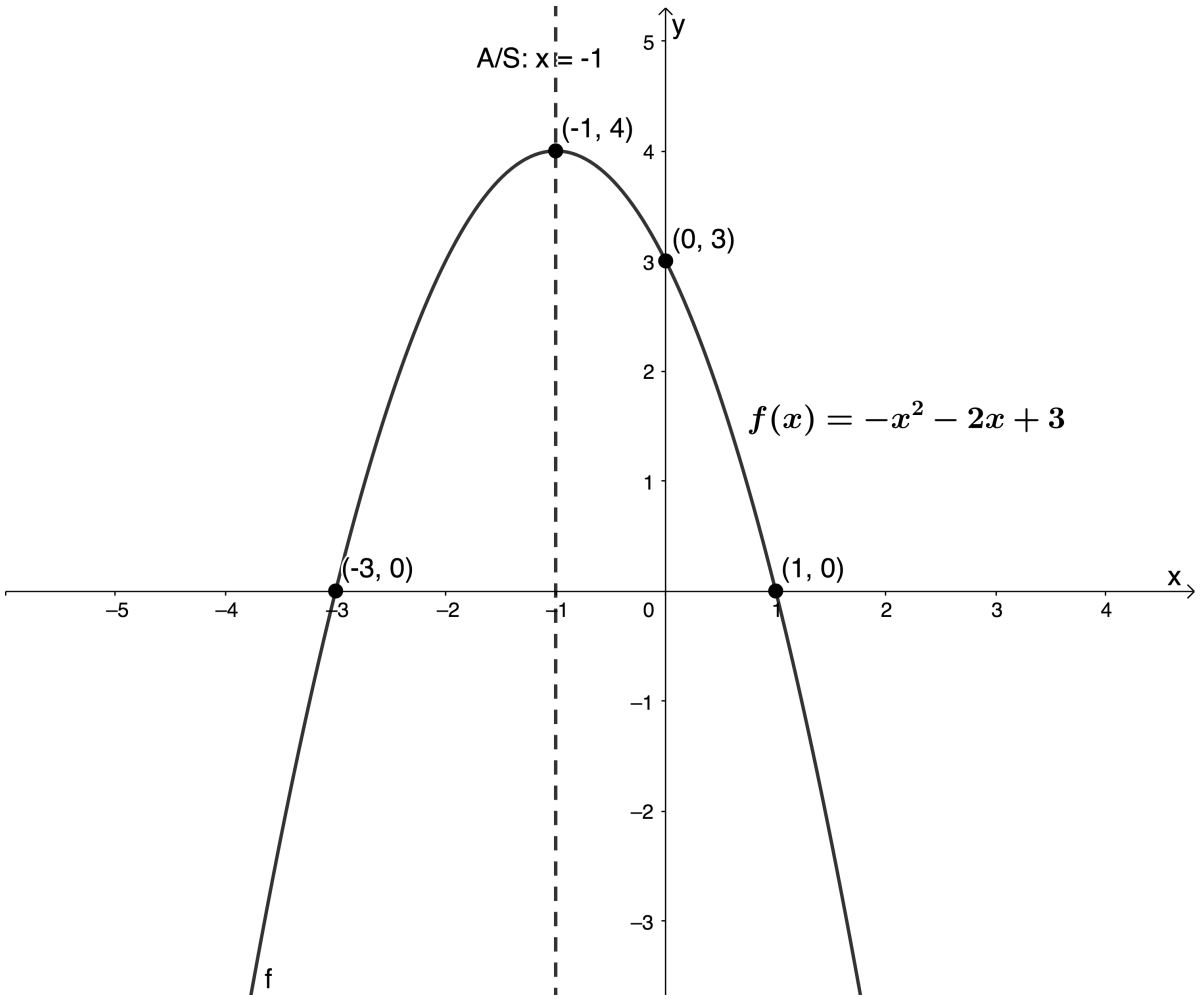
   or

   $$-\frac{b^2 - 4ac}{4a} = -\frac{(-2)^2 - 4(-1)(3)}{4(-1)}$$
   $$= -\frac{4 + 12}{(-4)} = 4$$

   A/S:

   $x = -\frac{b}{2a} = -1$

   $a < 0$. Therefore, the TP is a maximum.



b. A/S is the line $x = -1$.

c. Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

d. Range of $f(x)$ is $\{f(x) : f(x) \leq 4\}$ or $(-\infty, 4]$

2. $y = (x-2)^2 - 7$

x-intercepts: Let $y = 0$

$$(x-2)^2 - 7 = 0$$

$$\therefore x^2 - 4x + 4 - 7 = 0$$

$$\therefore x^2 - 4x - 3 = 0$$

Because we cannot factorise this quadratic, we need to solve it using the quadratic formula. If you need help solving quadratic equations with the quadratic formula, see unit 3 of subject outcome 2.3 in level 3.

The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

In $x^2 - 4x - 3 = 0$ $a = 1, b = -4$ and $c = -3$.

$$
\begin{aligned}
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)} \\
&= \frac{4 \pm \sqrt{16 + 12}}{2} \\
&= \frac{4 \pm \sqrt{28}}{2} \\
&= \frac{4 \pm 2\sqrt{7}}{2} \\
&= 2 \pm \sqrt{7}
\end{aligned}
$$

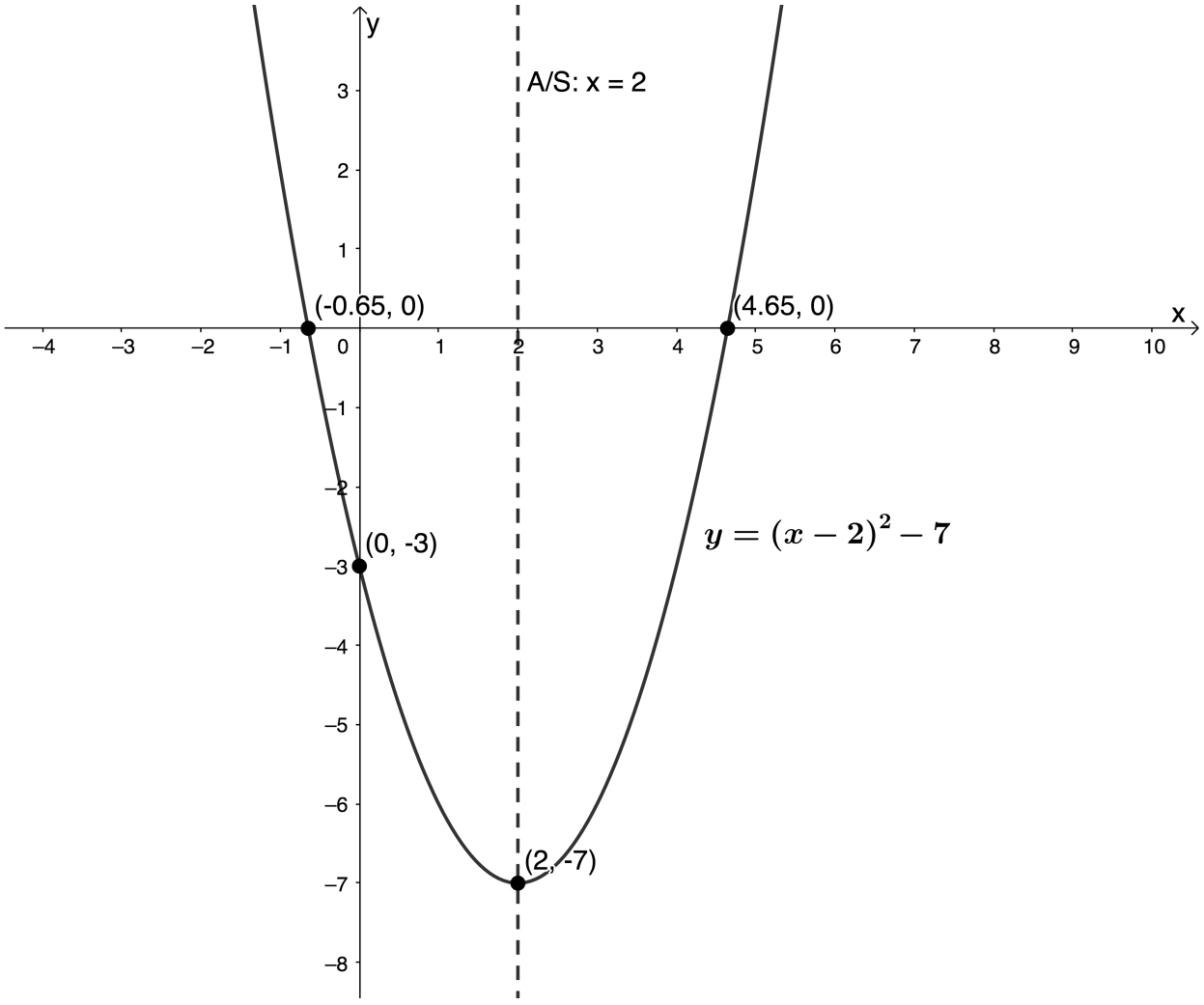
y-intercept: $x = 0$

$$
\begin{aligned}
y &= (0 - 2)^2 - 7 \\
&= 4 - 7 \\
&= -3
\end{aligned}
$$

TP:
$(-p, q) = (2, -7)$

A/S:
$x = 2$

$a > 0$. Therefore, the TP is a minimum.



3. $y = 4\left(x + \frac{7}{2}\right)^2 - 25$.

x-intercepts: Let $y = 0$

$$4\left(x + \frac{7}{2}\right)^2 - 25 = 0$$

$$\therefore 4\left(x^2 + 7x + \frac{49}{4}\right) - 25 = 0$$

$$\therefore 4x^2 + 28x + 49 - 25 = 0$$

$$\therefore 4x^2 + 28x + 24 = 0$$

$$\therefore x^2 + 7x + 6 = 0$$

$$\therefore (x + 1)(x + 6) = 0$$

$$\therefore x = -1 \text{ or } x = -6$$

y-intercept: Let $x = 0$

$$y = 4\left(0 + \frac{7}{2}\right)^2 - 25$$

$$= 4\left(\frac{49}{4}\right) - 25$$

$$= 49 - 25$$

$$= 24$$

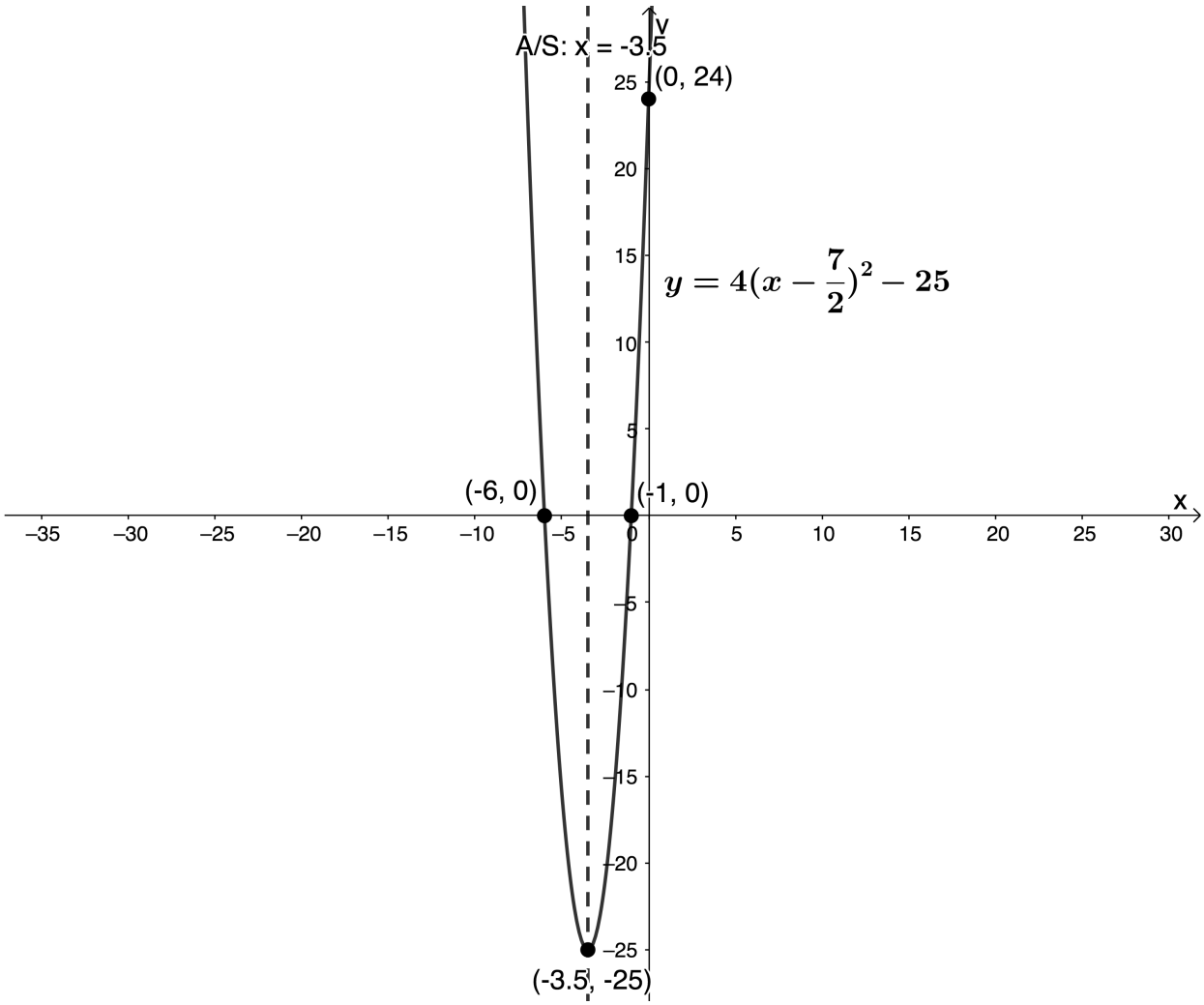
TP:

$(-p, q) = (-\frac{7}{2}, -25)$

A/S:

$x = -\frac{7}{2}$

$a > 0$. Therefore, the TP is a minimum.



Back to Unit 1: Assessment

Media Attributions

- Figure 4 © DHET is licensed under a CC BY-SA (Attribution ShareAlike) license
- activity1.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example 1.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example 1.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise 1.1A2f © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise 1.3A © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 2: Find the equation of quadratic functions

DYLAN BUSA



Unit 2 outcomes

By the end of this unit you will be able to:

- Find the equation of a quadratic function when given the x-intercepts and another point on the graph.
- Find the equation of a quadratic function when given the turning point and another point on the graph.

What you should know

Before you start this unit, make sure you can:

- Solve quadratic equations by:
  - Factorisation
  - Completing the square
  - Using the quadratic formula. Review unit 1 of subject outcome 2.3 for level 2 and unit 1, unit 2 and unit 3 of subject outcome 2.3 for level 3 for help with solving quadratic functions.
- Determine the x- and y-intercepts of a quadratic function.
- Plot quadratic functions of the form $y = a(x + p)^2 + q$ and $y = ax^2 + bx + c$. Review unit 1 of subject outcome 2.1 for level 3.
- Determine the turning point of a quadratic functions of the form $y = a(x + p)^2 + q$ and $y = ax^2 + bx + c$. Review unit 1 of subject outcome 2.1 for level 3.

Introduction

In the previous unit we learnt how to plot quadratic functions of the form $y = a(x + p)^2 + q$ and $y = ax^2 + bx + c$. Remember that $y = ax^2 + bx + c$ is the **general form** of the quadratic function and $y = a(x + p)^2 + q$ is the **turning point (TP)** form of the quadratic where the turning point of the parabola is given by $(-p, q)$.

In this unit, we are going to learn how to find the equations of quadratic functions from their graphs. There are two basic situations that we will come across:

- Where we are given the x-intercepts and one other point.
- Where we are given the turning point and one other point.

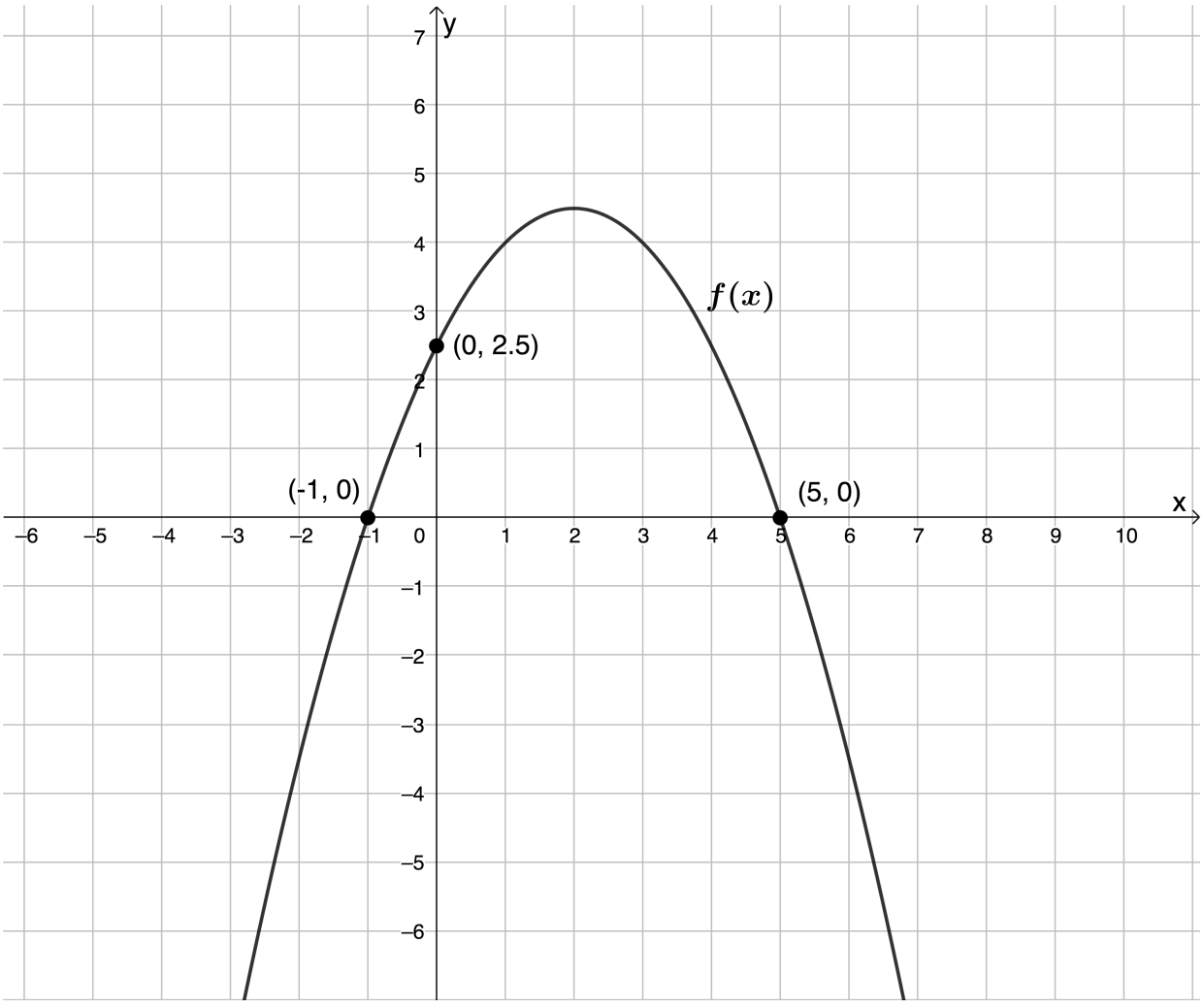
Given the x-intercepts and a point

When we are given the x-intercepts and one other point, we use a factorised form of the general form of the quadratic function $y = a(x - x_1)(x - x_2)$. We know that to find the x-intercepts we have to factorise the quadratic expression. Therefore, to find the expression from the x-intercepts we start with the factorised form and work backwards.

Work through Example 2.1 to see how this is done.

Example 2.1

Find the equation of the parabola of the function $f(x)$ from the graph.



Solution

From the graph we can see that we have been given the x-intercepts and the y-intercept. We know that when we are plotting a quadratic function, we find the x-intercepts by solving the quadratic equation $ax^2 + bx + c = 0$. When we factorise the left-hand side we get $a(x - x_1)(x - x_2) = 0$. We then use the zero product property to find the x-intercepts as x_1 and x_2.

If we are given the x-intercepts and we need to find the original equation, we have to reverse the process. We start with $y = a(x - x_1)(x - x_2)$ and substitute the given x-intercepts into the equation.

$$y = a\,(x-(-1))\,(x-5) \text{ Watch out for the negative sign in the first intercept}$$
$$\therefore y = a(x+1)(x-5)$$
$$\therefore y = a(x^2-4x-5)$$

But we still have an unknown of a in our expression. Therefore, we need one other point to substitute in for x and y in order to solve for a. In this example, that point is the y-intercept, but it could be any given point on the graph.

Substitute $(0, 2.5)$ into $y = a(x^2-4x-5)$.

$$2.5 = a\left(0^2-4(0)-5\right)$$
$$\therefore 2.5 = -5a$$
$$\therefore a = \frac{2.5}{-5} = -\frac{1}{2}$$

For a quick check, we can see from the graph that $a < 0$.

Our final expression for this quadratic function in standard form is:

$$y = -\frac{1}{2}(x^2-4x-5)$$
$$\therefore y = -\frac{1}{2}x^2+2x+\frac{5}{2}$$

Sometimes you may be asked to give the function in the turning point form in which case you will need to complete the square.

Note

You must always give the final quadratic function in the form requested in the question. If no form is specified, you can give it in either standard form or TP form.



Example 2.2

Find the equation of the parabola with an x-intercept of $(-7, 0)$ an axis of symmetry (A/S) $x = -\frac{11}{2}$ and passing through the point $(-2, 30)$.

Solution

In this case, it does not seem like we have enough information because we only have a single x-intercept. However, we are also given the A/S and we know that the x-intercepts are the same distance from the A/S, by definition. The A/S is the line the cuts a parabola into two mirror image halves.

The A/S is the line $x = -\frac{11}{2} = -5.5$ and one of the x-intercepts is $(-7, 0)$. This intercept is one and a half units to the left of the A/S. Therefore, the other x-intercept must be one and a half units to the right of the A/S. In other words, it is the point $(-4, 0)$. Now we have two x-intercepts and we can proceed as before by substituting these intercepts into $y = a(x-x_1)(x-x_2)$.

$$y = a\,(x-(-7))\,(x-(-4)) \qquad \text{Watch out for the negative signs}$$
$$\therefore y = a(x+7)(x+4)$$
$$\therefore y = a(x^2+11x+28)$$

Now substitute $(-2, 30)$ into $y = a(x^2+11x+28)$.

$$30 = a\left((-2)^2+11(-2)+28\right)$$
$$\therefore 30 = a(4-22+28)$$
$$\therefore 30 = 10a$$
$$\therefore a = \frac{30}{10} = 3$$

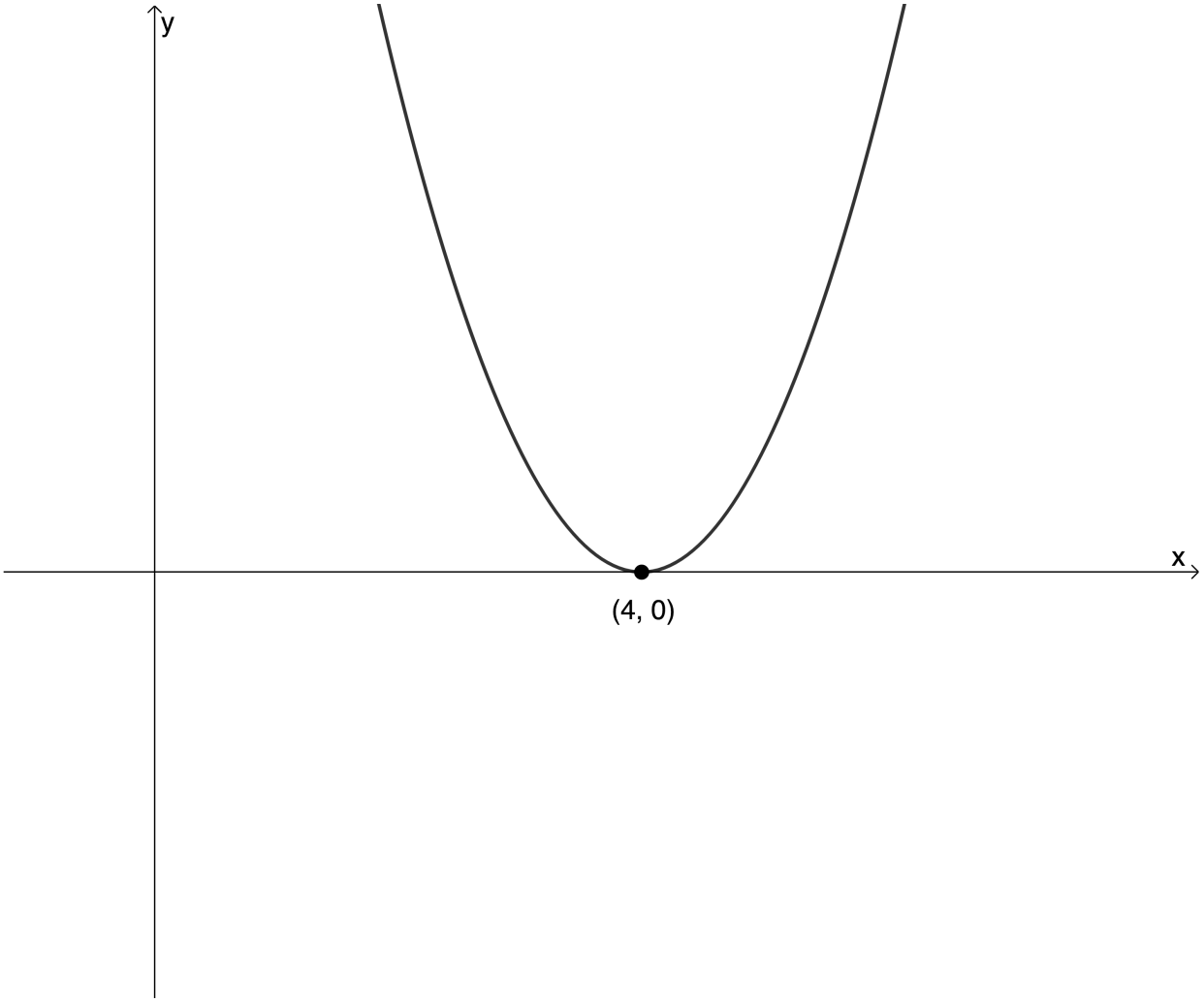
Our final expression for this quadratic function in standard form is:

$$y = 3(x^2+11x+28)$$
$$\therefore y = 3x^2+33x+84$$



Take note!

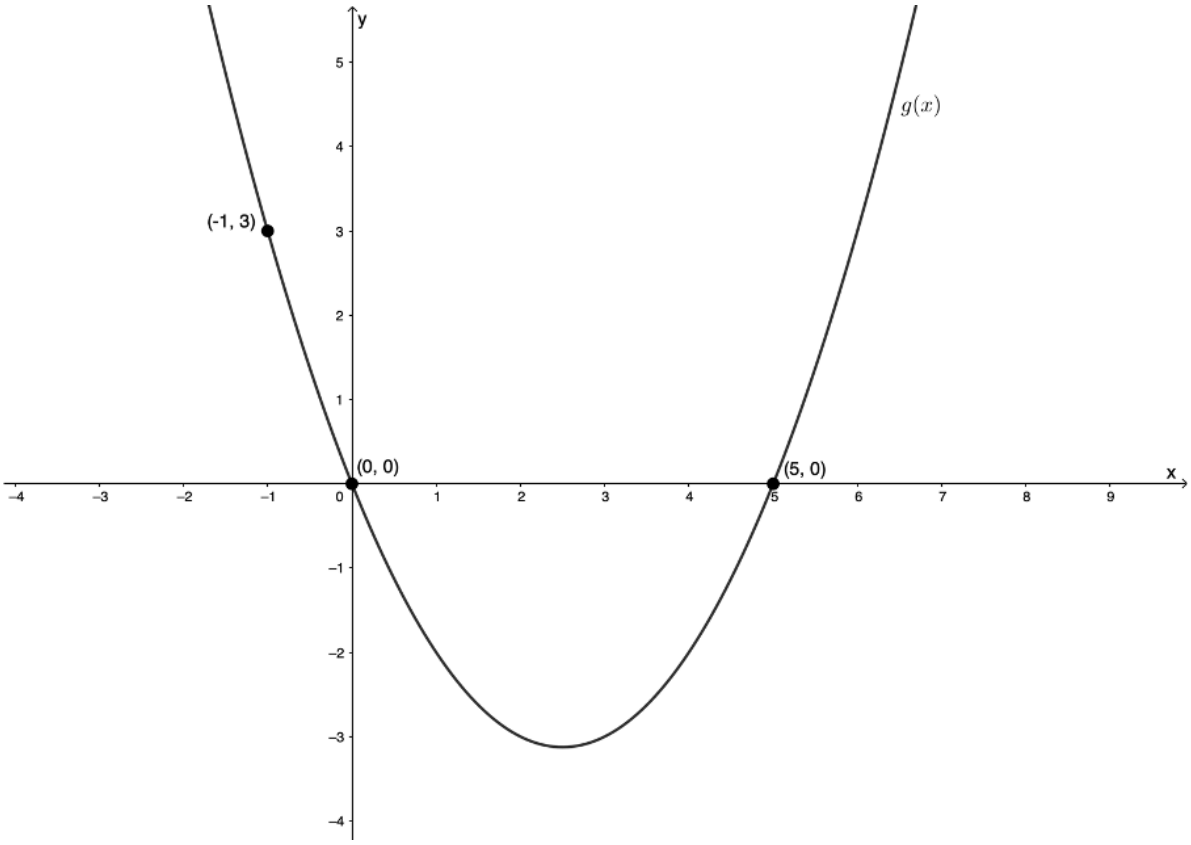
In cases where the parabola just touches the x-axis and there is only a single x-intercept, use the same value for both x_1 and x_2 in $y = a(x-x_1)(x-x_2)$.



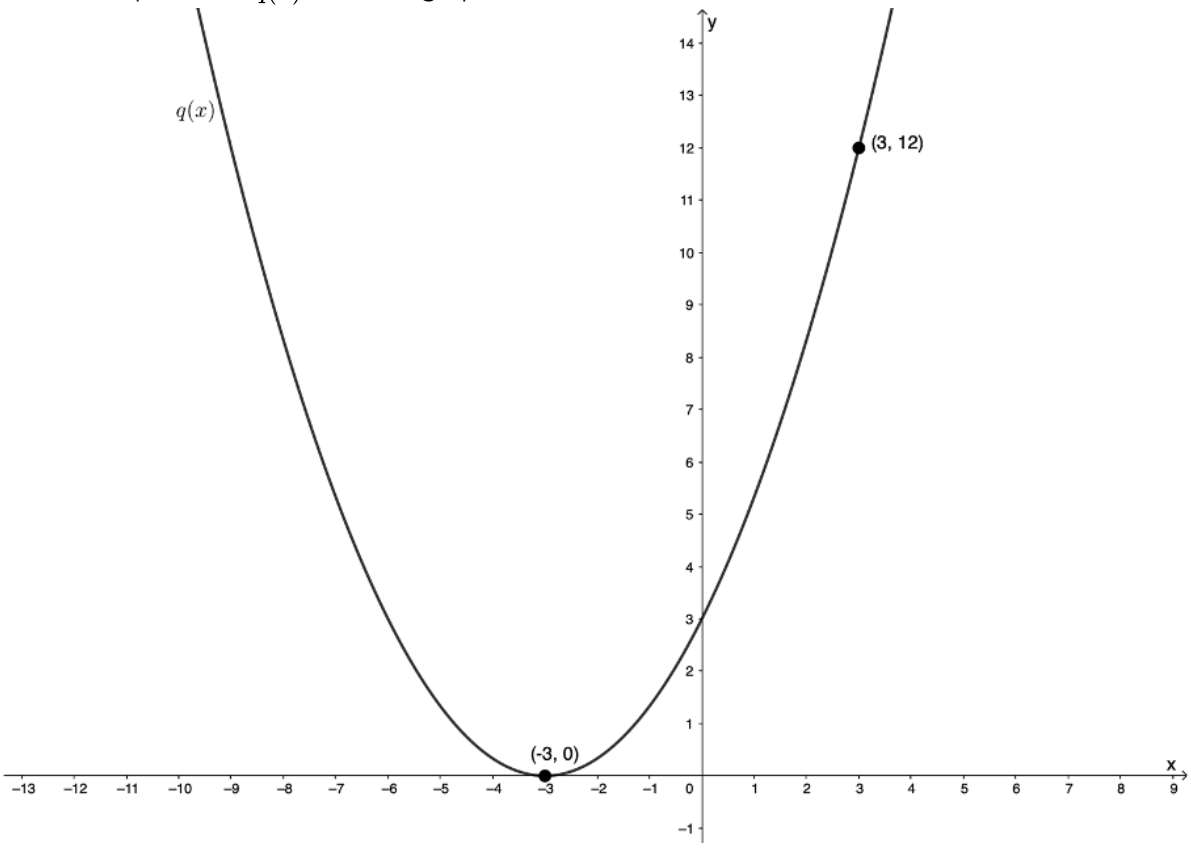
In the parabola above, both x-intercepts are the same point $(4, 0)$.

Exercise 2.1

1. Find the equation of $g(x)$ from the graph.



2. Find the equation of $q(x)$ from the graph.



The full solutions are at the end of the unit.

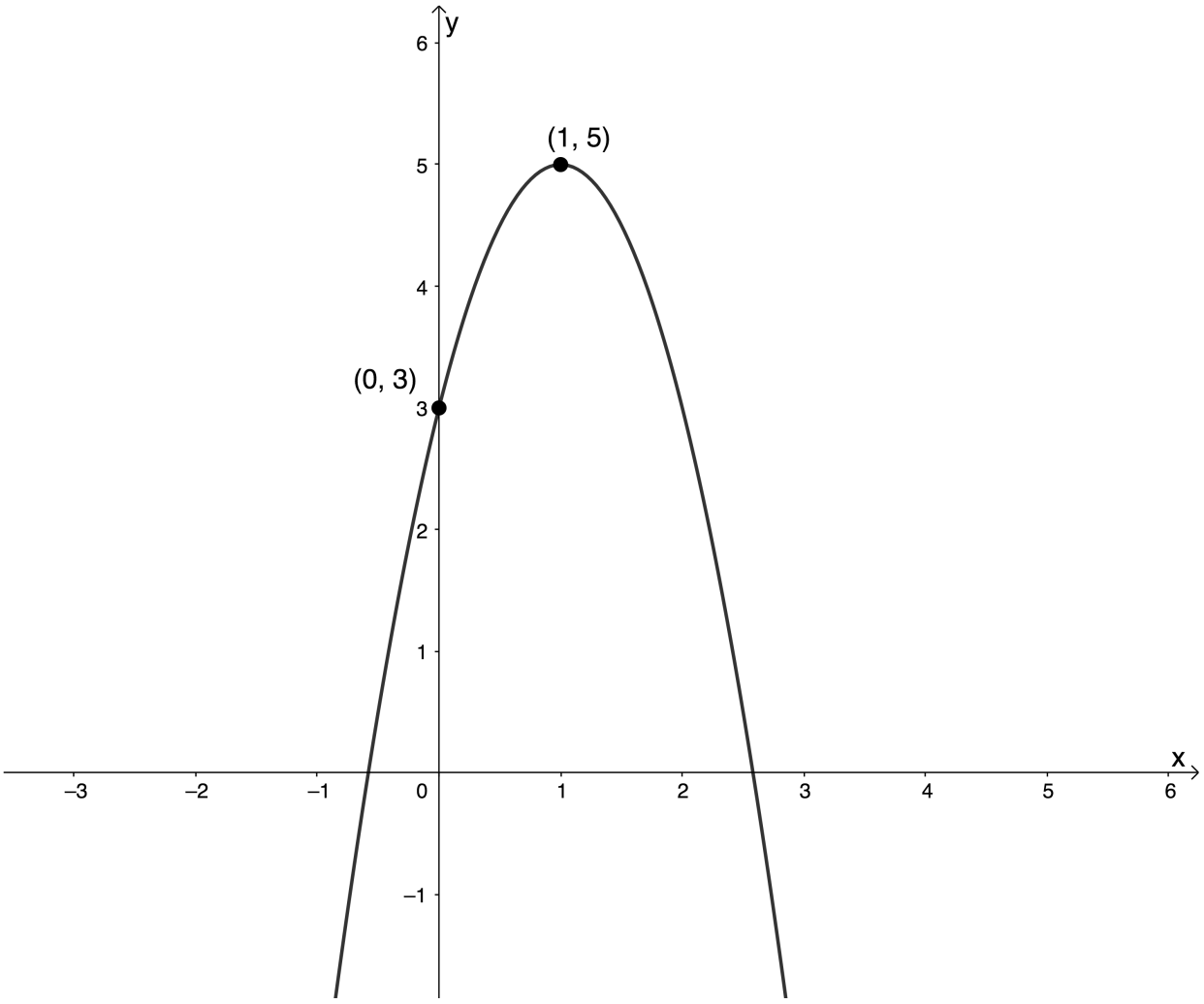
Given the turning point and a point

When we are given the TP of a parabola, and we need to find the function, we start by substituting these values into the TP form of the quadratic equation, $y = a(x + p)^2 + q$. Remember the TP is the point $(-p, q)$.

Work through Example 2.3 to see how.

Example 2.3

Find the equation of the parabola from the graph.



Solution

In this example, we have been given the turning point (TP) and another point on the graph. It just so happens that this other point is the y-intercept.

We know that in the turning point form of the quadratic function $y = a(x + p)^2 + q$, the TP is the point $(-p, q)$. We know what the TP is, therefore, we know what the values of p and q are and can substitute these into the TP form.

$$y = a(x - 1)^2 + 5$$
$$\therefore y = a(x^2 - 2x + 1) + 5$$

But we still have an unknown of a in our equation. Therefore, we need one other point to substitute for x and y in order to solve for a. In this example, that point is the y-intercept, but it could be any point on the graph.

Substitute $(0, 3)$ into $y = a(x^2 - 2x + 1) + 5$.

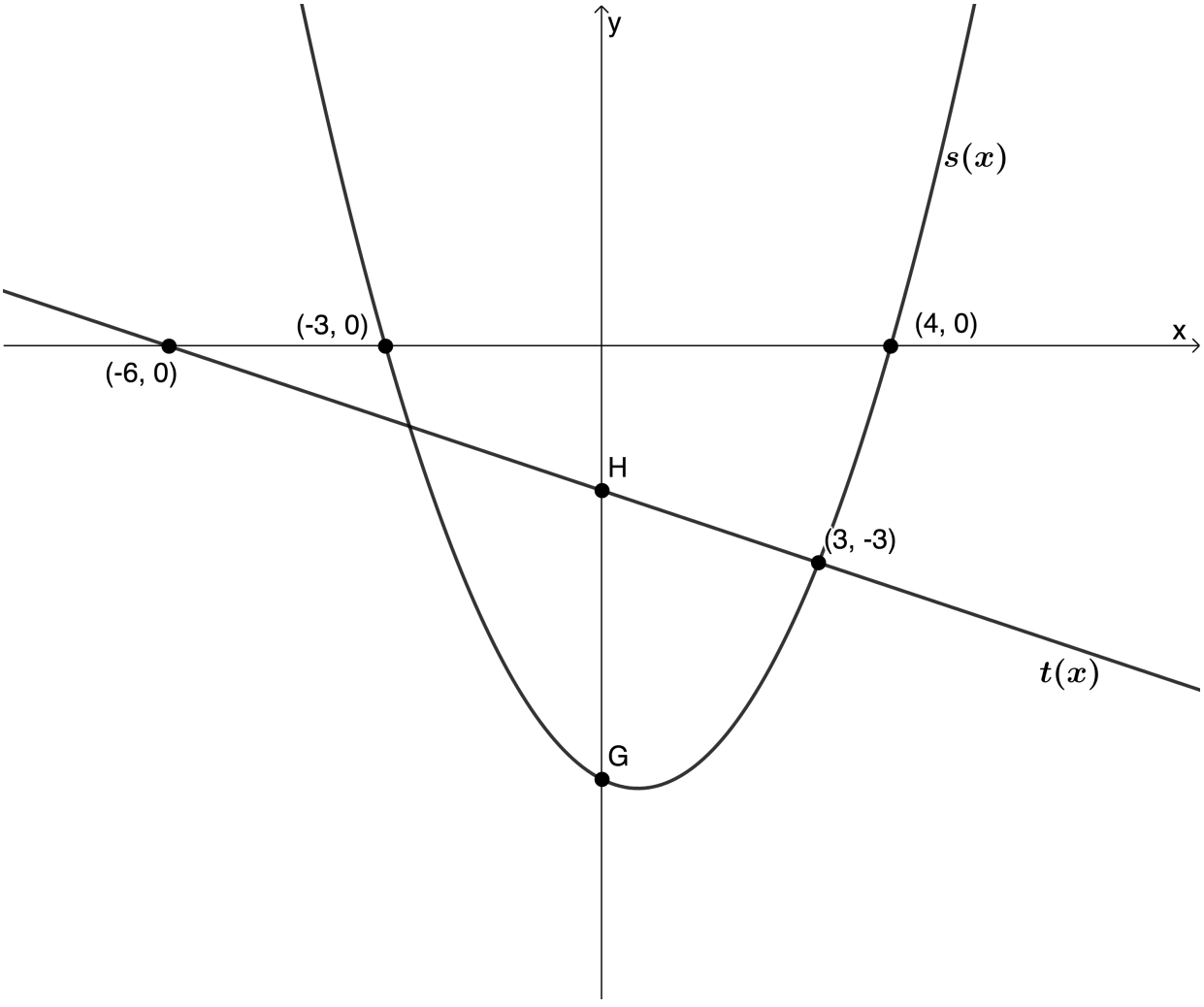
$$3 = a\left(0^2 - 2(0) + 1\right) + 5$$
$$\therefore 3 = a + 5$$
$$\therefore a = -2$$

Our final expression for this quadratic function in TP form is $y = -2(x - 1)^2 + 5$.

Sometimes you may be asked to give the quadratic function in its standard form in which case you just need to expand and simplify to get $y = -2x^2 + 4x + 3$.

Example 2.4

In the diagram below, $s(x)$ is a quadratic function with x-intercepts of -3 and 4. G is the y-intercept of $s(x)$. $t(x)$ is a linear function with an x-intercept of -6. $t(x)$ intersects $s(x)$ at $(3, -3)$. H is the y-intercept of $t(x)$.



1. Determine the equation of $s(x)$.
2. Determine the equation of $t(x)$.
3. Find the distance GH.

Solutions

1. $s(x) = a(x - x_1)(x - x_2)$ where $x_1 = -3$ and $x_2 = 4$.
   $s(x) = a(x + 3)(x - 4)$
   $\therefore s(x) = a(x^2 - x - 12)$ The point $(3, -3)$ lies on $s(x)$.

   $$-3 = a\left((3)^2 - (3) - 12\right)$$
   $$\therefore -3 = -6a$$
   $$\therefore a = \frac{1}{2}$$

   $$s(x) = \frac{1}{2}(x^2 - x - 12)$$
   $$= \frac{1}{2}x^2 - \frac{1}{2}x - 6$$

2. We know the equation of a straight line is given by $y = mx + c$. Therefore, $t(x) = mx + c$.
   We have two points on the straight line so we can calculate the gradient.
   $$m_{t(x)} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-3)}{-6 - 3} = \frac{3}{-9} = -\frac{1}{3}$$

   But $t(x) = -\frac{1}{3}x + c$ passes through the point $(3, -3)$ so we can substitute these values for x and y and solve for c.

   $$-3 = -\frac{1}{3}(3) + c$$
   $$\therefore -3 = -1 + c$$
   $$\therefore c = -2$$

   $$t(x) = -\frac{1}{3}x - 2$$

3. To find the vertical distance between two functions we say the top graph minus the bottom graph. So, it is the top graph's function value minus the bottom graph's function value.H is the y-intercept of $t(x)$. Find $t(0)$.

   $t(0) = -\frac{1}{3}(0) - 2 = -2$. Remember in $y = mx + c$, c is the y-intercept.

   G is the y-intercept of $s(x)$. Find $s(0)$.

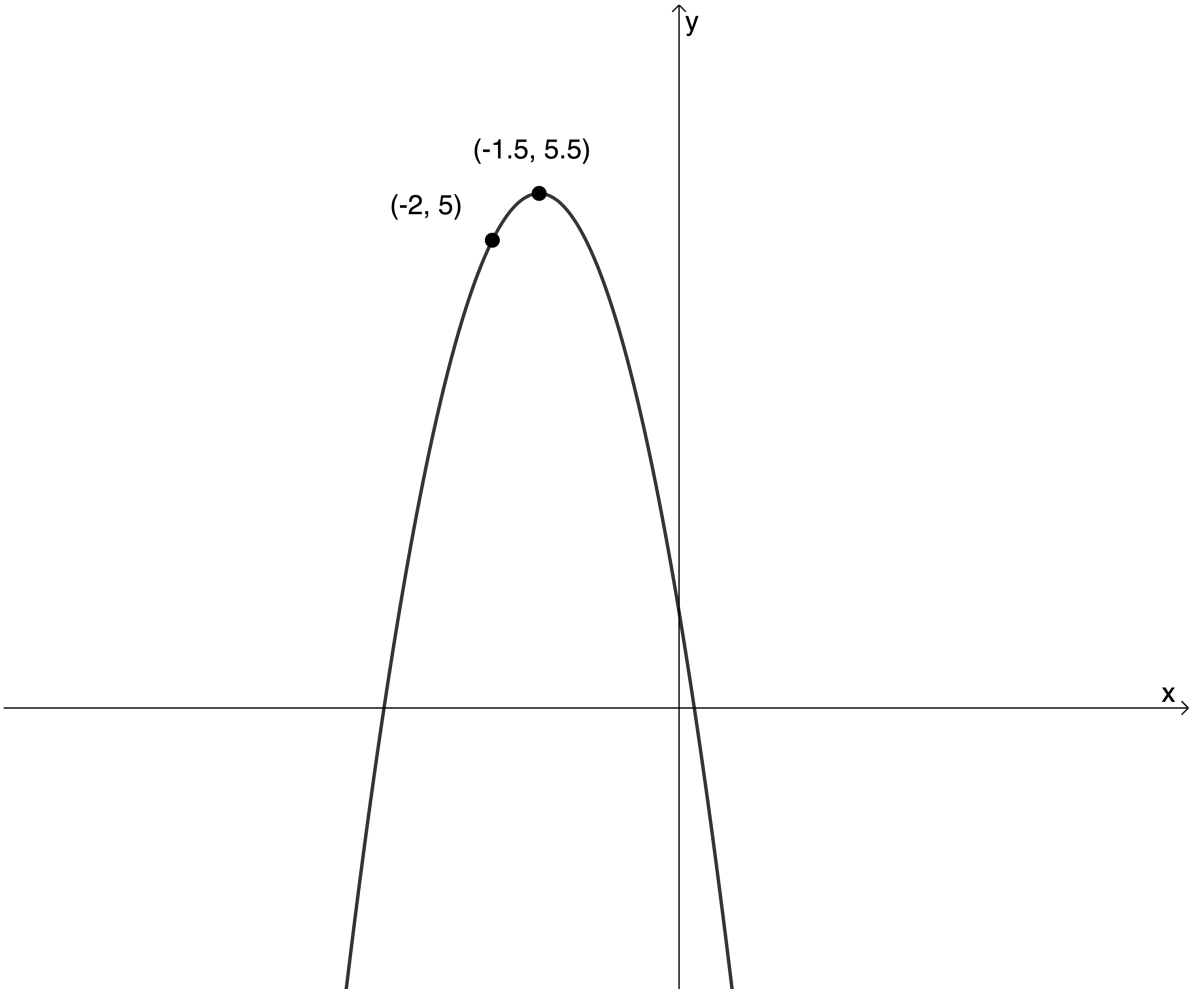
   $s(0) = \frac{1}{2}(0)^2 - \frac{1}{2}(0) - 6 = -6$. Remember that in $y = ax^2 + bx + c$, c is the y-intercept.

   Therefore

   $$\mathrm{GH} = -2 - (-6)$$
   $$= 4$$

Exercise 2.2

1. Find the equation of the parabola from the graph in TP form.



2. The quadratic function $k(x)$ has a turning point of $(2, -5)$ and a y-intercept of $(0, -4)$. What is the equation of the function in turning point form?

The full solutions are at the end of the unit.

Summary

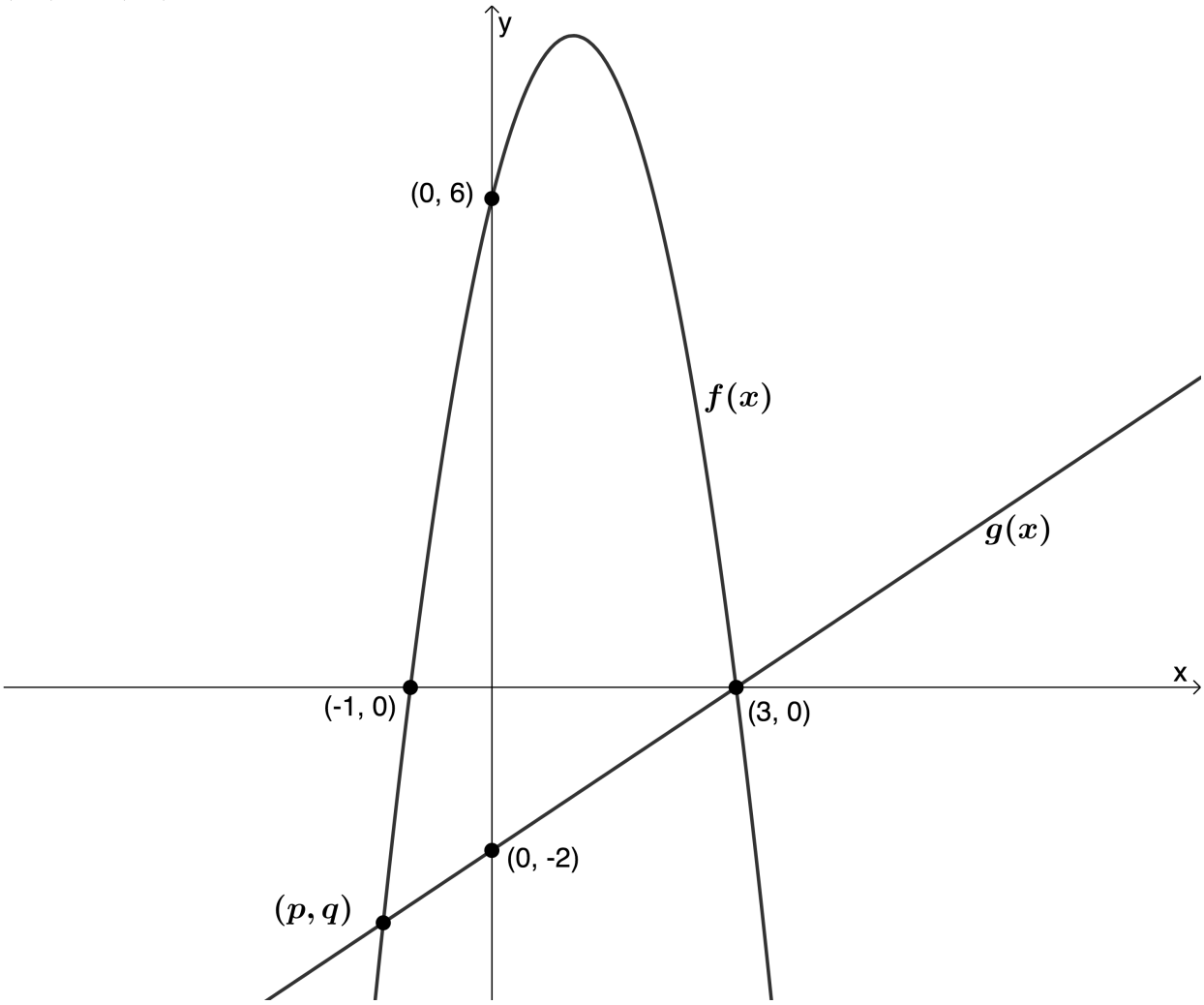
In this unit you have learnt the following:

- How to find the equation of a quadratic function when given the x-intercepts and one other point.
- How to find the equation of a quadratic function when given the turning point and one other point.

Unit 2: Assessment

Suggested time to complete: 40 minutes

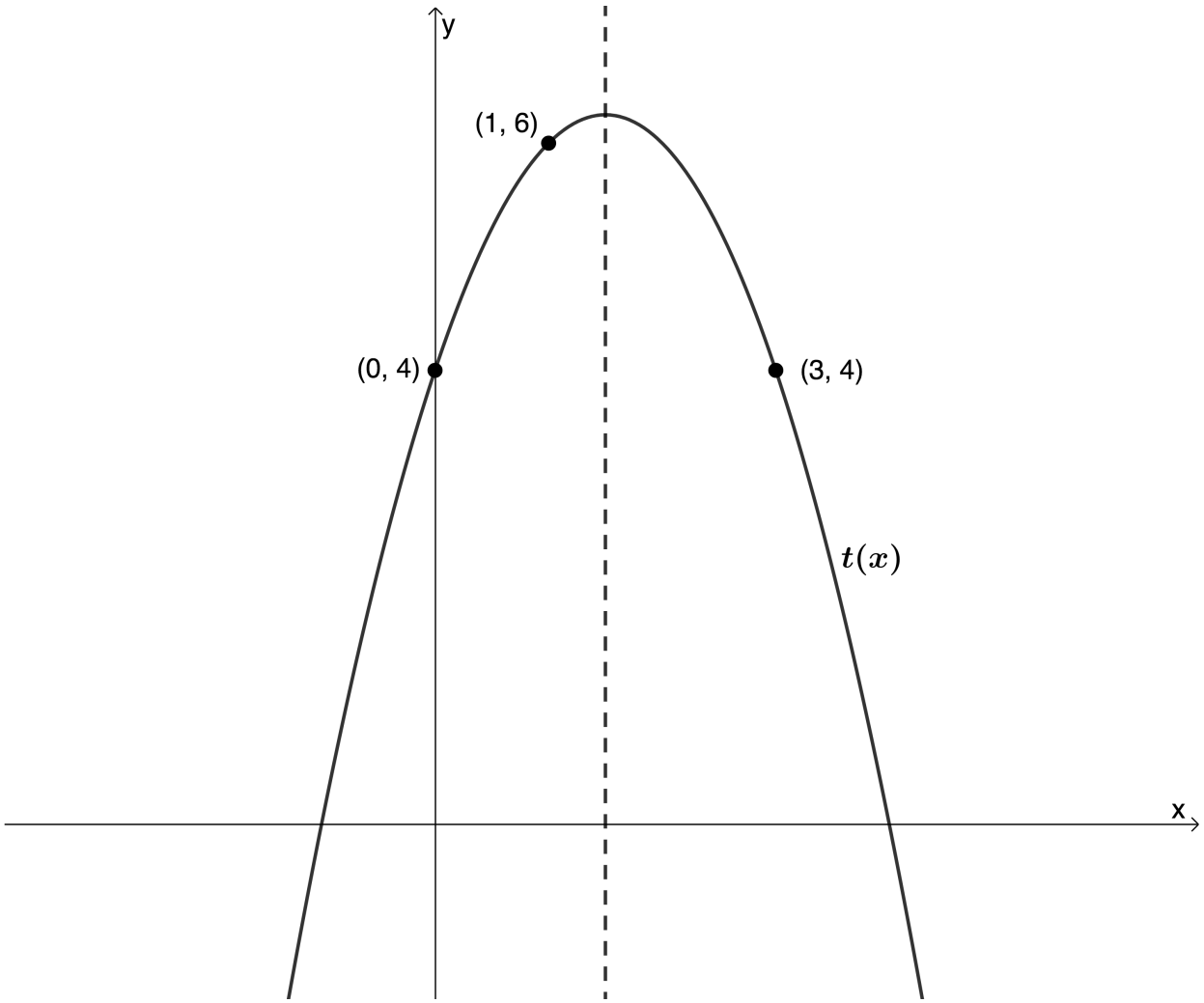
1. The system of axes given below shows the graphs of $f(x)$ and $g(x)$. The graph given as $f(x)$ is a parabola which cuts the x-axis at -1 and 3 and has a y-intercept of 6. The graph given as $g(x)$ is a straight line that passes through the points $(0, -2)$ and $(3, 0)$. The two graphs intersect at the points $(3, 0)$ and (p, q).



   a. Determine the equation of $f(x)$.
   b. Determine the equation of $g(x)$.
   c. Find the values of p and q.
   d. Use the graph to determine the values of x for which $f(x).\, g(x) \geq 0$.

2. What are the x-intercepts of the quadratic function that has a turning point of $(2, -2)$ and passes through the point $\left(3, \frac{3}{2}\right)$?

3. Determine the equation of $t(x)$ from the graph. HINT: Remember that the axis of symmetry divides the parabola into two mirror image halves. The standard form of the quadratic function is $y = ax^2 + bx + c$. The A/S is given by $x = -\frac{b}{2a}$.



The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1. $g(x) = a(x - x_1)(x - x_2)$ where $x_1 = 0$ and $x_2 = 5$.

   $g(x) = a(x - 0)(x - 5)$

   $\therefore g(x) = a(x)(x - 5)$

   $\therefore g(x) = a(x^2 - 5x)$

   The point $(-1, 3)$ lies on $g(x)$.

   $3 = a\left((-1)^2 - 5(-1)\right)$

   $\therefore 3 = 6a$

   $\therefore a = \frac{1}{2}$

   The quadratic function has the equation:

   $g(x) = \frac{1}{2}(x^2 - 5x)$

   $\therefore g(x) = \frac{1}{2}x^2 - \frac{5}{2}x$

2. $q(x) = a(x - x_1)(x - x_2)$ where $x_1 = -3$ and $x_2 = -3$.

$q(x) = a(x + 3)(x + 3)$

$\therefore q(x) = a(x^2 + 6x + 9)$

The point $(3, 12)$ lies on $q(x)$.

$12 = a\left((3)^2 + 6(3) + 9\right)$

$\therefore 12 = 36a$

$\therefore a = \frac{1}{3}$

The quadratic function has the equation:

$q(x) = \frac{1}{3}(x^2 + 6x + 9)$

$\therefore q(x) = \frac{1}{3}x^2 + 2x + 3$

Back to Exercise 2.1

Exercise 2.2

1. $y = a(x + p)^2 + q$ where the TP is $(-1.5, 5.5)$ or $\left(-\frac{3}{2}, \frac{11}{2}\right)$.

$y = a\left(x + \frac{3}{2}\right)^2 + \frac{11}{2}$

$\therefore y = a\left(x^2 + 3x + \frac{9}{4}\right) + \frac{11}{2}$

The point $(-2, 5)$ lies on the graph.

$5 = a\left((-2)^2 + 3(-2) + \frac{9}{4}\right) + \frac{11}{2}$

$\therefore 5 = a\left(4 - 6 + \frac{9}{4}\right) + \frac{11}{2}$

$\therefore 5 = \frac{1}{4}a + \frac{11}{2}$

$\therefore 20 = a + 22$

$\therefore a = -2$

The quadratic function has the equation:

$y = -2\left(x + \frac{3}{2}\right)^2 + \frac{11}{2}$ in TP form.

2. $y = a(x + p)^2 + q$ where the TP is $(2, -5)$.

$y = a(x - 2)^2 - 5$

$\therefore y = a\left(x^2 - 4x + 4\right) - 5$

The point $(0, -4)$ lies on the graph.

$-4 = a\left((0)^2 - 4(0) + 4\right) - 5$

$\therefore -4 = 4a - 5$

$\therefore 4a = 1$

$\therefore a = \frac{1}{4}$

The quadratic function has the equation:

$y = \frac{1}{4}(x - 2)^2 - 5$

Back to Exercise 2.2

Unit 2: Assessment

1. .

   a. $f(x) = a(x - x_1)(x - x_2)$ where $x_1 = -1$ and $x_2 = 3$.

   $f(x) = a(x + 1)(x - 3)$

   $\therefore f(x) = a(x^2 - 2x - 3)$

   The point $(0, 6)$ lies on the graph.

   $6 = a\left((0)^2 - 2(0) - 3\right)$

   $\therefore 6 = -3a$

   $\therefore a = -2$

   $f(x) = -2(x^2 - 2x - 3) = -2x^2 + 4x + 6$.

   b. $g(x) = mx + c$ where $c = -2$. $m_{g(x)} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2 - 0}{0 - 3} = \dfrac{2}{3}$

   $g(x) = \dfrac{2}{3}x - 2$.

   c. The point (p, q) is where $f(x)$ and $g(x)$ intersect.

   $f(x) = g(x)$

   $\therefore -2x^2 + 4x + 6 = \dfrac{2}{3}x - 2$

   $\therefore -6x^2 + 12x + 18 = 2x - 6$

   $\therefore -6x^2 + 10x + 24 = 0$

   $\therefore 3x^2 - 5x - 12 = 0$

   $\therefore (3x + 4)(x - 3) = 0$

   $\therefore x = -\dfrac{4}{3}$ or $x = 3$

   $p = -\dfrac{4}{3}$. Substitute into $x = -\dfrac{4}{3}$ into $g(x)$.

   $g\left(-\dfrac{4}{3}\right) = \dfrac{2}{3}\left(-\dfrac{4}{3}\right) - 2$

   $= -\dfrac{8}{9} - \dfrac{18}{9}$

   $= -\dfrac{26}{9}$

   $q = -\dfrac{26}{9}$

   d. $f(x).g(x) \geq 0$ when $f(x) \leq 0$ and $g(x) \leq 0$ or $f(x) \geq 0$ and $g(x) \geq 0$. Therefore $f(x).g(x) \geq 0$ for $x \in (-\infty, -1]$.

2. First, we need to find the equation of the quadratic function.

   $y = a(x + p)^2 + q$ where the TP is $(2, -2)$.

   $y = a(x - 2)^2 - 2$

   $\therefore y = a\left(x^2 - 4x + 4\right) - 2$

   The point $\left(3, -\dfrac{3}{2}\right)$ lies on the graph.

   $-\dfrac{3}{2} = a\left((3)^2 - 4(3) + 4\right) - 2$

   $\therefore -\dfrac{3}{2} = a - 2$

   $\therefore a = \dfrac{1}{2}$

   The quadratic function has the equation:

   $y = \dfrac{1}{2}(x - 2)^2 - 2$

   To find the x-intercepts, we need to expand the quadratic function into standard form, set the function

equal to zero and solve for x.

$$y = \frac{1}{2}(x-2)^2 - 2$$

$$\therefore y = \frac{1}{2}(x^2 - 4x + 4) - 2$$

$$\therefore y = \frac{1}{2}x^2 - 2x + 2 - 2$$

$$\therefore y = \frac{1}{2}x^2 - 2x$$

Let $y = 0$:

$$\frac{1}{2}x^2 - 2x = 0$$

$$\therefore x^2 - 4x = 0$$

$$\therefore x(x-4) = 0$$

$$\therefore x = 0 \text{ or } x = 4$$

The x-intercepts are 0 and 4.

3. We are given three points on the graph but none of them are x-intercepts or the TP. The point $(0, 4)$ is the y-intercept. From two of the points, we can work out what the A/S is. The points $(0, 4)$ and $(3, 4)$ are reflected about the A/S and so the A/S is halfway between them. Halfway between 0 and 3 is $\frac{3}{2}$. Therefore, the A/S is the line $x = \frac{3}{2}$.

   We know that the A/S is given by $\frac{-b}{2a}$. Therefore:

   $$\frac{-b}{2a} = \frac{3}{2}$$

   $$\therefore -2b = 6a$$

   $$\therefore b = -3a \quad \text{Equation } 1$$

   The graph passes through the point $(1, 6)$ so $6 = a(1)^2 + b(1) + c$. But we know that the y-intercept is 4. Therefore $c = 4$.

   $$6 = a(1)^2 + b(1) + 4$$

   $$\therefore 6 = a + b + 4$$

   $$\therefore a = -b + 2 \quad \text{Equation } 2$$

   Substitute equation 1 into equation 2:

   $$a = -(-3a) + 2$$

   $$\therefore a = 3a + 2$$

   $$\therefore 2a = -2$$

   $$\therefore a = -1$$

   Substitute $a = -1$ into equation 1:

   $$b = -3\,(-1)$$

   $$\therefore b = 3$$

   The equation of the quadratic function is $t(x) = -x^2 + 3x + 4$.

Back to Unit 2: Assessment

Media Attributions

- example2.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- takenote © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise2.1Q1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise2.1Q2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example2.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example2.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise2.2Q1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

- assessmentQ3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 3: Sketch hyperbolic functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Determine the domain and range of hyperbolic functions of the form $y = \frac{a}{(x+p)} + q$.
- Find the asymptotes of hyperbolic functions of the form $y = \frac{a}{(x+p)} + q$.
- Find the axes of symmetry of hyperbolic functions of the form $y = \frac{a}{(x+p)} + q$.
- Find the intercepts with the axes of hyperbolic functions of the form $y = \frac{a}{(x+p)} + q$.
- Sketch hyperbolic functions of the form $y = \frac{a}{(x+p)} + q$.

What you should know

Before you start this unit, make sure you can:

- Solve equations of the form $y = \frac{a}{x} + q$.
- Plot hyperbolic functions of the form $y = \frac{a}{x} + q$.
- Determine the x- and y-intercepts of a hyperbolic function of the form $y = \frac{a}{x} + q$.
- Determine the domain and range of a hyperbolic function of the form $y = \frac{a}{x} + q$.
- Determine the asymptotes of hyperbolic functions of the form $y = \frac{a}{x} + q$.

Refer to level 2 subject outcome 2.1 unit 3 if you require assistance with any of these skills.

Introduction

In level 2 we were introduced to the **hyperbolic function** of the form $y = \frac{a}{x} + q$ and the graph of the function called a **hyperbola**. We learnt that these types of functions and their graphs are widely used in the design of lenses and mirrors because of their ability to focus light to a single point.

We discovered that the hyperbola has two arms and the shape and position of these are affected by the values of a and q.

Take note!

The effect of q:

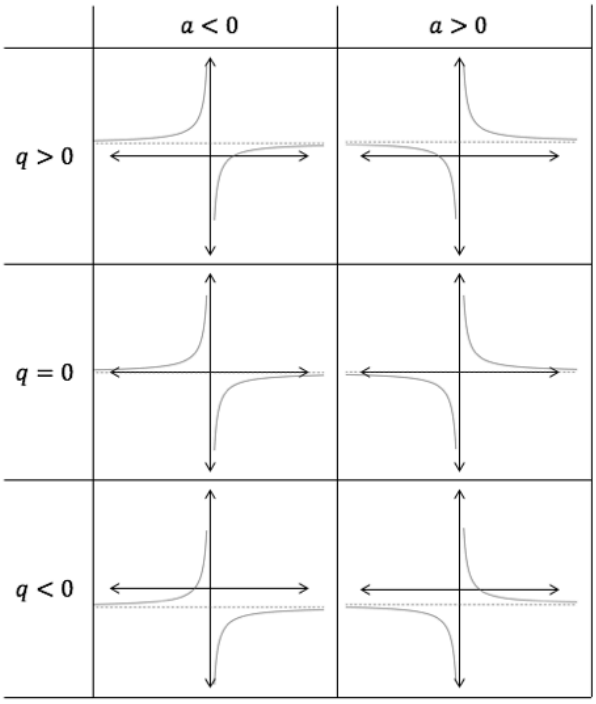
- if $q > 0$, the function is shifted **up** by q units
- if $q < 0$, the function is shifted **down by** q units
- **horizontal asymptote** is the line $y = q$
- **vertical asymptote** is the y-axis or the line $x = 0$
- **axes of symmetry** are $y = x + q$ and $y = -x + q$.

The effect of a:

if $a > 0$, the function lies in the first and third quadrants

if $a < 0$, the function lies in the second and fourth quadrants

as the value of a gets closer to zero (from the positive or negative side), the graph gets closer to the axes and/or asymptotes.



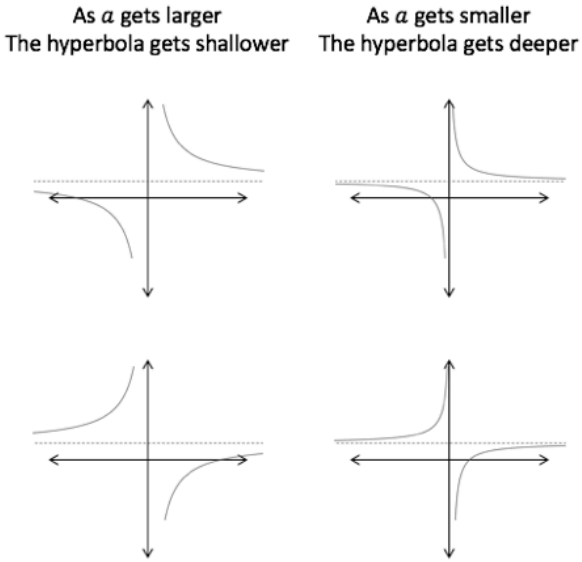


Figure 1: Summary of the characteristics of the function $y = \frac{a}{x} + q$

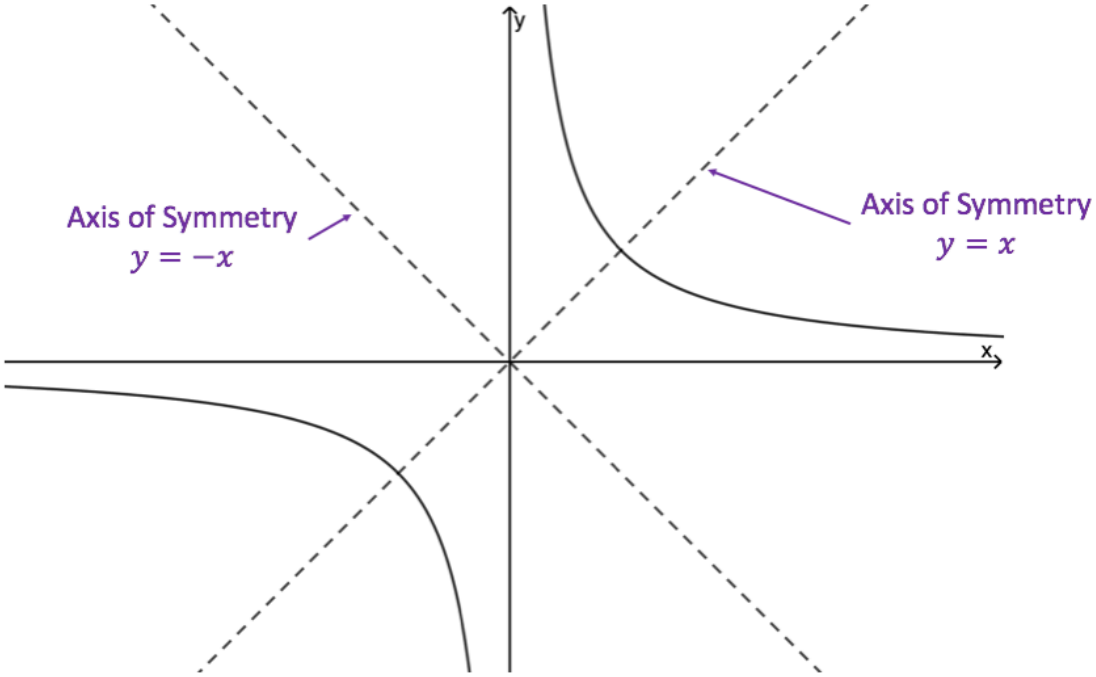


Figure 2: Axes of symmetry of the function $y = \frac{a}{x} + q$

Note

If you have access to the internet, visit this interactive simulation.

Here you will find a hyperbolic function of the form $y = \frac{a}{x} + q$ with sliders to change the values of a and q. Spend some time playing with it to make sure that you understand how changing the values of a and q affects the shape and location of the hyperbola of the form $y = \frac{a}{x} + q$.

The hyperbolic function of the form $y = \frac{a}{(x+p)} + q$

We know that the value of q in $y = \frac{a}{x} + q$ shifts the graph vertically up or down. Now, let's investigate the

effect of p in $y = \frac{a}{(x+p)} + q$. Based on your knowledge of the effect of p in the turning point form of the quadratic function $y = a(x+p)^2 + q$, What do you think this effect will be?

Figure 3 shows the graphs of three hyperbolic functions, $f(x) = \frac{2}{x}$, $g(x) = \frac{2}{x+2}$ and $h(x) = \frac{2}{x-3}$. We can see that because $a > 0$ in each case, all three graphs lie in the first and third quadrants and because $q = 0$, all three graphs have horizontal asymptotes of $y = 0$.

But what is different? Can you see that the vertical asymptote of $f(x)$ is the y-axis (the line $x = 0$) while the graph of $g(x)$ has been shifted two units to the **left** of $f(x)$ (its vertical asymptote is $x = -2$) and $h(x)$ has been shifted three unit to the **right** (its vertical asymptote is $x = 3$)?

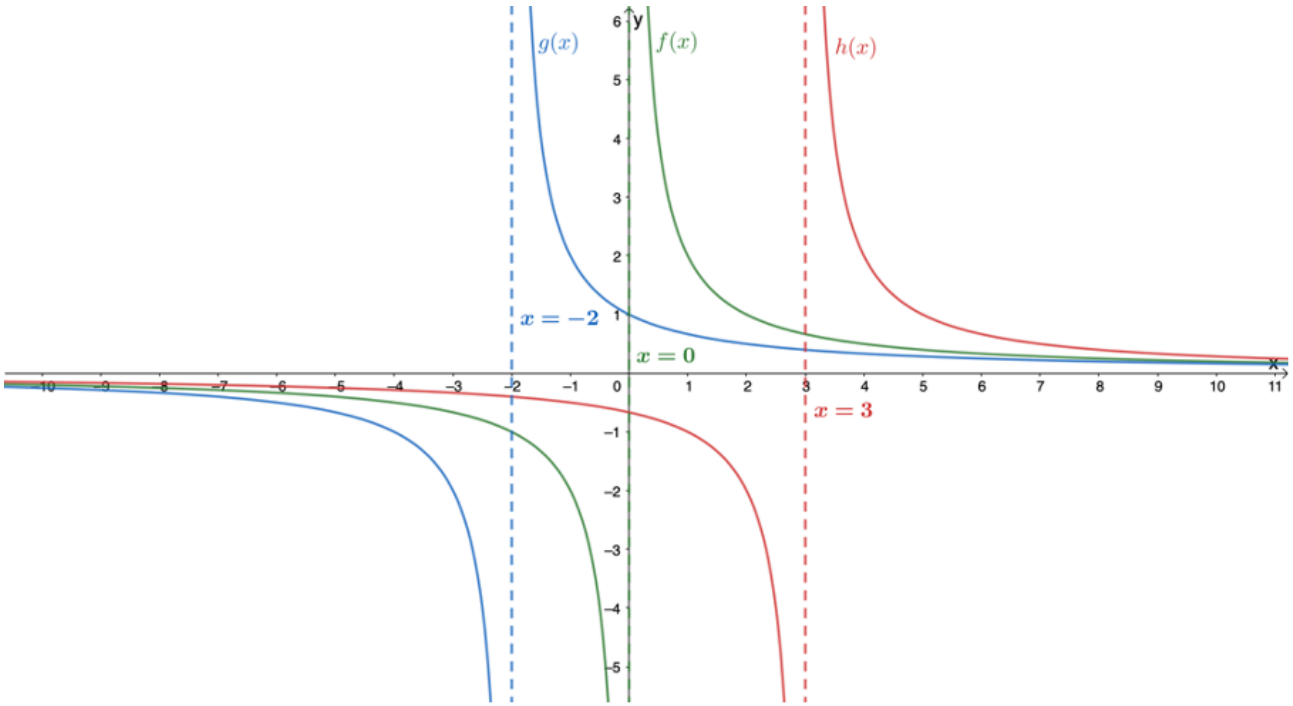


Figure 3: Graphs of $f(x) = \frac{2}{x}$, $g(x) = \frac{2}{(x+2)}$ *and* $h(x) = \frac{2}{(x-3)}$

The value of p in $y = \frac{a}{(x+p)} + q$ is therefore responsible for a horizontal shift of the graph.

- If $p > 0$, the graph is shifted p units to the left.
- If $p < 0$, the graph is shifted p units to the right.

This horizontal shift means a shift of the vertical asymptote of the graph. The vertical asymptote is the line $x = -p$.

Take note!

In the function $y = \frac{a}{(x+p)} + q$:

- the horizontal asymptote is the line $y = q$
- the vertical asymptote is the line $x = -p$.

Domain and range of the hyperbolic function of the form $y = \frac{a}{(x+p)} + q$

We can see that in $y = \frac{a}{(x+p)} + q$, $(x+p) \neq 0$ and therefore $x \neq -p$. Therefore, the domain is $\{x|x \in \mathbb{R}, x \neq -p\}$.

If $x \neq -p$, then:

$$y = \frac{a}{(x+p)} + q$$
$$\therefore y - q = \frac{a}{(x+p)}$$
$$\therefore (y-q)(x+p) = a$$
$$\therefore (x+p) = \frac{a}{(y-q)}$$
$$\therefore y - q \neq 0$$
$$\therefore y \neq q$$

So, the range is $\{y|y \in \mathbb{R}, y \neq q\}$



Take note!

For $y = \frac{a}{(x+p)} + q$:

- the domain is $\{x|x \in \mathbb{R}, x \neq -p\}$
- the range is $\{y|y \in \mathbb{R}, y \neq q\}$.

Note that the restrictions on the domain and range determine the vertical and horizontal asymptotes.



Example 3.1

Determine the domain, range and asymptotes of $s(x) = \frac{4}{(x-2)} + \frac{1}{2}$.

Solution

$x - 2 \neq 0$
$\therefore x \neq 2$
Domain is $\{x|x \in \mathbb{R}, x \neq 2\}$
Vertical asymptote is the line $x = 2$.
Let $s(x)$ be y:

$$y = \frac{4}{(x-2)} + \frac{1}{2}$$

$$\therefore y - \frac{1}{2} = \frac{4}{(x-2)}$$

$$\therefore x - 2 = \frac{4}{\left(y - \frac{1}{2}\right)}$$

$$\therefore y - \frac{1}{2} \neq 0$$

$$\therefore y \neq \frac{1}{2}$$

Range is $\{y | y \in \mathbb{R}, y \neq \frac{1}{2}\}$

Horizontal asymptote is the line $y = \frac{1}{2}$.

Take note!

The vertical asymptote of a hyperbola of the form $y = \frac{a}{x+p} + q$ is the line $x = -p$.

The horizontal asymptote of a hyperbola of the form $y = \frac{a}{x+p} + q$ is the line $y = q$.

Exercise 3.1

Determine the domain, range and asymptotes of each of the following functions:

1. $y = -\frac{3}{x-3} - 1$
2. $x = \frac{7}{y+4} + 1$
3. $(x+2)(y-3) = \frac{3}{2}$

The full solutions are at the end of the unit.

Intercepts

When the values of p and q in $y = \frac{a}{(x+p)} + q$ are zero, the graph has vertical and horizontal asymptotes that are the y-axis and x-axis respectively. In this case, the graph never crosses or even touches either of the axes and so there are no intercepts with the axes. However, when p and/or q are not equal to zero and the asymptotes shift, then there are intercepts with the axes.

To calculate these intercepts, we use the same techniques we have always used:

- to calculate the y-intercept, we let $x = 0$
- to calculate the x-intercept, we let $y = 0$.

Example 3.2

Determine the intercepts with the axes of $j(x) = -\frac{3}{x+4} - 3$.

Solution

Step 1: Find the x-intercepts
x-intercepts (let $y = 0$):

$$0 = -\frac{3}{x+4} - 3$$
$$\therefore \frac{3}{x+4} = -3$$
$$\therefore 3 = -3(x+4)$$
$$\therefore 3 = -3x - 12$$
$$\therefore 3x = -15$$
$$\therefore x = -5$$

The x-intercept is the point $(-5, 0)$.

Step 2: Find the y-intercepts
y-intercepts (let $x = 0$):

$$j(0) = -\frac{3}{0+4} - 3$$
$$\therefore j(0) = -\frac{3}{4} - 3$$
$$= -\frac{3}{4} - \frac{12}{4}$$
$$= -\frac{15}{4}$$

The y-intercept is the point $(0, -\frac{15}{4})$.

Exercise 3.2

Determine the intercepts with the axes of the following functions:

1. $f(x) = -\frac{2}{\left(x + \frac{1}{2}\right)} - 3$
2. $y - 7 = \frac{3}{x-1}$

3. $g(x) + \frac{1}{3} = \frac{4}{(x-2)}$

The full solutions are at the end of the unit.

Axes of symmetry

We know that the hyperbola has two axes of symmetry. One has a gradient of 1 and the other a gradient of -1. Therefore, they intersect each other at right angles. But where do they intersect each other?

Look at Figures 4, 5 and 6 which show the functions $f(x) = \frac{2}{x}$, $g(x) = \frac{2}{(x+2)} + 1$ and $h(x) = \frac{2}{(x-3)} - 1$. In each case, at what point do the axes of symmetry of the function intersect? How does this point relate to the values of p and q in each case?

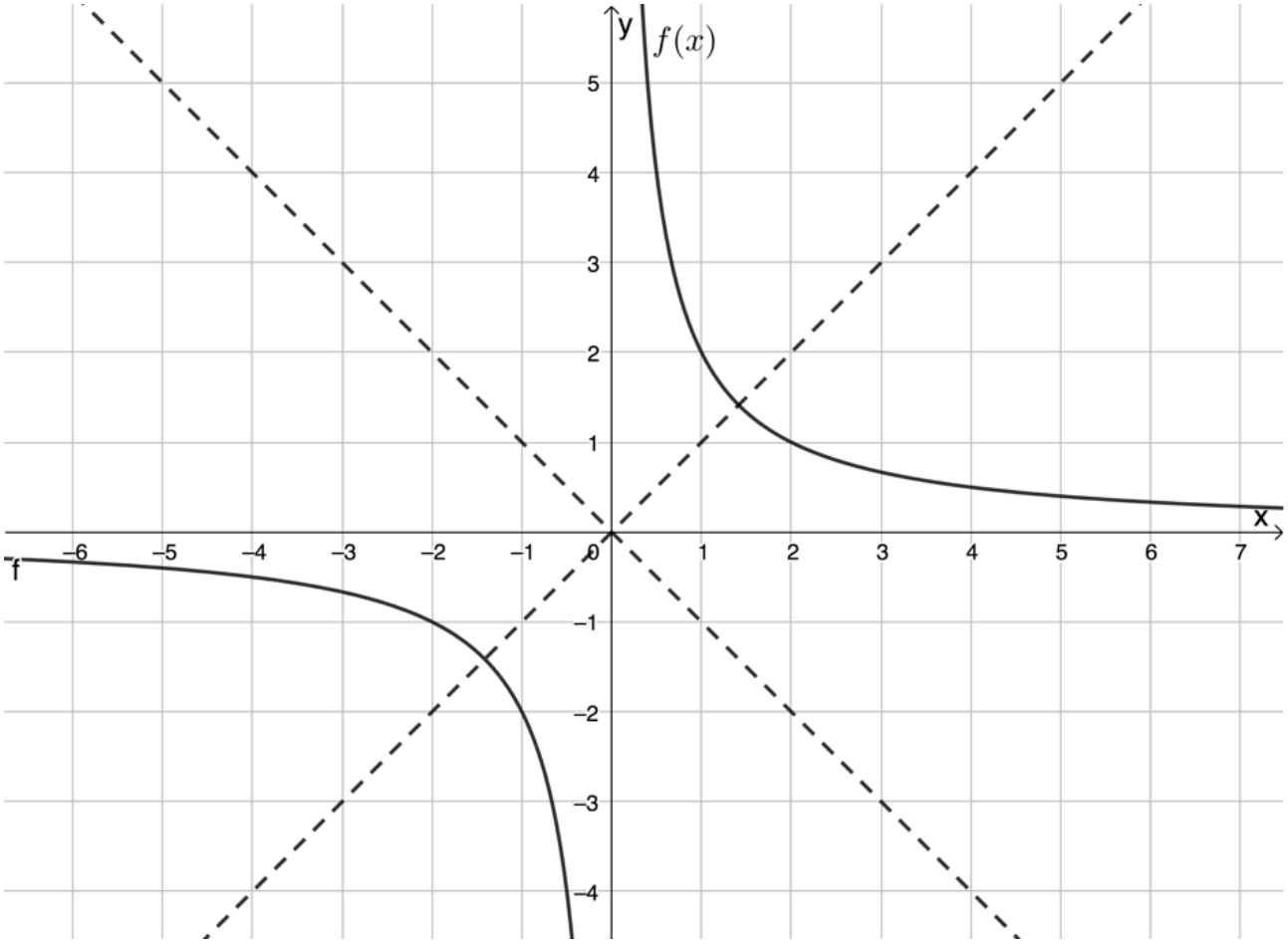


Figure 4: Graph of $f(x) = \frac{2}{x}$

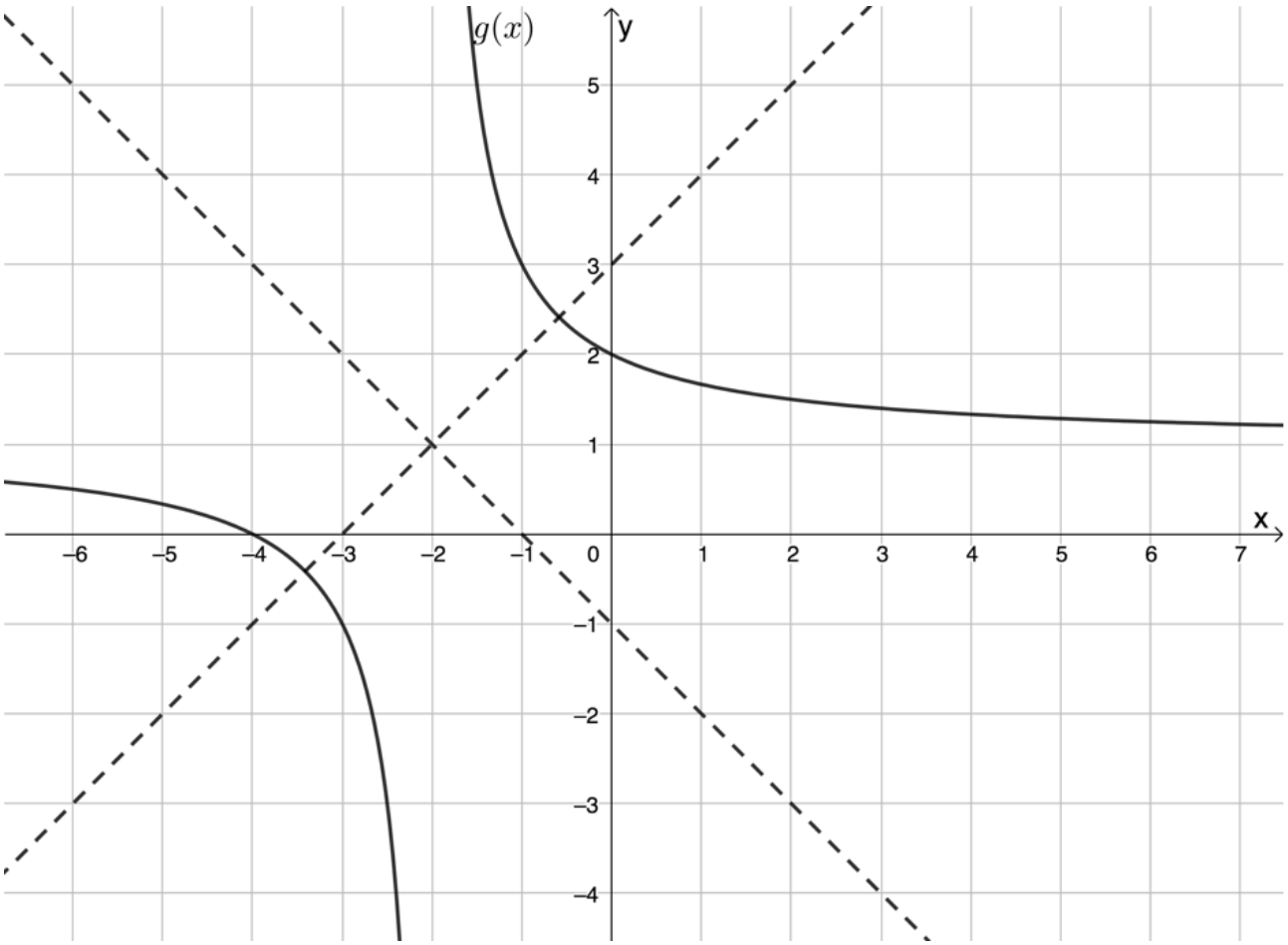


Figure 5: Graphs of $g(x) = \dfrac{2}{(x+2)} + 1$

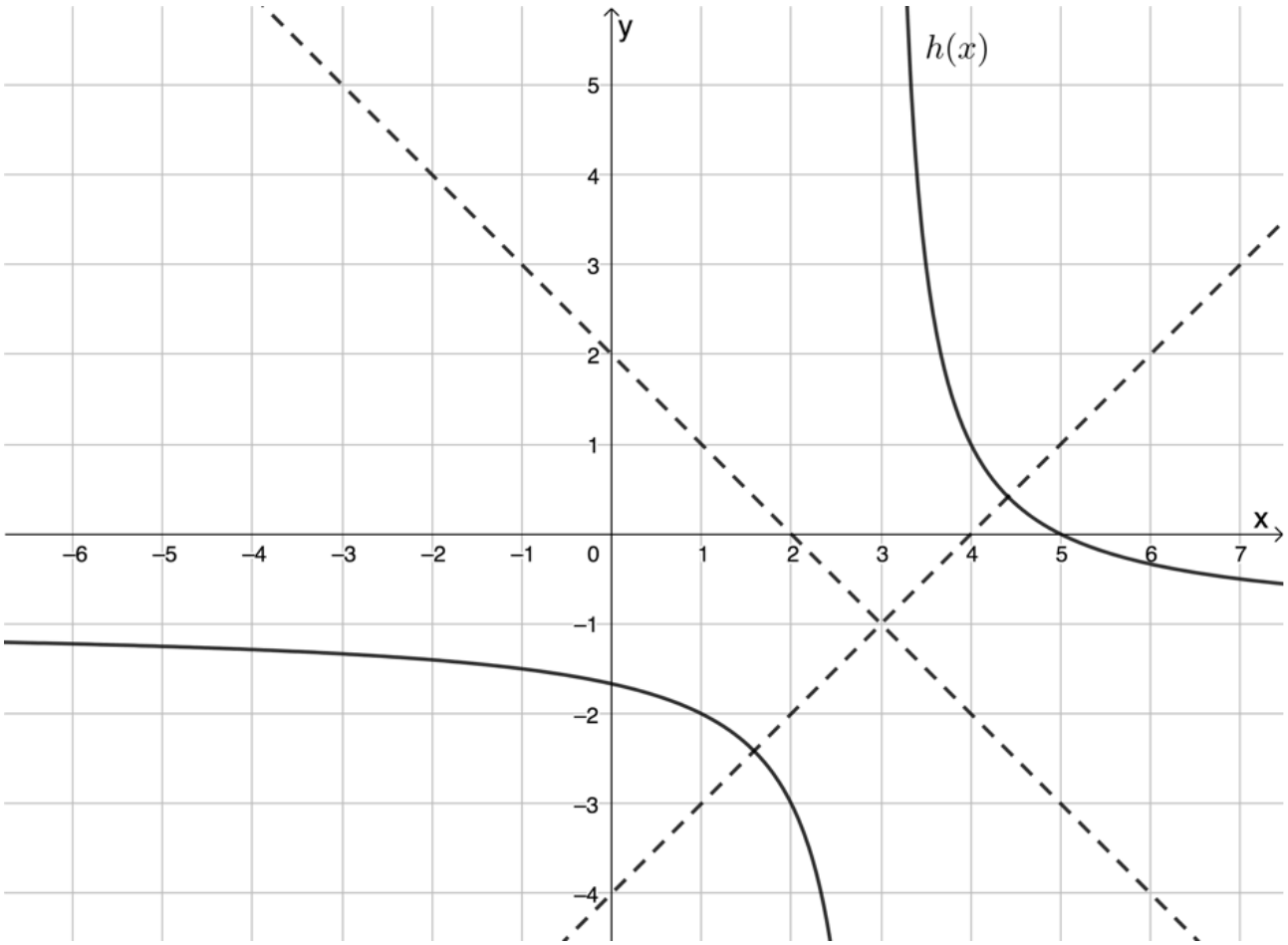


Figure 6: Graphs of $h(x) = \dfrac{2}{(x-3)} - 1$

In each case, we can see that the axes of symmetry intersect at the point $(-p, q)$. This is, in fact, the same point where the two asymptotes intersect and so the point of intersection can also be thought of in terms of the values of the asymptotes.

Now, if the axes of symmetry have gradients of 1 and -1, we know that their equations will be $y = x + c_1$ and $y = -x + c_2$ respectively. Therefore, all we need to do to find the values of c_1 and c_2 is substitute the point $(-p, q)$ into each equation.



Example 3.3

Determine the equations of the axes of symmetry of $f(x) = -\dfrac{3}{x-3} + 2$.

Solution

Step 1: Determine where the axes of symmetry intersect
The point where the axes of symmetry intersect is $(-(-3), 2)$ or $(3, 2)$.

Step 2: Substitute this point into the equations of the two axes of symmetry
Substitute this point into $y = x + c_1$:

$2 = 3 + c_1$
$\therefore c_1 = -1$
Substitute this point into $y = -x + c_2$:
$2 = -3 + c_2$
$\therefore c_2 = 5$

Step 3: Write the final solution
The equations of the axes of symmetry are $y = x - 1$ and $y = -x + 5$

Exercise 3.3

Determine the equations of the axes of symmetry of the following functions:

1. $b(x) = \frac{4}{x-1} + 2$
2. $(y-2)(x+3) = 2$

The full solutions are at the end of the unit.

Take note!

The axes of symmetry of a hyperbola of form $y = \frac{a}{x+p} + q$ are always straight lines with equations $y = x + c_1$ and $y = -x + c_2$.

The axes of symmetry of a hyperbola of form $y = \frac{a}{x+p} + q$ intersect at the point $(-p, q)$.

Sketch the hyperbolic function $y = \frac{a}{(x+p)} + q$

Now that we are able to determine all the characteristics of the hyperbolic function of the form $y = \frac{a}{(x+p)} + q$, we can sketch its graph.

Example 3.4

Draw a sketch of $f(x) = \frac{6}{x+2} - 1$.

Solution

First make sure that the function is in the standard form, i.e. in the form $y = \frac{a}{(x+p)} + q$.

To draw an accurate sketch of a hyperbola we need four key pieces of information:

- whether $a > 0$ or $a < 0$ to determine if the graph lies in the first and third or second and fourth quadrants
- the asymptotes
- the axes of symmetry
- the intercepts with the axes (if these exist).

Step 1: Determine the quadrants

$a > 0$. Therefore the graph will lie in the first and third quadrants.

Step 2: Find the asymptotes

$$f(x) = \frac{6}{x+2} - 1$$

Therefore the vertical asymptote is the line $x = -2$ and the horizontal asymptote is the line $y = -1$.

Step 3: Find the axes of symmetry

The axes of symmetry intersect at the point $(-2, -1)$.

Substitute $(-2, -1)$ into $y = x + c_1$:
$-1 = -2 + c_1$
$\therefore c_1 = 1$

The A/S is $y = x + 1$

Substitute $(-2, -1)$ into $y = -x + c_2$:
$-1 = 2 + c_2$
$\therefore c_2 = -3$

The A/S is $y = -x - 3$

Step 4: Find the intercepts with the axes (if these exist)

As neither of the asymptotes is an axis, both an x- and a y-intercept exist.

x-intercept (let $y = 0$):

$$0 = \frac{6}{x+2} - 1$$
$$\therefore \frac{6}{x+2} = 1$$
$$\therefore x + 2 = 6$$
$$\therefore x = 4$$

x-intercept is the point $(4, 0)$

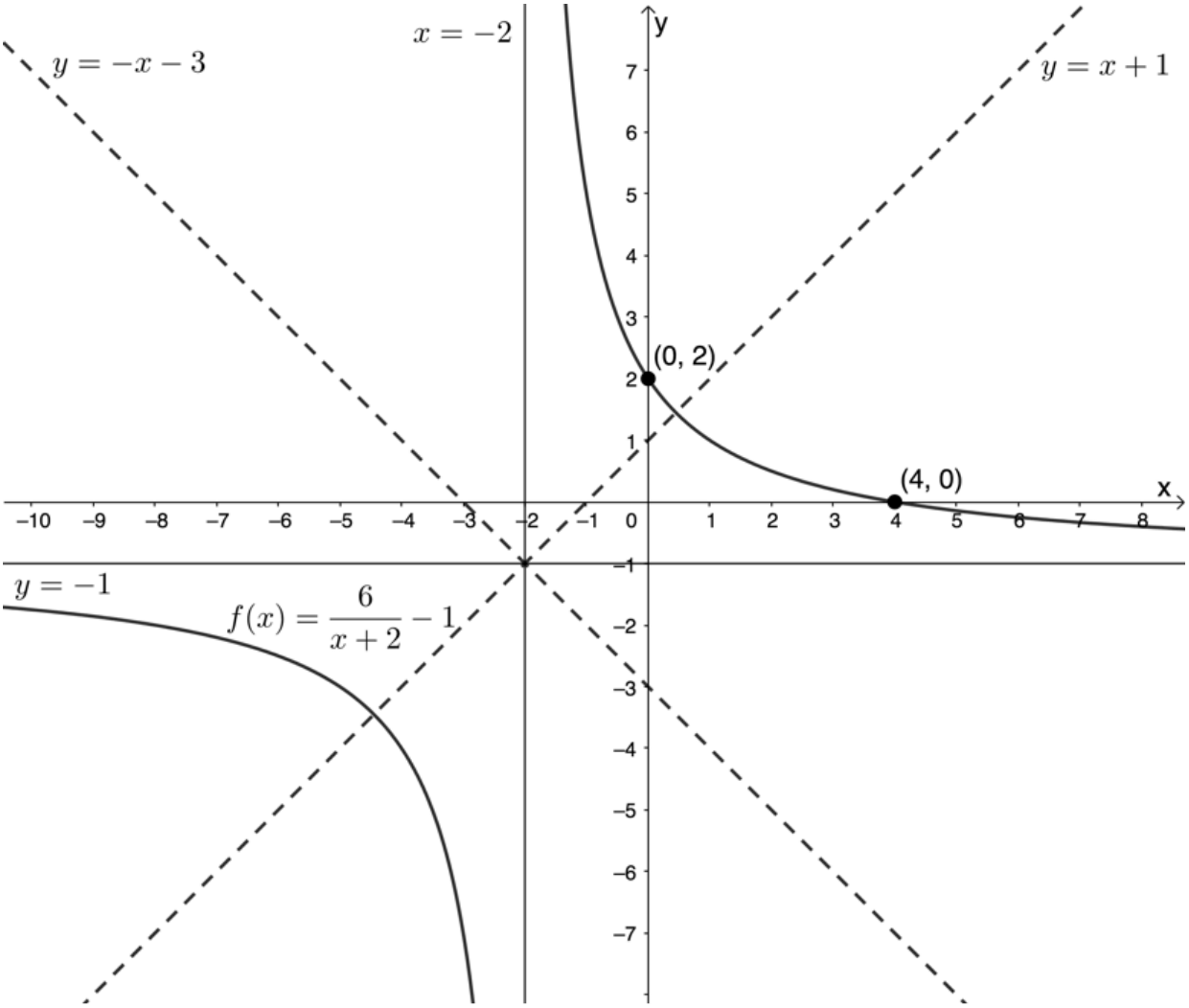
y-intercept (let $x = 0$):

$$f(0) = \frac{6}{0+2} - 1$$
$$\therefore f(0) = 2$$

y-intercept is the point $(0, 2)$

Note: If intercepts with the axes do not exist (i.e. the values of p and/or q are zero), find two other points on the graph to help you get the right shape.

Step 5: Make a neat sketch
Draw in and label the asymptotes and axes of symmetry. Mark the intercepts with the axes (if these exist) or the other points on the graph you found. Noting the value of a and the quadrants in which the graph exists, draw a smooth hyperbola passing through the points, symmetrical about the axes of symmetry and approaching the asymptotes.



Note

If you have an internet connection visit this online hyperbola simulation.

Here you will find a hyperbolic function of the form $y = \frac{a}{(x+p)} + q$ with sliders to change the values of a, p and q. This helps you see the effects of these on the shape and the position of the graph as well as the values of the asymptotes, axes of symmetry and intercepts with the axes.

Summary

In this unit you have learnt the following:

- The value of p in $y = \frac{a}{(x+p)} + q$ shifts the graph horizontally and also changes the value of the vertical asymptote.
- How to determine the asymptotes, domain and range of hyperbolic functions of the form $y = \frac{a}{(x+p)} + q$.
- How to determine the equations of the axes of symmetry of a hyperbolic function of the form $y = \frac{a}{(x+p)} + q$.
- How to determine the intercepts with the axes of a hyperbolic function of the form $y = \frac{a}{(x+p)} + q$.
- How to use the above information to make an accurate sketch of a hyperbolic function of the form $y = \frac{a}{(x+p)} + q$.

Unit 3: Assessment

Suggested time to complete: 30 minutes

1. Given $g(x) = \frac{6}{x-2} + 1$
   a. Determine the asymptote(s) of $g(x)$.
   b. For which values of x is the function $g(x)$ undefined?
   c. Draw the graph of $g(x)$.
   d. Determine $g(1)$.
   e. What is the domain and range of $g(x)$?
2. Given $f(x) - 2 = \frac{2}{x-4}$
   a. Determine the asymptote(s) of $f(x)$.
   b. Determine the x- and y-intercepts of $f(x)$.
   c. Draw the graph of $f(x)$.
   d. Determine $f(-1)$.
   e. What are the domain and range of $f(x)$?

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1. $y = -\dfrac{3}{x-3} - 1$
   Domain: $\{x | x \in \mathbb{R}, x \neq 3\}$
   Vertical asymptote is the line $x = 3$
   Range: $\{y | y \in \mathbb{R}, x \neq -1\}$
   Horizontal asymptote is the line $y = -1$
2. $$x = \frac{7}{y+4} + 1$$
   $$\therefore x - 1 = \frac{7}{y+4}$$
   $$\therefore y + 4 = \frac{7}{x-1}$$
   $$\therefore y = \frac{7}{x-1} - 4$$
   Domain: $\{x | x \in \mathbb{R}, x \neq 1\}$
   Vertical asymptote is the line $x = 1$
   Range: $\{y | y \in \mathbb{R}, x \neq -4\}$
   Horizontal asymptote is the line $y = -4$
3. $$(x+2)(y-3) = \frac{3}{2}$$
   $$\therefore y - 3 = \frac{\frac{3}{2}}{(x+2)}$$
   $$\therefore y = \frac{\frac{3}{2}}{(x+2)} + 3$$
   Domain: $\{x | x \in \mathbb{R}, x \neq -2\}$
   Vertical asymptote is the line $x = -2$
   Range: $\{y | y \in \mathbb{R}, x \neq 3\}$
   Horizontal asymptote is the line $y = 3$

Back to Exercise 3.1

Exercise 3.2

1. x-intercept (let $y = 0$):

$$0 = -\frac{2}{\left(x + \frac{1}{2}\right)} - 3$$

$$\therefore \frac{2}{\left(x + \frac{1}{2}\right)} = -3$$

$$\therefore x + \frac{1}{2} = -\frac{2}{3}$$

$$\therefore x = -\frac{2}{3} - \frac{1}{2}$$

$$= \frac{-4 - 3}{6}$$

$$= -\frac{7}{6}$$

x-intercept is the point $(-\frac{7}{6}, 0)$

y-intercept (let $x = 0$):

$$f(0) = -\frac{2}{\left(0 + \frac{1}{2}\right)} - 3$$

$$\therefore f(0) = -\frac{2}{\frac{1}{2}} - 3$$

$$= -4 - 3$$

$$= -7$$

y-intercept is the point $(0, -7)$

2. x-intercept (let $y = 0$):

$$0 - 7 = \frac{3}{x - 1}$$

$$\therefore x - 1 = -\frac{3}{7}$$

$$\therefore x = -\frac{3}{7} + 1$$

$$= \frac{-3 + 7}{7}$$

$$= \frac{4}{7}$$

x-intercept is the point $(\frac{4}{7}, 0)$

y-intercept (let $x = 0$):

$$y - 7 = \frac{3}{0 - 1}$$

$$\therefore y = -3 + 7$$

$$= 4$$

y-intercept is the point $(0, 4)$

3. x-intercept (let $y = 0$s):

$$0 + \frac{1}{3} = \frac{4}{(x - 2)}$$

$$\therefore x - 2 = \frac{4}{\frac{1}{3}}$$

$$= 12$$

$$\therefore x = 14$$

x-intercept is the point $(14, 0)$

y-intercept (let $x = 0$):

$$g(0) + \frac{1}{3} = \frac{4}{(0-2)}$$
$$\therefore g(0) = -2 - \frac{1}{3}$$
$$= -2\frac{1}{3}$$

y-intercept is the point $(0, -2\frac{1}{3})$

Back to Exercise 3.2

Exercise 3.3

1.

$b(x) = \frac{4}{x-1} + 2$

Axes of symmetry intersect at $(1, 2)$.

Substitute $(1, 2)$ into $y = x + c_1$:

$2 = 1 + c_1$

$\therefore c_1 = 1$

The A/S is $y = x + 1$

Substitute $(1, 2)$ into $y = -x + c_2$:

$2 = -1 + c_2$

$\therefore c_2 = 3$

The A/S is $y = -x + 3$

2.

$(y-2)(x+3) = 2$

$$\therefore y - 2 = \frac{2}{(x+3)}$$
$$\therefore y = \frac{2}{(x+3)} + 2$$

Axes of symmetry intersect at $(-3, 2)$.

Substitute $(-3, 2)$ into $y = x + c_1$:

$2 = -3 + c_1$

$\therefore c_1 = 5$

The A/S is $y = x + 5$

Substitute $(-3, 2)$ into $y = -x + c_2$:

$2 = -(-3) + c_2$

$\therefore c_2 = -1$

The A/S is $y = -x - 1$

Back to Exercise 3.3

Unit 3: Assessment

1.

a. $g(x) = \frac{6}{x-2} + 1$

Vertical asymptote is the line $x = 2$.

Horizontal asymptote is the line $y = 1$.

b. The line $x = 2$ is an asymptote. Therefore, the function is undefined at $x = 2$.

c. We know the asymptotes but to make an accurate sketch, we also need to know the axes of symmetry and the intercepts with the axes.

The axes of symmetry intersect at $(2, 1)$.
Substitute $(2, 1)$ into $y = x + c_1$:
$1 = 2 + c_1$
$\therefore c_1 = -1$
Substitute $(2, 1)$ into $y = -x + c_2$:
$1 = -2 + c_2$
$\therefore c_2 = 3$
The axes of symmetry are $y = x - 1$ and $y = -x + 3$.
x-intercept (let $y = 0$):

$$0 = \frac{6}{x - 2} + 1$$

$$\therefore \frac{6}{x - 2} = -1$$

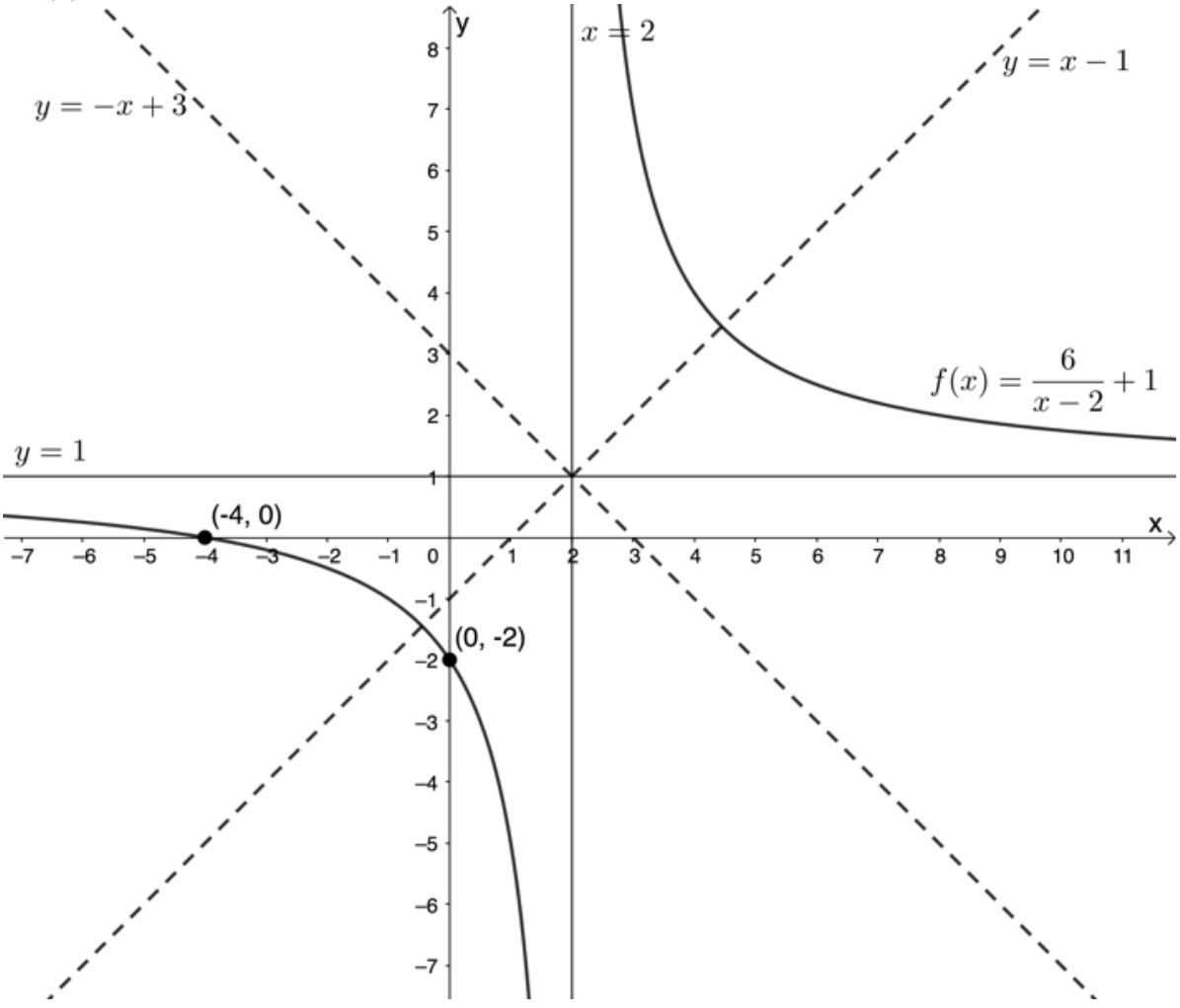
$$\therefore x - 2 = -6$$

$$\therefore x = -4$$

y-intercept (let $x = 0$):

$$g(x) = \frac{6}{0 - 2} + 1$$

$$\therefore g(x) = -2$$



d. $g(1) = \frac{6}{1 - 2} + 1 = -5$

e. Domain: $\{x | x \in \mathbb{R}, x \neq 2\}$
Range: $\{g(x) | g(x) \in \mathbb{R}, g(x) \neq 1\}$

2.

a.

$$f(x) - 2 = \frac{2}{x-4}$$

$$\therefore f(x) = \frac{2}{x-4} + 2$$

Vertical asymptote is the line $x = 4$.

Horizontal asymptote is the line $y = 2$.

b. x-intercept (let $y = 0$):

$$0 = \frac{2}{x-4} + 2$$

$$\therefore \frac{2}{x-4} = -2$$

$$\therefore x - 4 = -1$$

$$\therefore x = 3$$

y-intercept (let $x = 0$):

$$f(0) = \frac{2}{0-4} + 2$$

$$\therefore f(0) = \frac{3}{2}$$

c. To draw an accurate sketch, we need to find the axes of symmetry.

The axes of symmetry intersect at $(4, 2)$.

Substitute $(4, 2)$ into $y = x + c_1$:

$$2 = 4 + c_1$$

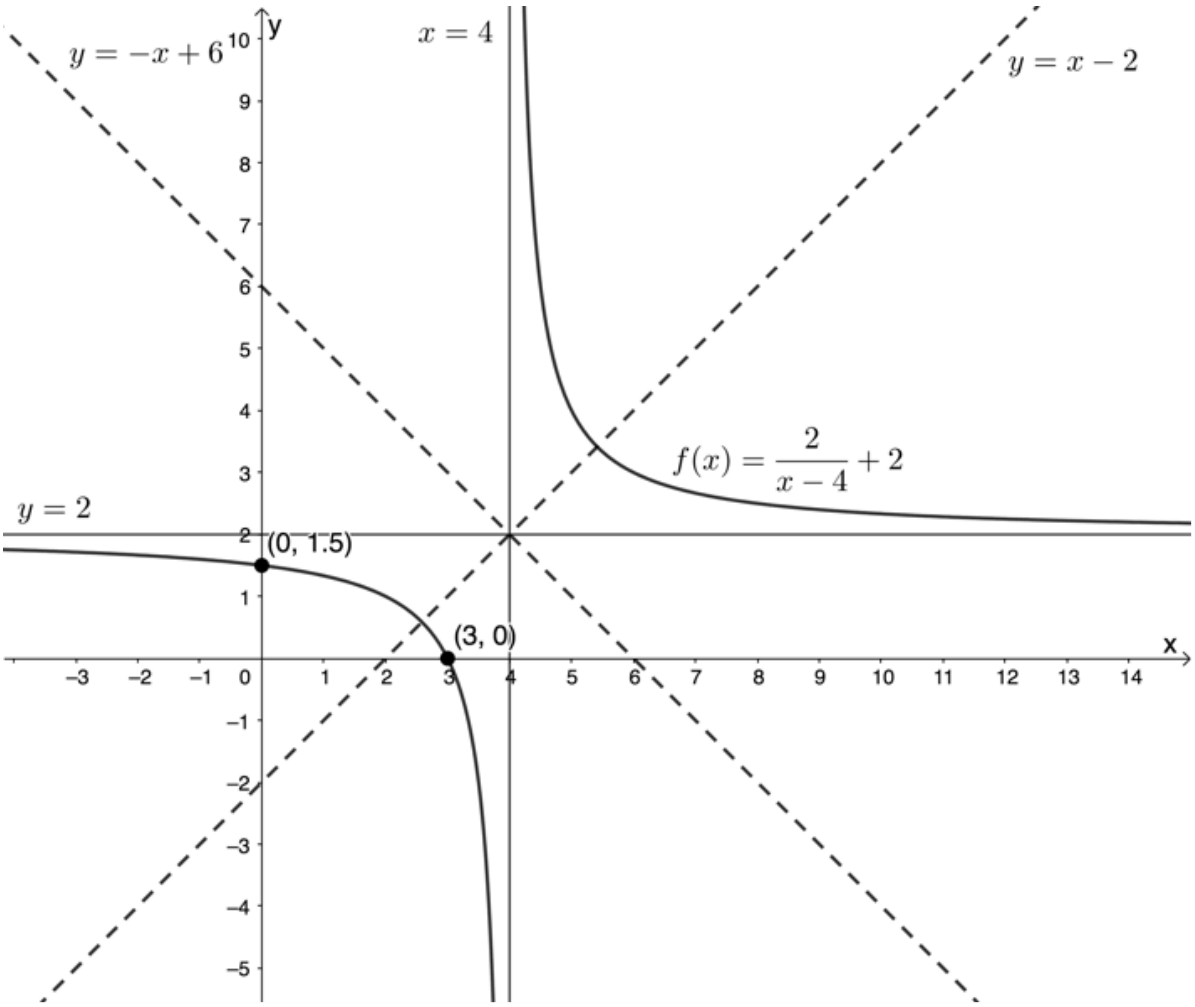
$$\therefore c_1 = -2$$

Substitute $(4, 2)$ into $y = -x + c_2$:

$$2 = -4 + c_2$$

$$\therefore c_2 = 6$$

The axes of symmetry are $y = x - 2$ and $y = -x + 6$.



d. $f(-1) = \frac{2}{(-1)-4} + 2 = \frac{8}{5}$

e. Domain: $\{x | x \in \mathbb{R}, x \neq 4\}$
Range: $\{f(x) | f(x) \in \mathbb{R}, f(x) \neq 2\}$

Back to Unit 3: Assessment

Media Attributions

- figure18 © DHET is licensed under a CC BY (Attribution) license
- figure4 © DHET is licensed under a CC BY (Attribution) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure6 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example3.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 4: Find the equation of hyperbolic functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the equation of a hyperbola in the form $y = \frac{a}{x+p} + q$.

What you should know

Before you start this unit, make sure you can:

- Solve linear simultaneous equations. Refer to level 2 subject outcome 2.3 unit 4 if you need assistance.
- Determine the domain and range of hyperbolic functions of the form $y = \frac{a}{x+p} + q$. Refer to unit 3 in this subject outcome if you need assistance.
- Find the asymptotes of hyperbolic functions of the form $y = \frac{a}{x+p} + q$. Refer to unit 3 in this subject outcome if you need assistance.
- Find the axes of symmetry of hyperbolic functions of the form $y = \frac{a}{x+p} + q$. Refer to unit 3 in this subject outcome if you need assistance.
- Find the intercepts with the axes of hyperbolic functions of the form $y = \frac{a}{x+p} + q$. Refer to unit 3 in this subject outcome if you need assistance.

Introduction

In the previous unit we learnt how to determine various characteristics of hyperbolic functions of the form $y = \frac{a}{x+p} + q$ in order to sketch their graphs, called hyperbolae.

These characteristics of a hyperbola include:

- the domain and range
- the asymptotes
- the axes of symmetry
- and the intercepts with the axes.

In this unit, we will see how we can use these characteristics from a hyperbola to determine the equation of the hyperbolic function. The best way to learn how to do this, is to work through a number of different examples.

Finding the equation of a hyperbola

To find the equation of a hyperbolic function of the form $y = \frac{a}{x+p} + q$ from its graph, we need to find the values of a, p and q.

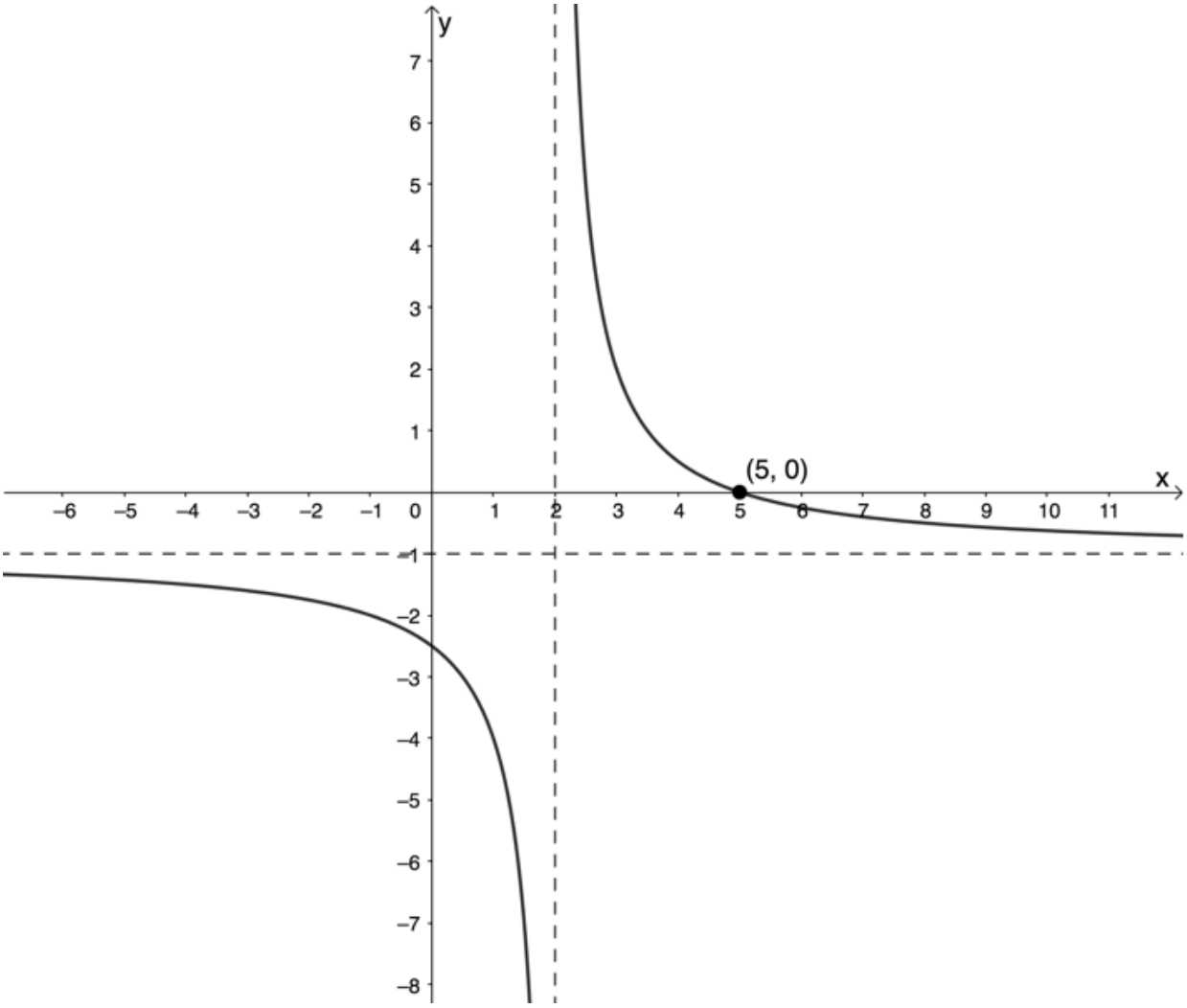
We can always determine whether $a > 0$ or $a < 0$ by looking at which quadrants the graph lies in. We can determine the values of p and q by looking at the:

- asymptotes
- point where the axes of symmetry intersect.

Because we have three unknowns to find, we require three pieces of information.

Example 4.1

The graph below is a hyperbola of the form $y = \frac{a}{x+p} + q$. Determine the equation of the function.



Solution

We have been given three pieces of information. We know the asymptotes and we have a point on the graph which happens to be the x-intercept.

We know that the asymptotes of a hyperbola are given by the lines $x = -p$ and $y = q$. The asymptotes are the lines $x = 2$ and $y = -1$. Therefore, $p = -2$ and $q = -1$.

The equation of the hyperbola is $y = \frac{a}{x-2} - 1$. We can substitute the point $(5, 0)$ into this equation to find the value of a.

$$0 = \frac{a}{5-2} - 1$$
$$\therefore \frac{a}{3} = 1$$
$$\therefore a = 3$$

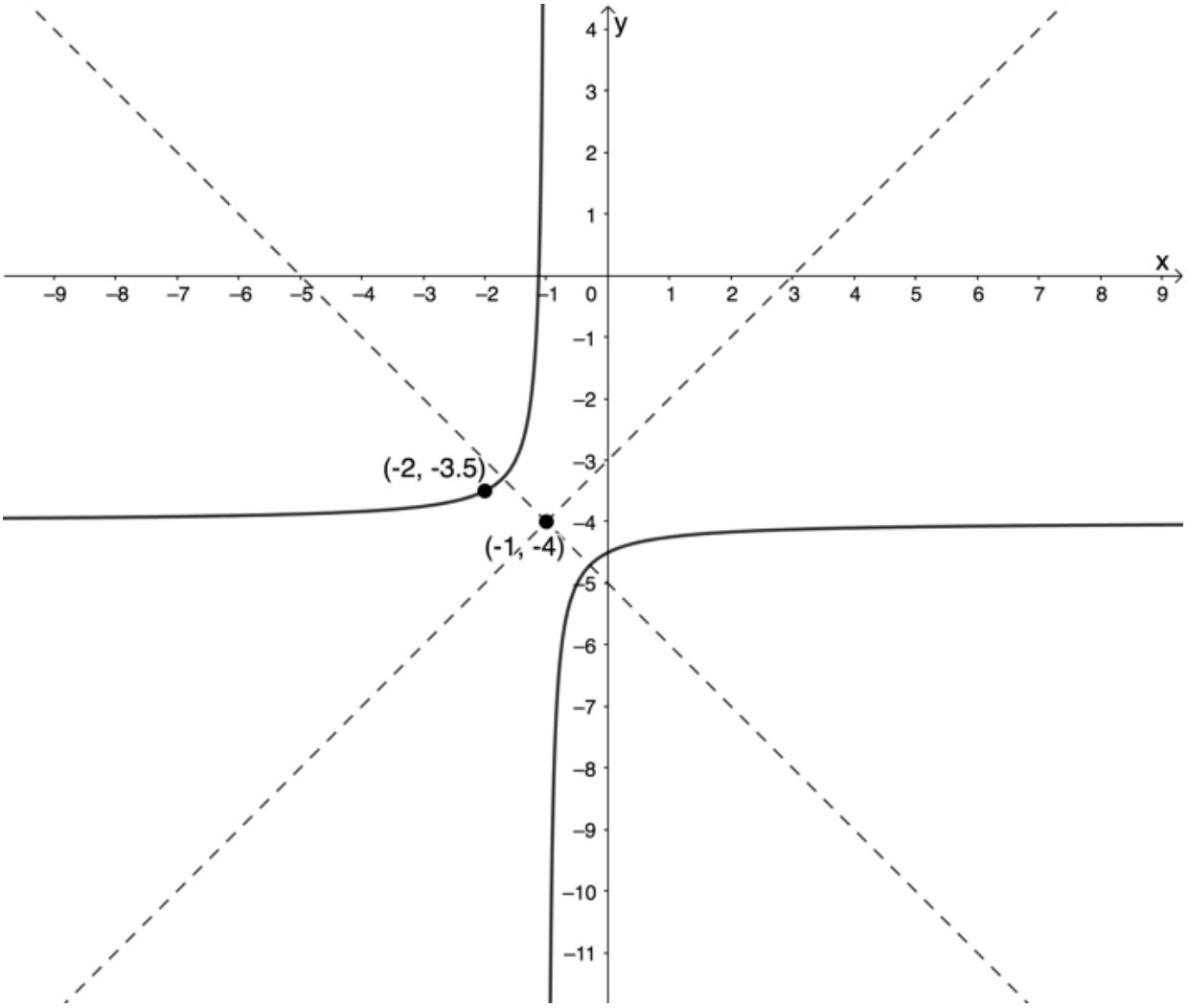
We know that $a > 0$ because the graph lies in the first and third quadrants.

The equation of the function is $y = \frac{3}{x-2} - 1$.



Example 4.2

The graph below is a hyperbola of the form $y = \frac{a}{x+p} + q$. Determine the equation of the function if the axes of symmetry intersect at the point $(-1, -4)$ and the point $(-2, -3.5)$ lies on the graph.



Solution

Again, we have been given three pieces of information. We know the equations of the axes of symmetry (not directly but we could easily work them out using the point of intersection) and we have a point on the graph.

We know that the point where the axes of symmetry meet is the point $(-p, q)$. Therefore, $p = 1$ and $q = -4$.

The equation of the hyperbola is $y = \frac{a}{x+1} - 4$. We can substitute the point $(-2, -3.5)$ into the equation to find the value of a.

$$-\frac{7}{2} = \frac{a}{(-2)+1} - 4$$
$$\therefore a = -4 + \frac{7}{2}$$
$$= -\frac{1}{2}$$

We know that $a < 0$ because the graph lies in the second and fourth quadrants.

The equation of the function is $y = -\frac{\frac{1}{2}}{x+1} - 4$.

Example 4.3

A hyperbola of the form $y = \frac{a}{x+p} + q$ has axes of symmetry of $y = x - 1$ and $y = -x + 5$. It has a y-intercept of $(0, \frac{8}{3})$. Find the equation of the function.

Solution

We have been given three pieces of information – the equations of the axes of symmetry and a point that lies on the graph.

We cannot determine the values of p or q directly from the axes of symmetry. We need to determine their point of intersection by solving these two linear equations simultaneously.

$y = x - 1$ (1)
$y = -x + 5$ (2)

Substitute (2) into (1):
$-x + 5 = x - 1$
$\therefore 2x = 6$
$\therefore x = 3$

Substitute $x = 3$ into (2):
$y = -3 + 5$
$= 2$

The point of intersection is $(3, 2)$. The point of intersection of the axes of symmetry is the point $(-p, q)$. Therefore, $p = -3$ and $q = 2$.

The equation of the hyperbola is $y = \frac{a}{x-3} + 2$. We can substitute the point $(0, \frac{8}{3})$ into the equation to find the value of a.

$$\frac{8}{3} = \frac{a}{0-3} + 2$$
$$\therefore \frac{a}{3} = 2 - \frac{8}{3} = -\frac{2}{3}$$
$$\therefore a = -2$$

The equation of the function is $y = -\frac{2}{x-3} + 2$.

Exercise 4.1

1. The axes of symmetry of a hyperbola $f(x)$ intersect at the point $(-3, 2)$ and the graph has an x-intercept of $(-2, 0)$. Determine the equation of $f(x)$.
2. $g(x) = \frac{a}{x+p} + q$ has a domain of $\{x | x \in \mathbb{R}, x \neq -1\}$ and a range of $\{g(x) | g(x) \in \mathbb{R}, g(x) \neq -2\}$. If $g(-2) = -6$, determine the equation of $g(x)$.

The full solutions are at the end of the unit.

Summary

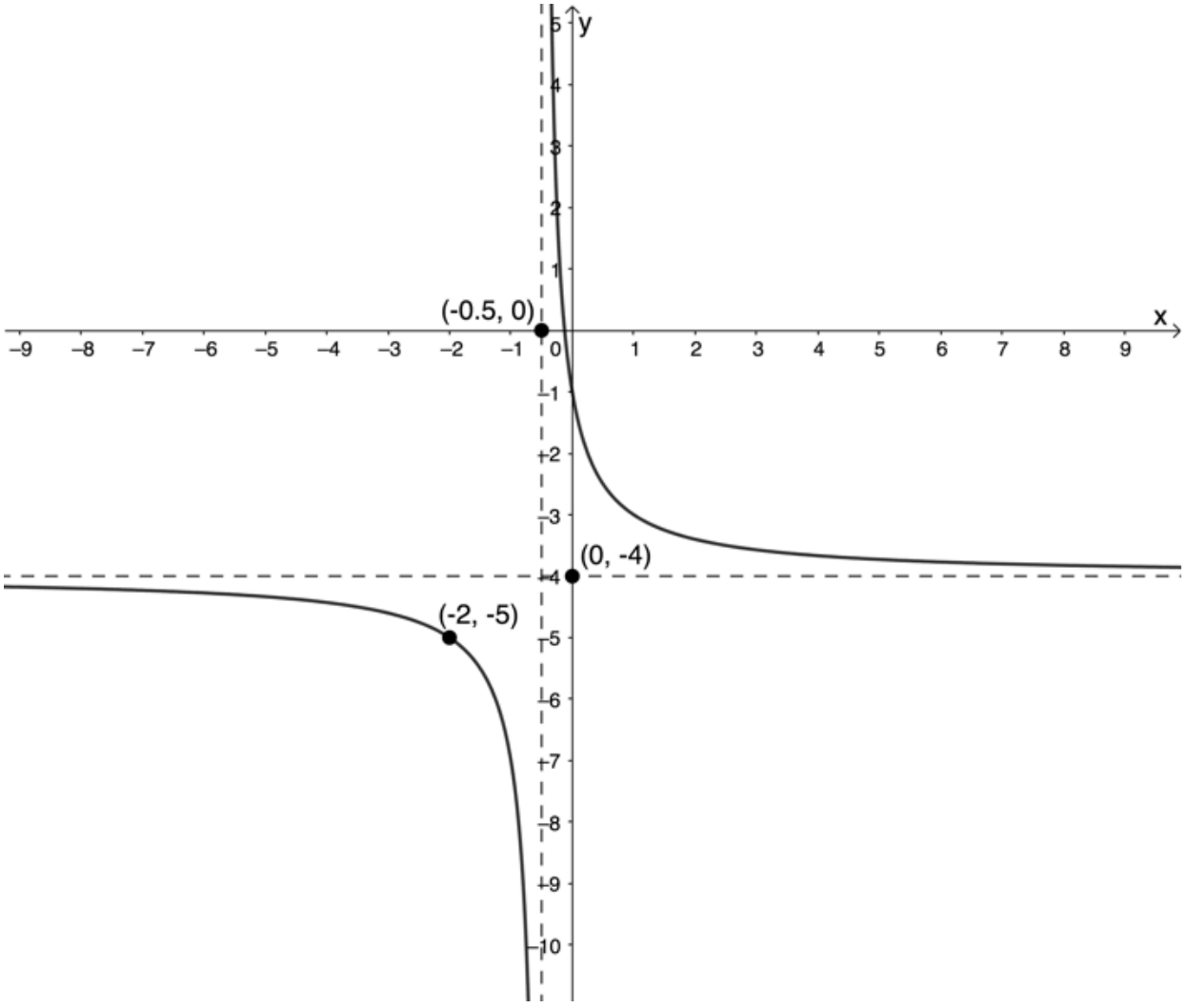
In this unit you have learnt the following:

- How to find the equation of a hyperbola of the form $y = \frac{a}{x+p} + q$.

Unit 4: Assessment

Suggested time to complete: 20 minutes

1. The graph shows the graph of a hyperbolic function of the form $y = \frac{a}{x+p} + q$. Find the equation of the function.



2. The axes of symmetry of a hyperbola $s(x) = \frac{a}{x+p} + q$ are $y = x + 2$ and $y = -x + 6$. If $s(0) = 6$, determine the equation of $s(x)$.

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. The axes of symmetry of $f(x)$ intersect at $(-3, 2)$. Therefore, $p = 3$ and $q = 2$.
   $f(x) = \dfrac{a}{x+3} + 2$. Substitute in $(-2, 0)$.
   $$0 = \frac{a}{(-2)+3} + 2$$
   $$\therefore a = -2$$
   $$f(x) = -\frac{2}{x+3} + 2$$
2. Domain of $g(x)$ $\{x|x \in \mathbb{R}, x \neq -1\}$. Therefore, the vertical asymptote is the line $x = -1$. Therefore, $p = 1$.
   Range of $g(x)$ is $\{g(x)|g(x) \in \mathbb{R}, g(x) \neq -2\}$. Therefore, the horizontal asymptote is $y = -2$. Therefore $q = -2$.
   $g(x) = \dfrac{a}{x+1} - 2$
   $g(-2) = -6$. Therefore, the point $(-2, -6)$ lies on the graph. Substitute this into $g(x)$.
   $$-6 = \frac{a}{(-2)+1} - 2$$
   $$\therefore a = 4$$
   $$g(x) = \frac{4}{x+1} - 2$$

Back to Exercise 4.1

Unit 4: Assessment

1. The vertical asymptote is given as $x = -\dfrac{1}{2}$. Therefore, $p = \dfrac{1}{2}$.
   The horizontal asymptote is given as $y = -4$. Therefore $q = -4$.
   $y = \dfrac{a}{x + \frac{1}{2}} - 4$. Substitute in the point $(-2, -5)$.
   $$-5 = \frac{a}{(-2) + \frac{1}{2}} - 4$$
   $$\therefore \frac{a}{\frac{3}{2}} = 1$$
   $$\therefore a = \frac{3}{2}$$
   $$y = \frac{\frac{3}{2}}{x + \frac{1}{2}} - 4$$
2. To determine the point $(-p, q)$, we need to find the point where the axes of symmetry intersect.
   $y = x + 2$ (1)
   $y = -x + 6$ (2)
   Substitute (2) into (1):
   $$-x + 6 = x + 2$$
   $$\therefore -2x = -4$$
   $$\therefore x = 2$$
   Substitute 2 into (2):

$y = -2 + 6$

$= 4$

$(-p, q) = (2, 4)$. Therefore, $p = -2$ and $q = 4$

$s(x) = \frac{a}{x - 2} + 4$

But $s(0) = 6$. Therefore, substitute in $(0, 6)$.

$6 = \frac{a}{0 - 2} + 4$

$\therefore \frac{a}{2} = -2$

$\therefore a = -4$

$s(x) = -\frac{4}{x - 2} + 4$

Back to Unit 4: Assessment

Media Attributions

- example4.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example4.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 5: Sketch exponential functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Determine the domain and range of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- Find the asymptote of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- Find the intercepts with the axes of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- Sketch exponential functions of the form $y = a.b^{x+p} + q, b > 0$.

What you should know

Before you start this unit, make sure you can:

- Solve exponential equations.
- Plot exponential functions of the form $y = a.b^x + q$.
- Determine the x- and y-intercepts of exponential functions of the form $y = a.b^x + q$.
- Determine the domain and range of exponential functions of the form $y = a.b^x + q$.
- Determine the asymptote of exponential functions of the form $y = a.b^x + q$.

Refer to level 2 subject outcome 2.1 unit 4 if you require assistance with any of these skills.

Introduction

In level 2 we were introduced to the exponential function of the form $y = a.b^x + q, b > 0$. We learnt that the term 'exponential growth' is sometimes used to refer to something that happens very quickly. We also learnt that in Mathematics 'exponential' has a very specific meaning and exponential functions are specific kinds of functions.

We saw that there are many processes in nature and finance where the relationship between the inputs and outputs is exponential. One example is how bacteria reproduce. Bacteria reproduce by simply splitting into two new bacteria. If one bacterium splits into two and then each of these splits into two and then each of these splits into two, etc. pretty soon we have a whole lot of bacteria!

We learnt that these functions behave quite similarly to the other functions we have explored so far. Therefore, much of what we already know about linear, quadratic and hyperbolic functions can be applied to exponential functions as well.

Take note!

The effect of a and b:

if $b > 1$, the function value increases to positive infinity (if $a > 1$) or negative infinity (if $a < 1$) as x **increases**

if $0 < b < 1$, the function value increases to positive infinity (if $a > 1$) or negative infinity (if $a < 1$) as x **decreases**.

The effect of q:

if $q > 0$, the function is shifted **up** by q units

if $q < 0$, the function is shifted **down** by q units

Horizontal asymptote is the line $y = q$.

If $a > 1$, the graph lies **above** the asymptote.

If $a < 1$, the graph lies **below** the asymptote.

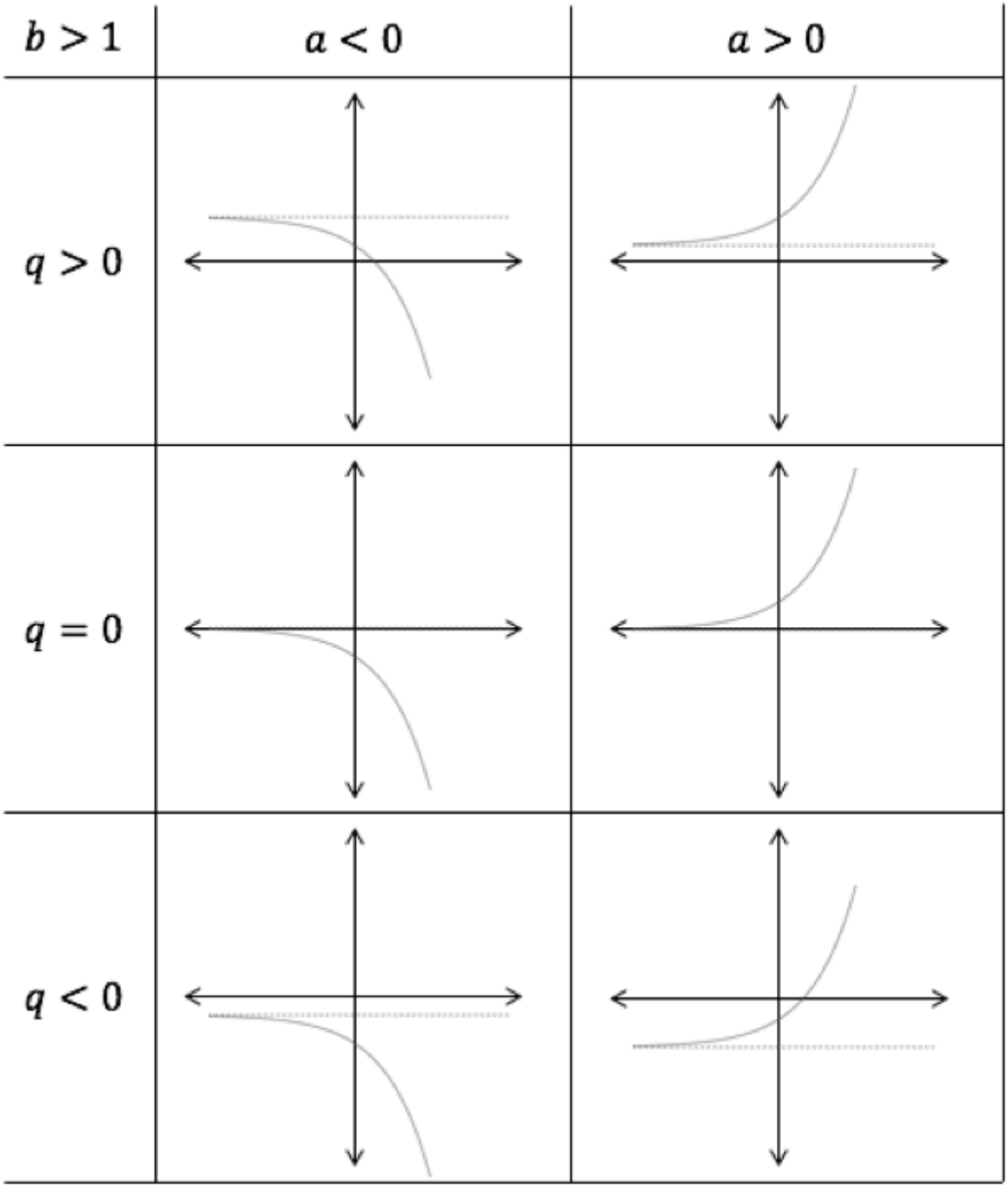


Figure 1: Summary of the characteristics of the function $y = a.b^x + q, b > 0$ *for* $b > 1$

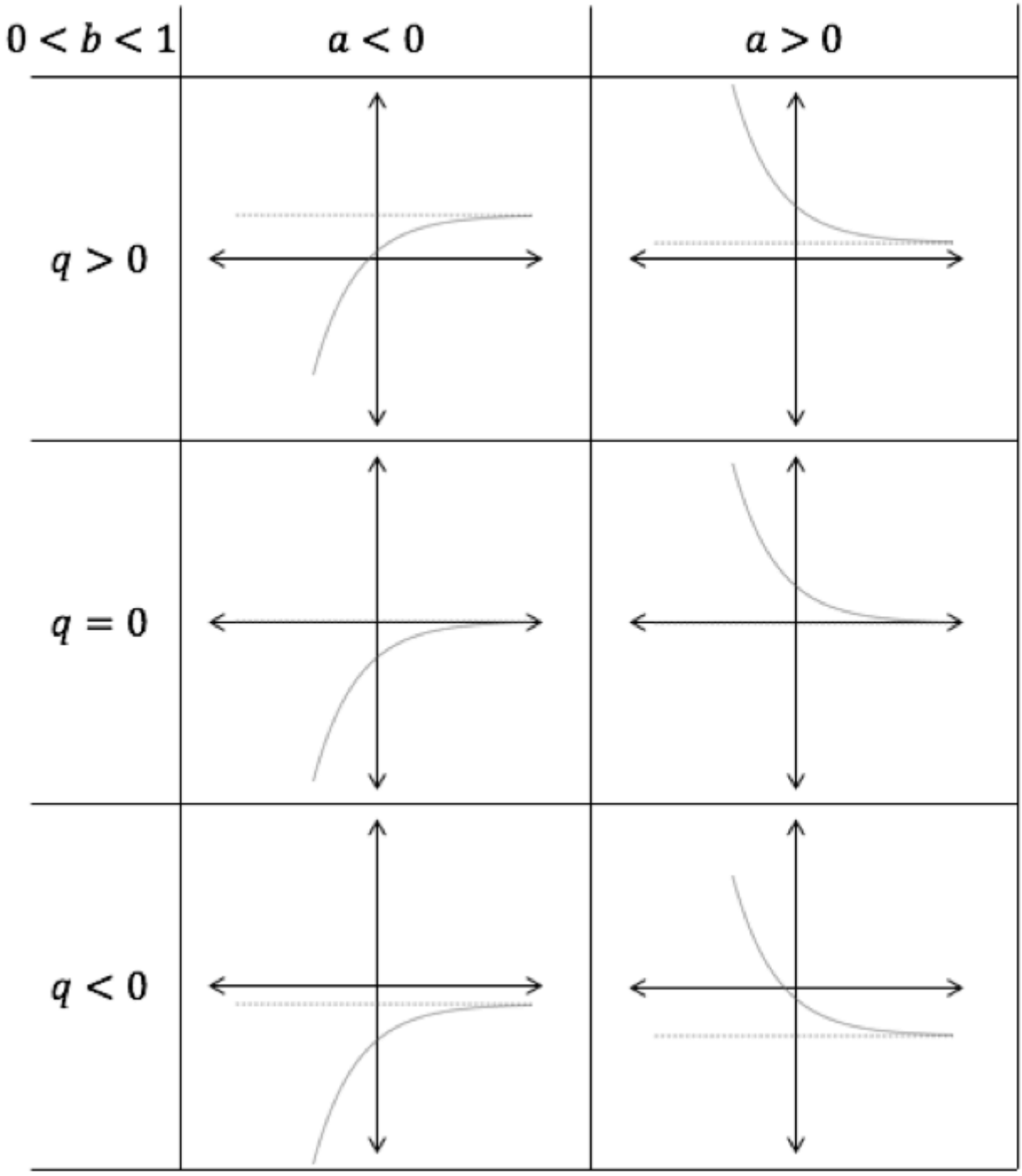


Figure 2: Summary of the characteristics of the function $y = a.b^x + q, b > 0$ *for* $0 < b < 1$

Note: It is important to remember that the value of b must be greater than zero. Negatives of b do not produce a continuous graph.

Note

If you have access to the internet, visit this interactive simulation.

Here you will find an exponential function of the form $y = a.b^x + q, b > 0$ with sliders to change the values of a, b and q. Spend some time playing with it to make sure that you understand how changing the values of a, b and q affect the shape and location of the exponential function of the form $y = a.b^x + q, b > 0$.

The exponential function of the form $y = a.b^{x+p} + q, b > 0$

We know that the value of q in $y = a.b^x + q$ shifts the graph vertically up or down. Now, let's investigate the effect of p in $y = a.b^{x+p} + q$. Based on your knowledge of the effect of p in the turning point form of the quadratic function $y = a(x + p)^2 + q$, and the hyperbolic function $y = \frac{a}{x+p} + q$, what do you think this effect will be?

Figure 3 shows the graphs of three exponential functions, $f(x) = 1 \times 2^x$, $g(x) = 1 \times 2^{(x+2)}$ and $h(x) = 1 \times 2^{(x-3)}$. We can see that because $a > 0$ and $b > 1$ in each case, all three functions increase in value as x increases. Because $q = 0$, all three functions have a horizontal asymptote of $y = 0$ or the x-axis.

But what is different? Have a look at the various points that have been labelled on each graph. Can you see the difference between the points with the same y-coordinates? What is this difference and how does this difference relate to the equation of each function?

We can see that the point $(0, 1)$ on $f(x)$ has been shifted **two units to the left** and is now the point $(-2, 1)$ on $g(x)$. This same point on $f(x)$ has been shifted **three units to the right** and is now the point $(3, 1)$ on $h(x)$.

The same horizontal shifting pattern can be seen for the points $(1, 2)$ and $(2, 4)$ on $f(x)$. These are shifted two units to the left on $g(x)$ and three units to the right on $h(x)$.

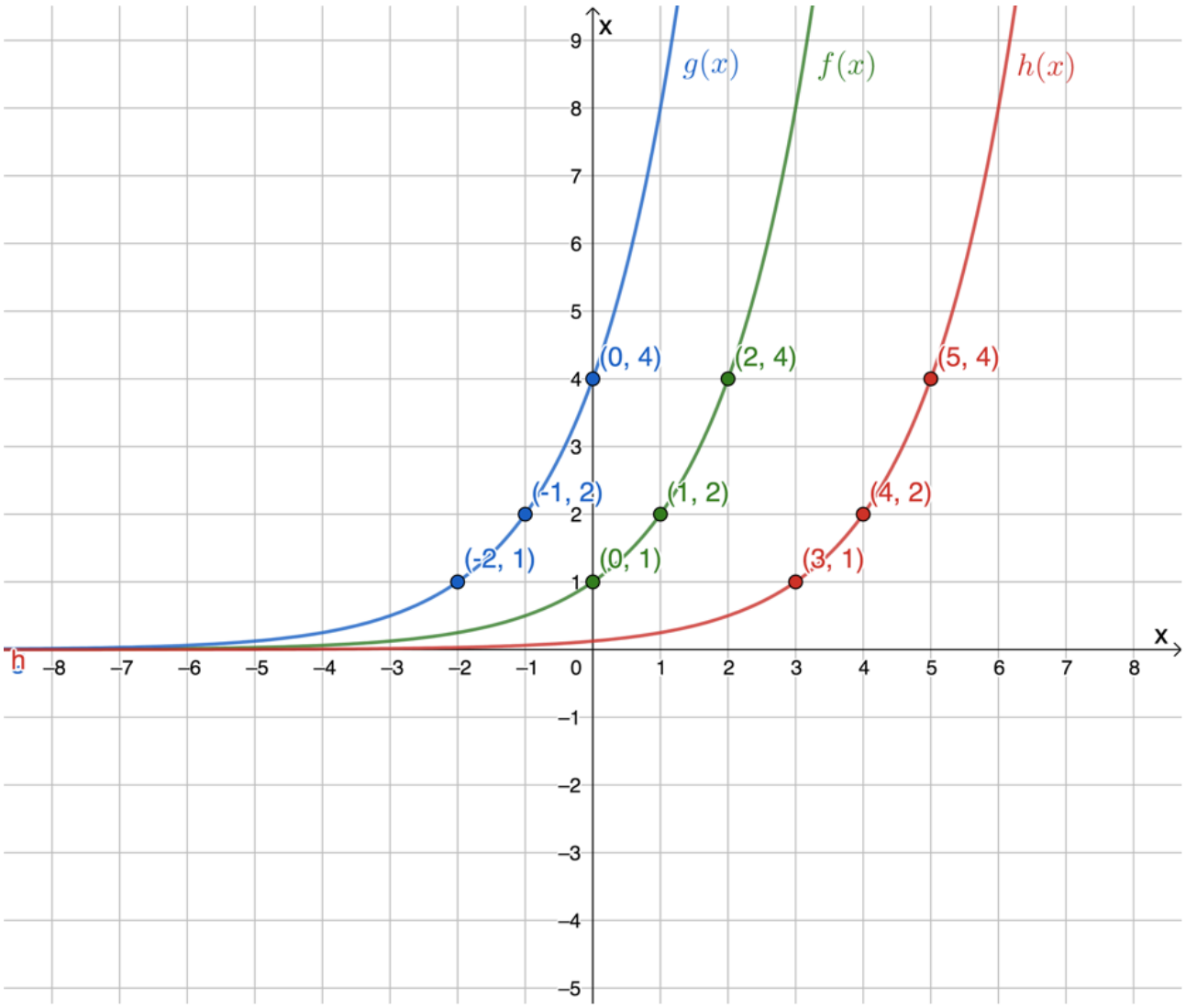


Figure 3: Graphs of $f(x) = 1 \times 2^x$, $g(x) = 1 \times 2^{(x+2)}$ *and* $h(x) = 1 \times 2^{(x-3)}$

Therefore, the effect of p on the function $y = a.b^{x+p} + q, b > 0$ is a **horizontal shift** because all the points on the graph are moved the same distance in the same direction. In other words, the entire graph shifts to the left or to the right.

- If $p > 0$ the graph is shifted p units to the left.
- If $p < 0$ the graph is shifted p units to the right.

Domain and range of the hyperbolic function of the form $y = a.b^{x+p} + q, b > 0$

The domain of $y = a.b^{x+p} + q, b > 0$is $\{x|x \in \mathbb{R}\}$ because there is no value of x for which the function is undefined.

As we have seen, the range depends on the value of a. If $a > 0$, the graph increases to positive infinity as x increases (when $b > 1$) or decreases (when $0 < b < 1$).
If $a > 0$, then:

$$b^{x+p} > 0$$
$$\therefore ab^{x+p} > 0$$
$$\therefore ab^{x+p} + q > q$$
$$\therefore f(x) > q$$

So, the range is $\{y|y \in \mathbb{R}, y > q\}$.

If $a < 0$, then:

$$b^{x+p} > 0$$
$$\therefore ab^{x+p} < 0$$
$$\therefore ab^{x+p} + q < q$$
$$\therefore f(x) < q$$

So, the range is $\{y|y \in \mathbb{R}, y < q\}$.

Take note!

For $y = a.b^{x+p} + q, b > 0$:

- the domain is $\{x|x \in \mathbb{R}\}$
- the range is $\{y|y \in \mathbb{R}, y > q\}$ if $a > 0$
- the range is $\{y|y \in \mathbb{R}, y < q\}$ if $a < 0$.

Note that the restrictions on the range determine the horizontal asymptote.

Example 5.1

Given $j(x) = 5 \times 3^{x-2} + 2$, determine the domain and range of $j(x)$.

Solution

The domain of $j(x) = 5 \times 3^{x-2} + 2$ is $\{x|x \in \mathbb{R}\}$ because there is no value for x for which the function is undefined.

$a > 0$ therefore the range is $\{j(x)|j(x) \in \mathbb{R}, y > q\}$. As $q = 2$, the range is $\{j(x)|j(x) \in \mathbb{R}, y > 2\}$.

The range can also be determined as follows:

$$3^{x-2} > 0$$
$$\therefore 5 \times 3^{x-2} > 0$$
$$\therefore 5 \times 3^{x-2} + 2 > 2$$
$$\therefore j(x) > 2$$

This means that the graph has a horizontal asymptote of $y = 2$.

Take note!

The horizontal asymptote of an exponential function of the form $y = a.b^{x+p} + q, b > 0$ is $y = q$. If $a > 0$, the graph lies **above** the asymptote. If $a < 0$, the graph lies **below** the asymptote.



Exercise 5.1

Determine the domain, range and asymptotes of each of the following functions:

1. $y = -4^{x+3} - 5$
2. $y + 3 = 3^{x-3}$
3. $\frac{y}{-2} = 3^{x+1} - 1$
4. $f(x) = \left(\frac{1}{2}\right)^{x+3}$

The full solutions are at the end of the unit.

Intercepts

The graph of $y = a.b^{x+p} + q, b > 0$ always has a y-intercept which we find by letting the function value be zero.

When the value of q in $y = a.b^{x+p} + q, b > 0$ is zero, the graph has a horizontal asymptote that is the x-axis. In this case, the graph never crosses or even touches the x-axis and so there is no intercept with the x-axis. However, when q is not equal to zero and the asymptote shifts up or down, then there **may** be an intercept with the x-axis. If $a > 0$ (and the asymptote is a minimum), the graph will intercept the x-axis if $q < 0$ (see Figure 4). If $a < 0$ (and the asymptote is a maximum), the graph will intercept the x-axis if $q > 0$ (see Figure 5).

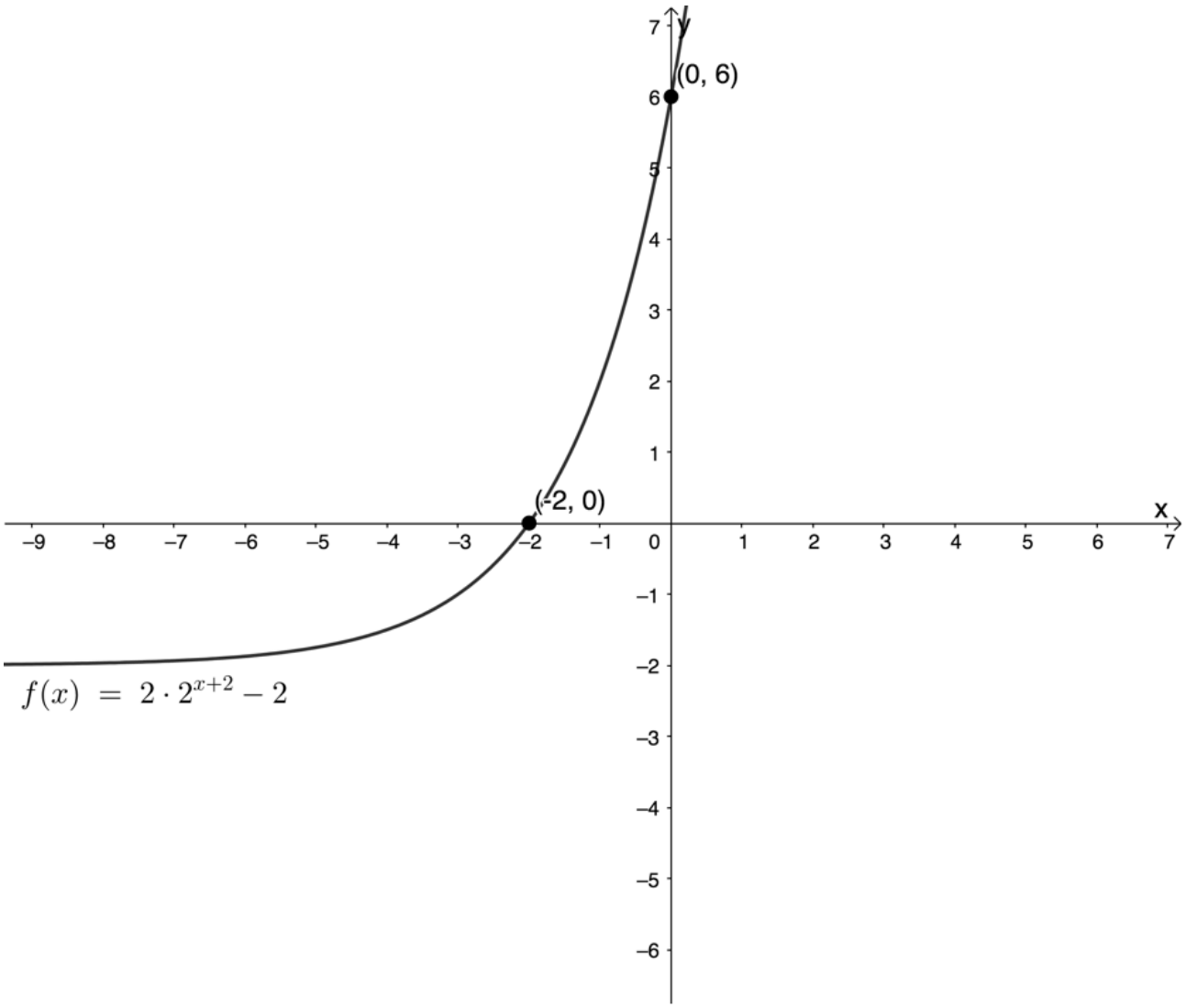


Figure 4: $a > 0$ and $q > 0$ so an x-intercept exists

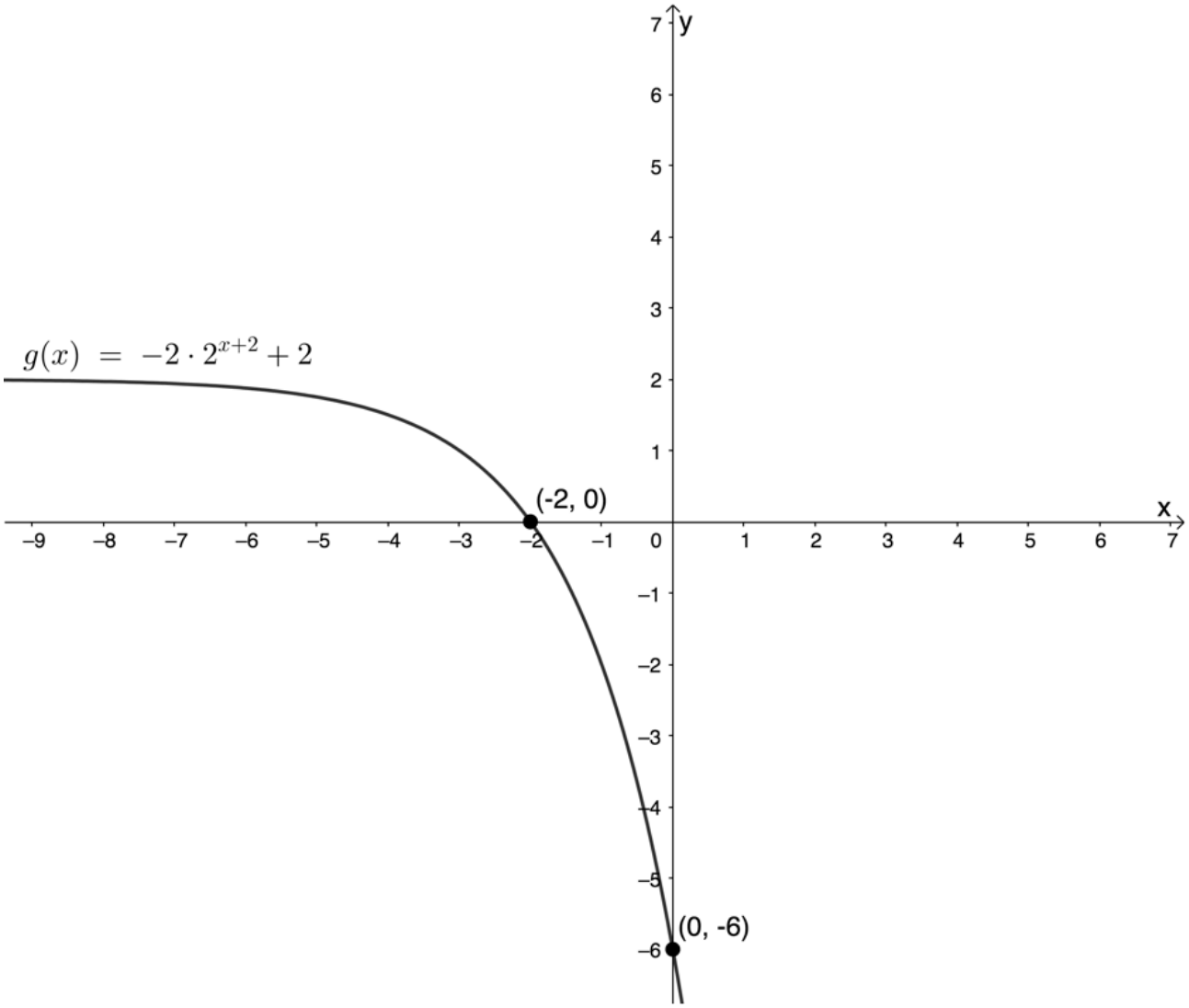


Figure 5: $a < 0$ and $q > 0$ so an x-intercept exists

To calculate the x-intercept (if it exists), we let $x = 0$.

Example 5.2

Determine the intercepts with the axes of $k(x) = 2 \times 2^{x-4} - 4$.

Solution

Step 1: Find the y-intercept
y-intercept (let $x = 0$)

$$
\begin{aligned}
k(0) &= 2 \times 2^{-4} - 4 \\
&= 2 \times \frac{1}{2^4} - 4 \\
&= 2 \times \frac{1}{16} - 4 \\
&= \frac{1}{8} - 4 \\
&= -3\frac{3}{8}
\end{aligned}
$$

The y-intercept is the point $(0, -3\frac{3}{8})$.

Step 2: Find the x-intercept (if it exists)
$a > 0$ and $q < 0$. Therefore, we expect to find an x-intercept.

x-intercepts (let $y = 0$):

$$
\begin{aligned}
0 &= 2 \times 2^{x-4} - 4 \\
\therefore 2 \times 2^{x-4} &= 4 \\
\therefore 2^{x-4} &= 2 \\
\therefore x - 4 &= 1 \\
\therefore x &= 5
\end{aligned}
$$

The x-intercept is the point $(5, 0)$.



Example 5.3

Determine the intercepts with the axes of $m(x) = -2 \times 2^{x-4} - 1$.

Solution

Step 1: Find the y-intercept
y-intercept (let $x = 0$)

$$
\begin{aligned}
m(0) &= -2 \times 2^{-4} - 1 \\
&= -2 \times \frac{1}{2^4} - 1 \\
&= -2 \times \frac{1}{16} - 1 \\
&= -\frac{1}{8} - 1 \\
&= -1\frac{1}{8}
\end{aligned}
$$

The y-intercept is the point $(0, -1\frac{1}{8})$.

Step 2: Find the x-intercept (if it exists)
$a < 0$ and $q < 0$. Therefore, we **do not** expect to find an x-intercept.

x-intercepts (let $y = 0$):

$$0 = -2 \times 2^{x-4} - 1$$
$$\therefore 2 \times 2^{x-4} = -1$$
$$\therefore 2^{x-4} = -\frac{1}{2} = -2^{-1}$$

There are no real solutions. Therefore there is no x-intercept.



Exercise 5.2

Determine the intercepts with the axes of the following functions:

1. $f(x) = 2^{x+1} - 8$
2. $y + 18 = 2 \times 3^{x+4}$
3. $g(x) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{x-\frac{1}{2}} + 4$

The full solutions are at the end of the unit.

Sketch the hyperbolic function $y = a.b^{x+p} + q, b > 0$

Now that we are able to determine all the characteristics of the exponential function of the form $y = a.b^{x+p} + q, b > 0$, we can sketch its graph.

In order to sketch the exponential function we need to know the following:

- the basic shape
- the domain and range
- the horizontal asymptote
- the y-intercept
- the x-intercept (if it exists).



Example 5.4

Draw a sketch of $2 \times f(x) = 10 \times 2^{x+1} - 5$.

Solution

First make sure that the function is in the standard form i.e. in the form $y = a.b^{x+p} + q, b > 0$.

$$2 \times f(x) = 10 \times 2^{x+1} - 10$$
$$\therefore f(x) = 5 \times 2^{x+1} - 5$$

Step 1: Determine the basic shape

$a > 0$ and $b > 1$. Therefore the graph **increases** as x **increases**.

Step 2: Find the domain and range
Domain: $\{x|x \in \mathbb{R}\}$
Range: $a = 5$ therefore $a > 0$ so range is $\{y|y \in \mathbb{R}, y > q\}$. $q = -5$ therefore the range is $\{y|y \in \mathbb{R}, y > -5\}$

Step 3: Find the horizontal asymptote
From the range, we can see that the asymptote is the line $y = -5$. The graph lies above the asymptote.

Step 4: Find the intercepts with the axes
As neither of the asymptotes is an axis, both an x- and a y-intercept exist.
y-intercept (let $x = 0$):

$$f(0) = 5 \times 2^1 - 5$$
$$\therefore f(0) = 5$$

y-intercept is the point $(0, 5)$

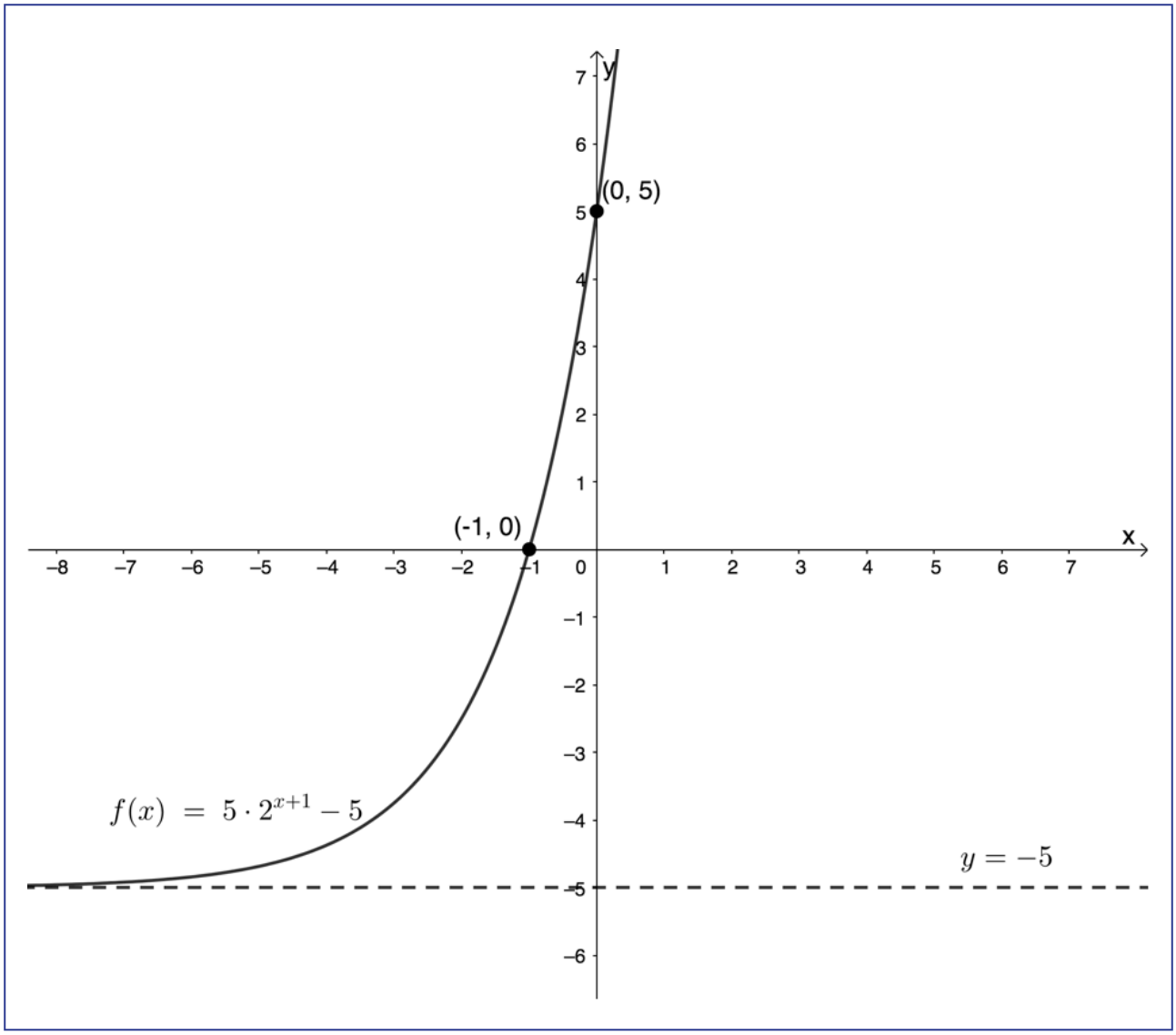
x-intercept (let $y = 0$) ($a > 0$ and $q < 0$) so we expect to find an x-intercept):

$$0 = 5 \times 2^{x+1} - 5$$
$$\therefore 5 \times 2^{x+1} = 5$$
$$\therefore 2^{x+1} = 1$$
$$\therefore x + 1 = 0$$
$$\therefore x = -1$$

x-intercept is the point $(-1, 0)$

Note: If the x-intercept does not exist, find one other point on the graph to help you get the right shape.

Step 5: Make a neat sketch
Draw in and label the asymptote and mark the intercepts with the axes or the other point on the graph you found. Noting the values of a, b and the general shape of the graph, draw a smooth exponential graph passing through the points, and approaching the asymptote.



Note

If you have an internet connection visit this online exponential graph simulation.

Here you will find an exponential function of the form $y = a.b^{x+p} + q, b > 0$ with sliders to change the values of a, b, p and q . You can see the effects of these on the shape and the position of the graph as well as on the values of the asymptote, and intercepts with the axes.

Summary

In this unit you have learnt the following:

- That the value of p in $y = a.b^{x+p} + q, b > 0$ shifts the graph horizontally.
- How to determine the asymptote, domain and range of exponential functions of the form $y = a.b^{x+p} + q, b > 0$.
- How to determine the intercepts with the axes of an exponential function of the form $y = a.b^{x+p} + q, b > 0$.
- How to use the above information to make an accurate sketch of an exponential function of the form $y = a.b^{x+p} + q, b > 0$.

Unit 5: Assessment

Suggested time to complete: 30 minutes

1. Given $g(x) = -2.2^{x-1} + 1$:
   a. What is the domain and range of $g(x)$?
   b. Determine the asymptote of $g(x)$.
   c. For which values of x is the function $g(x)$ undefined?
   d. Draw the graph of $g(x)$.
   e. Determine $g(4)$.
2. Given $f(x) = \left(\frac{1}{4}\right)^{x} + 1$:
   a. What are the domain and range of $f(x)$?
   b. Determine the asymptote of $f(x)$.
   c. Determine the x- and y-intercepts of $f(x)$.
   d. Draw the graph of $f(x)$.
   e. Determine $f(-1)$.

The full solutions are at the end of the unit.

Unit 5: Solutions

Exercise 5.1

1. $y = -4^{x+3} - 5$
   Domain: $\{x|x \in \mathbb{R}\}$
   Range: $a = -1$ therefore $a < 0$ so range is $\{y|y \in \mathbb{R}, y < q\}$. $q = -5$ therefore range is $\{y|y \in \mathbb{R}, y < -5\}$
   Horizontal asymptote is the line $y = -5$
2. $y + 3 = 3^{x-3}$
   $\therefore y = 3^{x-3} - 3$
   Domain: $\{x|x \in \mathbb{R}\}$
   Range: $a = 1$ therefore $a > 0$ so range is $\{y|y \in \mathbb{R}, y > q\}$. $q = -3$ therefore range is $\{y|y \in \mathbb{R}, y > -3\}$
   Horizontal asymptote is the line $y = -3$

3. $\frac{y}{-2} = 3^{x+1} - 1$
   $\therefore y = -2 \times 3^{x+1} + 2$
   Domain: $\{x|x \in \mathbb{R}\}$
   Range: $a = -2$ therefore $a < 0$ so range is $\{y|y \in \mathbb{R}, y < q\}$. $q = 2$ therefore range is $\{y|y \in \mathbb{R}, y < 2\}$
   Horizontal asymptote is the line $y = 2$
4. $f(x) = \left(\frac{1}{2}\right)^{x+3}$
   Domain: $\{x|x \in \mathbb{R}\}$
   Range: $a = 1$ therefore $a > 0$ so range is $\{y|y \in \mathbb{R}, y > q\}$. $q = 0$ therefore range is $\{y|y \in \mathbb{R}, y > 0\}$
   Horizontal asymptote is the line $y = 0$

Back to Exercise 5.1

Exercise 5.2

1. y-intercept (let $x = 0$)
   $f(0) = 2^1 - 8$
   $= -6$
   y-intercept is the point $(0, -6)$.
   x-intercept (let $y = 0$) ($a > 0, q < 0$ so we expect to find an x-intercept)
   $0 = 2^{x+1} - 8$
   $\therefore 2^{x+1} = 8 = 2^3$
   $\therefore x + 1 = 3$
   $\therefore x = 2$
   x-intercept is the point $(2, 0)$.
2. $y + 18 = 2 \times 3^{x+4}$
   $\therefore y = 2 \times 3^{x+4} - 18$
   y-intercept (let $x = 0$)
   $y = 2 \times 3^4 - 18$
   $= 2 \times 81 - 18$
   $= 144$
   y-intercept is the point $(0, 144)$.
   x-intercept (let $y = 0$) ($a > 0, q < 0$ so we expect to find an x-intercept)
   $0 = 2 \times 3^{x+4} - 18$
   $\therefore 2 \times 3^{x+4} = 18$
   $\therefore 3^{x+4} = 9 = 3^2$
   $\therefore x + 4 = 2$
   $\therefore x = -2$
   x-intercept is the point $(-2, 0)$.
3. y-intercept (let $x = 0$)

$$g(0) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{-\frac{1}{2}} + 4$$
$$= -\frac{1}{2} \times \frac{1}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} + 4$$
$$= -\frac{1}{2} \times \frac{1}{\sqrt{\frac{1}{2}}} + 4$$
$$= -\frac{1}{\left(\frac{2}{\sqrt{2}}\right)} + 4$$
$$= -\frac{\sqrt{2}}{2} + 4$$

y-intercept is the point $(0, -\frac{\sqrt{2}}{2} + 4)$.

x-intercept (let $y = 0$) ($a < 0, q > 0$ so we expect to find an x-intercept)

$$0 = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{x-\frac{1}{2}} + 4$$
$$\therefore \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{x-\frac{1}{2}} = 4$$
$$\therefore \left(\frac{1}{2}\right)^{x-\frac{1}{2}} = 8 = 2^3$$
$$\therefore 2^{-\left(x-\frac{1}{2}\right)} = 2^3$$
$$\therefore -x + \frac{1}{2} = 3$$
$$\therefore x = -\frac{5}{2}$$

x-intercept is the point $(-\frac{5}{2}, 0)$.

Back to Exercise 5.2

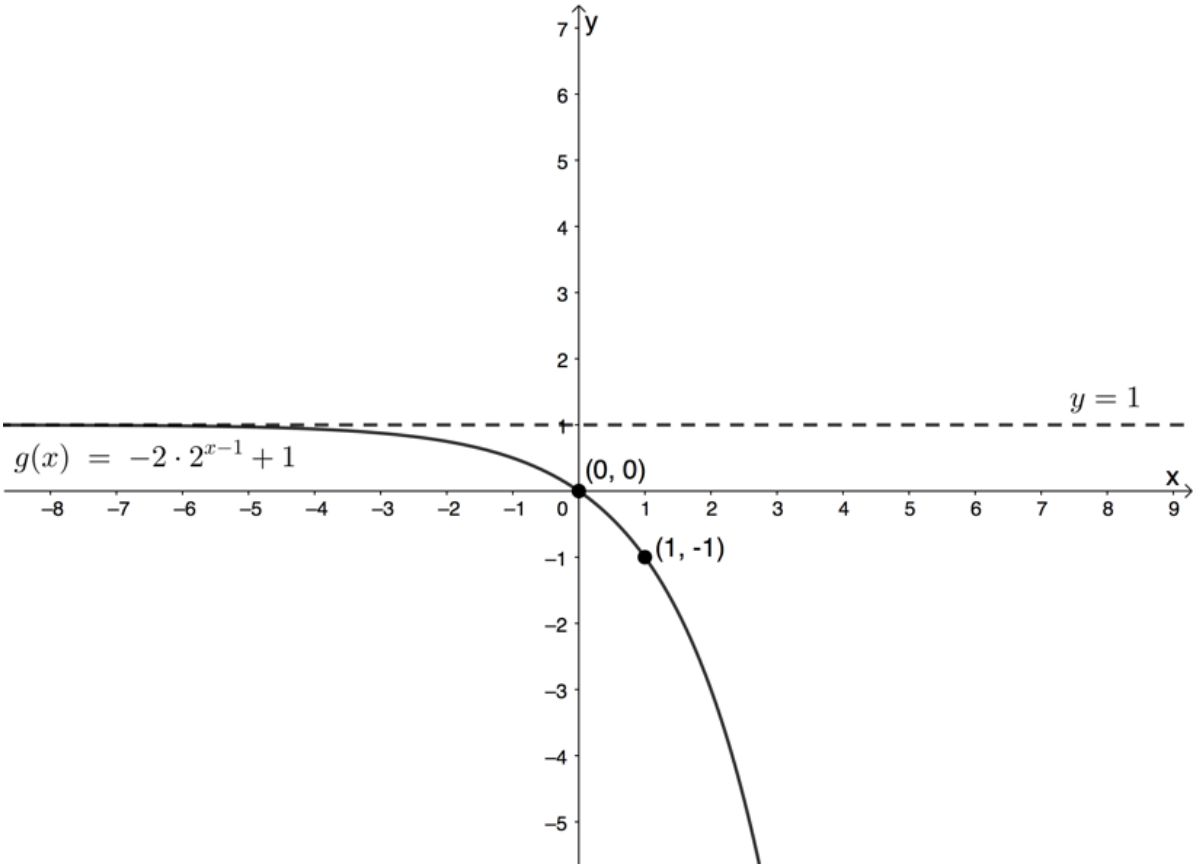
Unit 5: Assessment

1. $g(x) = -2.2^{x-1} + 1$
   a. Domain: $\{x | x \in \mathbb{R}\}$
   Range: $a = -2$ therefore $a < 0$ so range is $\{y | y \in \mathbb{R}, y < q\}$. $q = 1$ therefore range is $\{y | y \in \mathbb{R}, y < 1\}$
   b. $y = 1$
   c. The function is not undefined for any values of x.
   d. $a < 0$ and $b > 1$. Therefore, the graph decreases as x increases.
   Before we can draw the graph, we need to find the intercepts on the axes.
   y-intercept (let $x = 0$)
   $g(0) = -2.2^{-1} + 1$
   $= -2.\frac{1}{2} + 1$
   $= 0$
   y-intercept is the point $(0, 0)$.

The graph passes through the origin. Therefore, we need to find one other point on the graph to help get the right shape.

$g(1) = -2.2^{1-1} + 1$
$= -2.2^{0} + 1$
$= -2 + 1$
$= -1$

The point $(1, -1)$ lies on the graph.



e.

$g(4) = -2 \cdot 2^{x-1} + 1$
$= -2 \cdot 2^{3} + 1$
$= -16^{1}$
$= -15$

2. $f(x) = \left(\frac{1}{4}\right)^{x} + 1$

a. Domain: $\{x | x \in \mathbb{R}\}$
Range: $a = 1$ therefore $a > 0$ so range is $\{y | y \in \mathbb{R}, y > q\}$. $q = 1$ therefore range is $\{y | y \in \mathbb{R}, y > 1\}$

b. $y = 1$

c. y-intercept (let $x = 0$)

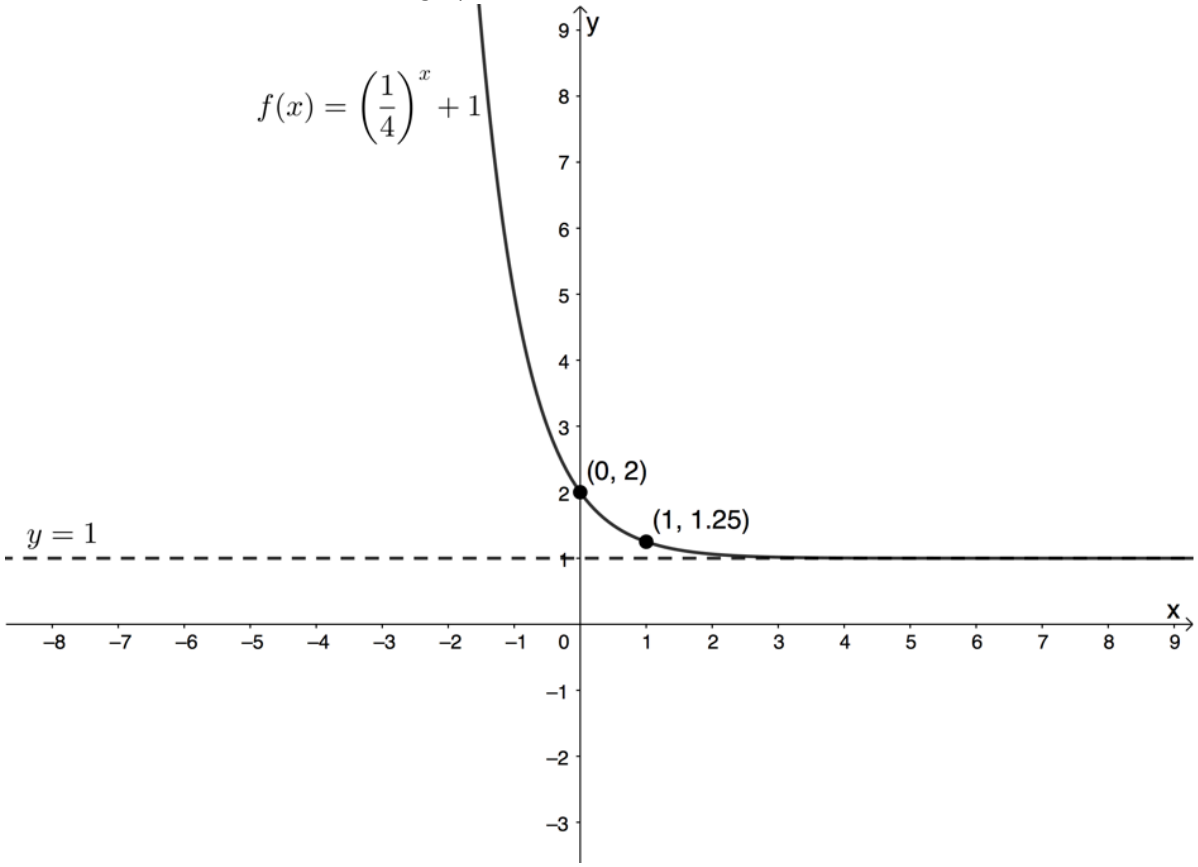
$f(0) = \left(\frac{1}{4}\right)^{0} + 1$
$= 1 + 1$
$= 2$

y-intercept is the point $(0, 2)$.
x-intercept (let $y = 0$)
$a > 0$ and $q > 1$. Therefore, there is no x-intercept. We choose another x-value to find another point on the graph to help with the drawing. Let $x = 1$.

$$f(1) = \left(\frac{1}{4}\right)^1 + 1$$
$$= \frac{5}{4}$$

The point $(1, \frac{5}{4})$ lies on the graph.

d. $a > 0$ and $0 < b < 1$. Therefore, the graph increases as x decreases.



e.

$$f(-1) = \left(\frac{1}{4}\right)^{-1} + 1$$
$$= 4 + 1$$
$$= 5$$

Back to Unit 5: Assessment

Media Attributions

- figure10 © DHET is licensed under a CC BY (Attribution) license
- figure11 © DHET is licensed under a CC BY (Attribution) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1d © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2d © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 6: Find the equation of an exponential function

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the equation of exponential graphs in the form $y = a.b^{x+p} + q,\ b > 0$.
- Analyse exponential functions.

What you should know

Before you start this unit, make sure you can:

- Solve linear simultaneous equations. Refer to level 2 subject outcome 2.3 unit 4 if you need assistance.
- Determine the domain and range of exponential functions of the form $y = a.b^{x+p} + q,\ b > 0$. Refer to unit 5 in this subject outcome if you need assistance.
- Find the asymptote of exponential functions of the form $y = a.b^{x+p} + q,\ b > 0$. Refer to unit 5 in this subject outcome if you need assistance.
- Find the intercepts with the axes of exponential functions of the form $y = a.b^{x+p} + q,\ b > 0$. Refer to unit 5 in this subject outcome if you need assistance.

Introduction

In the previous unit we learnt how to determine various characteristics of exponential functions of the form $y = a.b^{x+p} + q,\ b > 0$ in order to sketch their graphs.

These characteristics included:

- the general shape of the graph, affected by the values of a and b
- the domain and range
- the asymptote
- the intercepts with the axes.

In this unit, we will see how we can use these characteristics from the graph of an exponential function to determine the equation of the exponential function.

Find the equation of an exponential function

To find the equation of an exponential function of the form $y = a.b^{x+p} + q,\ b > 0$ from its graph, we need to find the values of a, b, p and q.

We can always determine whether $a > 0$ or $a < 0$ and whether $b > 1$ or $0 < b < 1$ by looking at shape of the graph. Does it increase or decrease as x increases or decreases?

We can determine the values of p and q by looking at:

- the asymptote (remember the horizontal asymptote is the line $y = q$)
- the intercepts with the axes.

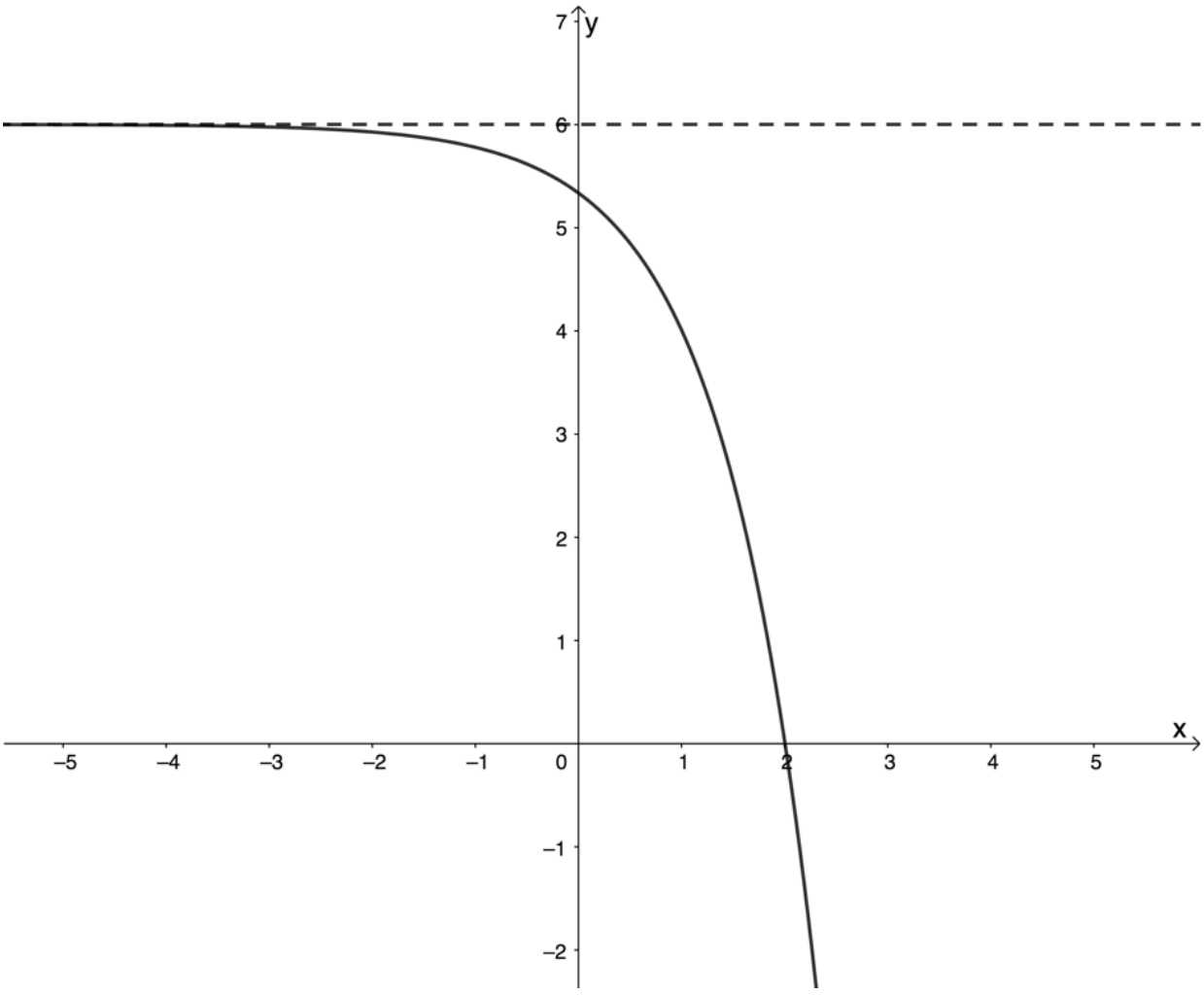
Because we have four unknowns to find, we require four pieces of information. Very often, you will be given one, or even two, of the values of a, b, p and q.

Let's look at some examples.

Example 6.1

Question taken from Siyavula Grade 11 Mathematics Worked Example 17 pg 192.

The graph below is an exponential function of the form $y = -2 \times 3^{x+p} + q$. Determine the equation of the function.



Solution

We are given that $a = -2$ and $b = 3$. We can see this in the shape of the graph. It decreases as x increases.

We have also been given that the horizontal asymptote is the line $y = 6$. Therefore, we know that $q = 6$. Therefore, the equation of the exponential function is $y = -2 \times 3^{x+p} + 6$.

We are also given that the x-intercept is the point $(2, 0)$. We can substitute this point into the equation to solve for p.

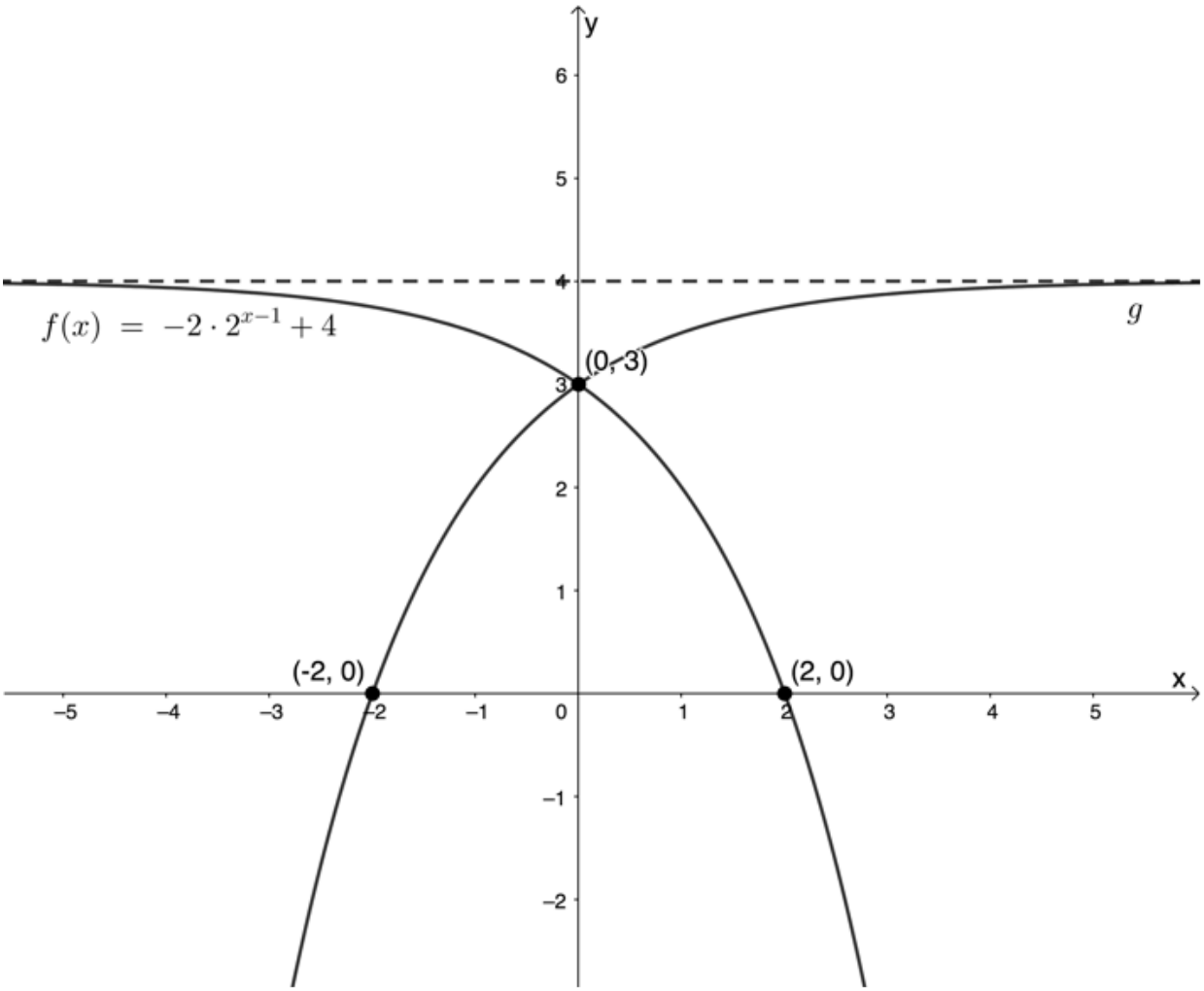
$$\begin{aligned} 0 &= -2 \times 3^{2+p} + 6 \\ \therefore 2 \times 3^{2+p} &= 6 \\ \therefore 3^{2+p} &= 3 \\ \therefore 2 + p &= 1 \\ \therefore p &= -1 \end{aligned}$$

The equation of the function is $y = -2 \times 3^{x-1} + 6$.



Example 6.2

Sketched below are graphs of $f(x) = -2.2^{x-1} + 4$ and $g(x) = a.b^{x+p} + q,\ b < 0$ where $g(x)$ is symmetrical to $f(x)$ about the y-axis. Determine the values of a, b, p and q.



Solution

We saw in level 2 subject outcome 2.1 unit 4 that the graphs of $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$ are symmetrical about the y-axis. Remember that symmetry about the y-axis means that every x-value is replaced with a $-x$. In other words, the point $(2, 0)$ becomes the point $(-2, 0)$ and $(-3, 2)$ becomes the point $(3, 2)$.

If we replace the x in $y = b^x$ with a $-x$ we get $y = b^{-x} = \frac{1}{b^x} = \left(\frac{1}{b}\right)^x$.

Therefore $g(x) = f(-x)$. This means that:

$$\begin{aligned} g(x) &= -2 \times 2^{-x-1} + 4 \\ &= -2 \times 2^{-(x+1)} + 4 \\ &= -2 \times \left(\frac{1}{2}\right)^{x+1} + 4 \end{aligned}$$

Example 6.3

An exponential function of the form $y = a.b^x + q,\ b > 0$ has a horizontal asymptote of $y = 0$, a y-intercept of 1 and passes through the point $(1, \frac{1}{3})$. Determine the equation of the function.

Solution

The first thing to notice is that we have been told that $p = 0$. The equation is of the form $y = a.b^x + q,\ b > 0$.

We are also told that the asymptote is the line $y = 0$. Therefore, $q = 0$.

The equation of the function is $y = a.b^x$. We need to find the values of a and b. In order to find two unknowns, we need two pieces of information. We have been given two points on the graph, $(0, 1)$ and $(1, \frac{1}{3})$.

Substitute $(0, 1)$:

$$1 = a.b^0$$
$$\therefore 1 = a.1$$
$$\therefore a = 1$$

Therefore, $y = b^x$. Substitute $(1, \frac{1}{3})$:

$$\frac{1}{3} = b^1$$
$$\therefore b = \frac{1}{3}$$

The equation of the function is $y = \left(\frac{1}{3}\right)^x$.

Exercise 6.1

1. The horizontal asymptote of $f(x) = 3^{x+p} + q$ is $y = -1$ and the graph has an x-intercept of $(-1, 0)$ and a y-intercept of $(0, 2)$. Determine the equation of $f(x)$.
2. $n(x) = 3.\left(\frac{1}{3}\right)^{x+p} + q$ has a domain of $\{x|x \in \mathbb{R}\}$ and a range of $\{n(x)|n(x) \in \mathbb{R}, n(x) > 2\}$. If $n(0) = 11$, determine the equation of $n(x)$.

The full solutions are at the end of the unit.

Summary

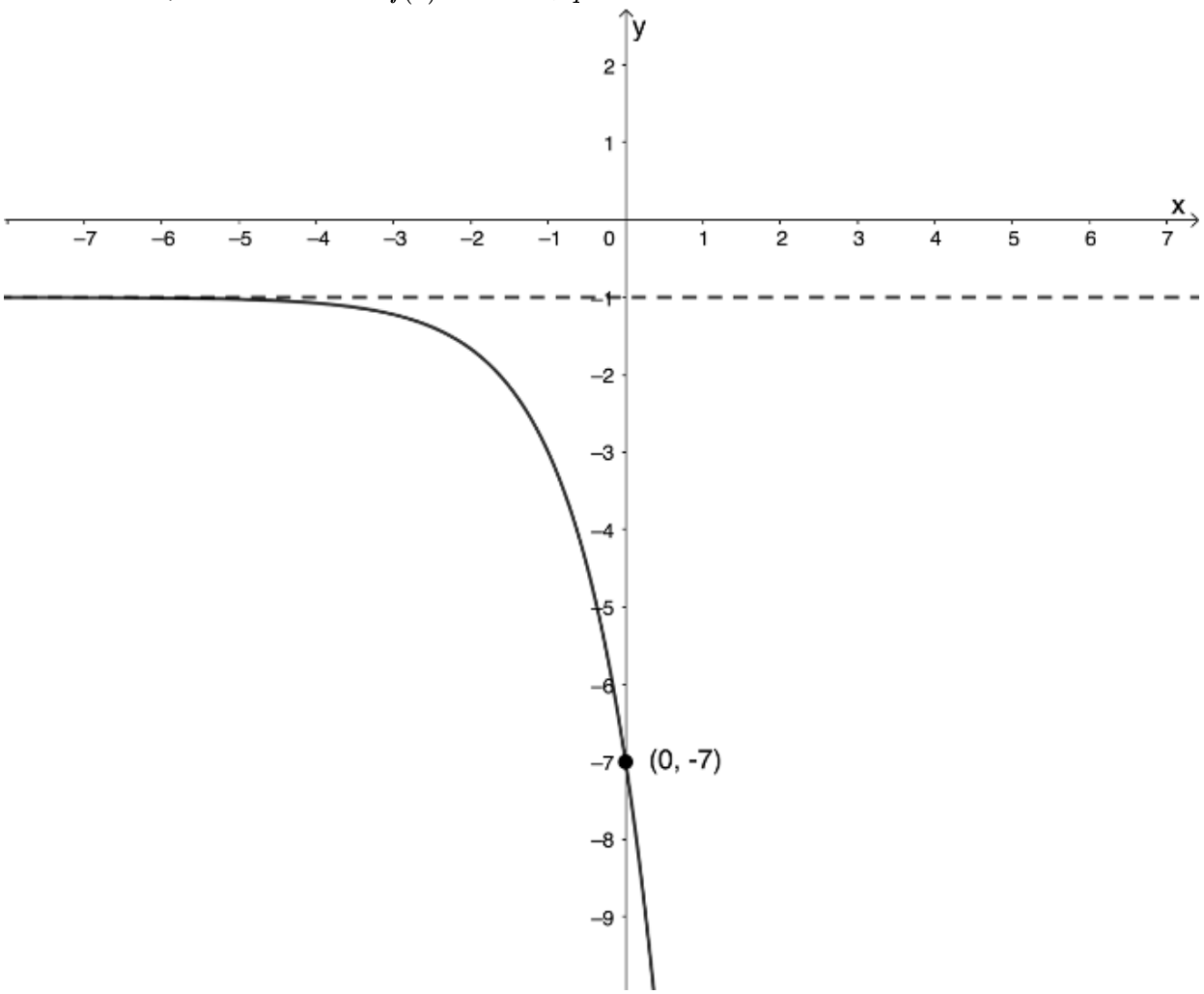
In this unit you have learnt the following:

- How to find the equation of an exponential function of the form $y = a.b^{x+p} + q,\ b > 0$.

Unit 6: Assessment

Suggested time to complete: 45 minutes

1. Below is an exponential function $f(x) = a.3^{x+1} + q$. Find the values of a and q.



2. The range of the exponential function $j(x) = \frac{1}{3}b^x + q$ is $\{j(x)|j(x) \in \mathbb{R}, j(x) > 1\}$ and $j(-2) = 2$. Determine the equation of $j(x)$.
3. An exponential function of the form $f(x) = \frac{1}{2}.2^{x+p} + q$ shares a horizontal asymptote with $g(x) = \frac{2}{x} - 4$ and intercepts with the axes with $h(x) = 2x - 2$.
   a. Determine the full equation of $f(x)$.
   b. Sketch the graphs of $f(x)$, $g(x)$ and $h(x)$ on the same set of axes.

The full solutions are at the end of the unit.

Unit 6: Solutions

Exercise 6.1

1. $f(x) = 3^{x+p} + q$
   $a = 1, b = 3$
   Horizontal asymptote is $y = -1$. Therefore, $q = -1$ and $f(x) = 3^{x+p} - 1$.
   Substitute $(0, 2)$:
   $2 = 3^p - 1$
   $\therefore 3^p = 3$
   $\therefore p = 1$
   $f(x) = 3^{x-1} - 1$

2. $n(x) = 3.\left(\frac{1}{3}\right)^{x+p} + q$ has a range of $\{n(x)|n(x) \in \mathbb{R}, n(x) > 2\}$. Therefore, the line $y = 2$ is a horizontal asymptote and $q = 2$ and $n(x) = 3.\left(\frac{1}{3}\right)^{x+p} + 2$.
   $n(0) = 11$. Therefore the x-intercept is the point $(0, 11)$. Substitute $(0, 11)$:
   $11 = 3.\left(\frac{1}{3}\right)^{p} + 2$
   $\therefore 3.\left(\frac{1}{3}\right)^{p} = 9$
   $\therefore \left(\frac{1}{3}\right)^{p} = 3$
   $\therefore 3^{-p} = 3^1$
   $\therefore p = -1$
   $n(x) = 3.\left(\frac{1}{3}\right)^{x-1} + 2$

Back to Exercise 6.1

Unit 6: Assessment

1. $f(x) = a.3^{x+1} + q$
   $b = 3$
   The horizontal asymptote is $y = -1$. Therefore, $q = -1$ and $f(x) = a.3^{x+1} - 1$.
   Substitute $(0, -7)$:
   $-7 = a.3^1 - 1$
   $\therefore 3a = -6$
   $\therefore a = -2$
   $f(x) = -2.3^{x+1} - 1$

2. $j(x) = \frac{1}{3}b^x + q$
   $a = \frac{1}{3}$
   Range is $\{j(x)|j(x) \in \mathbb{R}, j(x) > 1\}$ and $j(-2) = 2$. Therefore, the horizontal asymptote is $y = 1$ and $q = 1$.
   Therefore $j(x) = \frac{1}{3}b^x + 1$.
   $j(-2) = 2$. Therefore, the point $(-2, 2)$ lies on the graph. Substitute $(-2, 2)$:

$$2 = \frac{1}{3}b^{-2} + 1$$
$$\therefore \frac{1}{3}b^{-2} = 1$$
$$\therefore b^{-2} = 3$$
$$\therefore \frac{1}{b^2} = 3$$
$$\therefore b^2 = \frac{1}{3}$$
$$\therefore b = \pm\frac{1}{\sqrt{3}}$$

But for exponential functions $b > 0$. Therefore, $b = \frac{1}{\sqrt{3}}$.

$$j(x) = \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^x + 1$$

3. $f(x) = \frac{1}{2}.2^{x+p} - 4$, $g(x) = \frac{2}{x} - 4$, $h(x) = 2x - 2$

   a. $g(x) = \frac{2}{x} - 4$ has a horizontal asymptote of $y = -4$. Therefore $q = -4$ and $f(x) = \frac{1}{2}.2^{x+p} - 4$.

   x-intercept of $h(x) = 2x - 2$ (let $y = 0$):

   $$0 = 2x - 2$$
   $$\therefore 2x = 2$$
   $$\therefore x = 1$$

   y-intercept of $h(x) = 2x - 2$ (let $x = 0$):

   $h(0) = -2$

   $f(x)$ has an x-intercept of $(1, 0)$ and a y-intercept of $(0, -2)$

   Substitute in $(0, -2)$:

   $$f(0) = \frac{1}{2}.2^p - 4 = -2$$
   $$\therefore \frac{1}{2}.2^p = 2$$
   $$\therefore 2^p = 4$$
   $$\therefore p = 2$$
   $$f(x) = \frac{1}{2}.2^{x+2} - 4$$

   b. $f(x) = \frac{1}{2}.2^{x+2} - 4$

   $a > 0$ and $b > 1$. Therefore, the graph increases as x increases.

   $g(x) = \frac{2}{x} - 4$

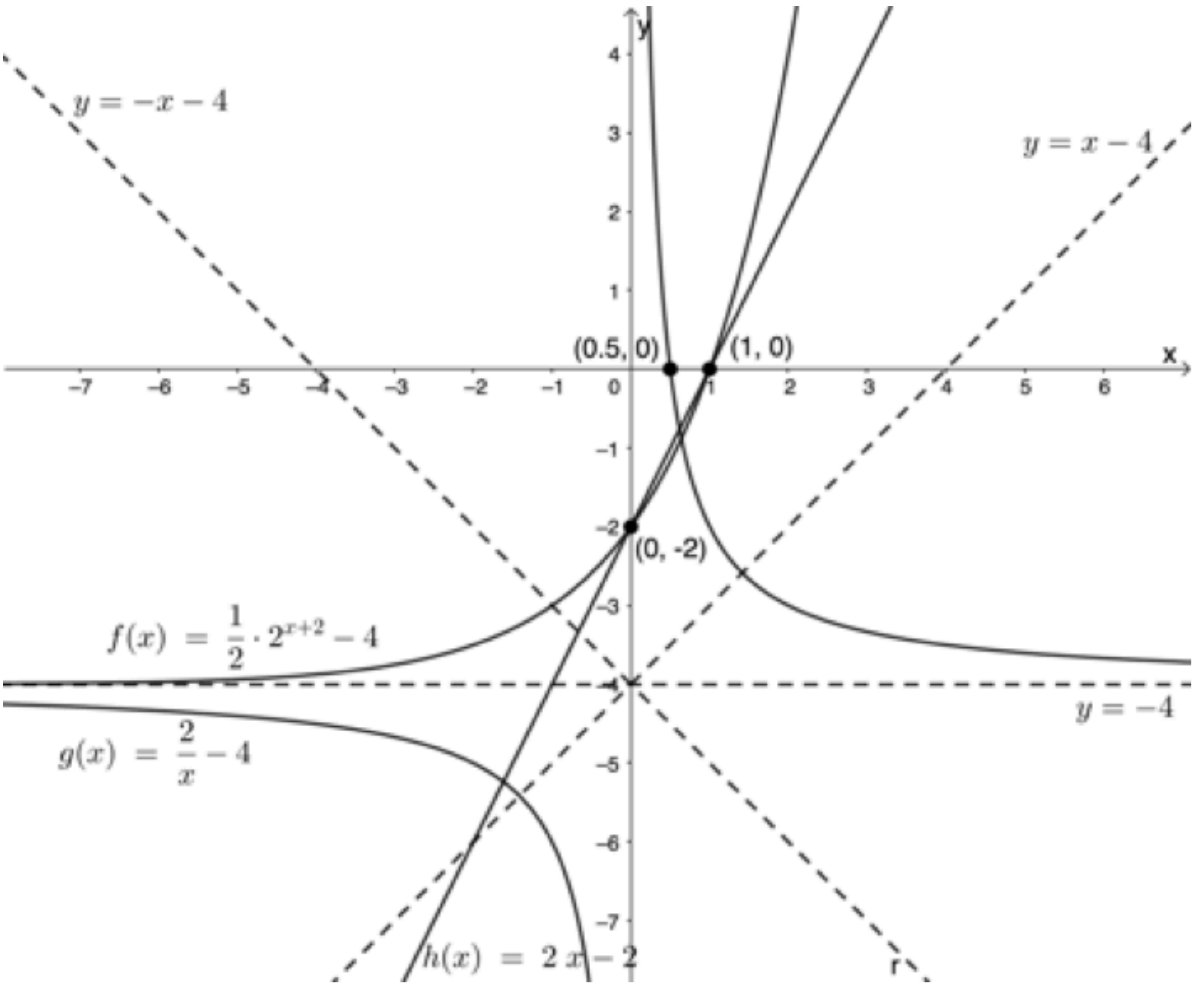
   $p = 0$. Therefore, the vertical asymptote is the y-axis and there are no y-intercepts. $a > 0$. Therefore, the graph lies in the first and third quadrants.

   x-intercept (let $y = 0$):

   $$0 = \frac{2}{x} - 4$$
   $$\therefore \frac{2}{x} = 4$$
   $$\therefore x = \frac{1}{2}$$

   Axes of symmetry:

   $y = x - 4$ and $y = -x - 4$



Back to Unit 6: Assessment

Media Attributions

- example6.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example6.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA3b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 7: The sine function

DYLAN BUSA

Unit 7 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = a \sin k\theta$.
- Determine the effects of a and k on the sine graph of the form $y = a \sin k\theta$.
- Find the values of a and k from a given sine graph of the form $y = a \sin k\theta$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \sin x$ and $y = \sin \theta$ are the same function.

What you should know

Before you start this unit, make sure you can:

- Sketch sine functions of the form $y = a \sin \theta + q$. Refer to level 2 subject outcome 2.1 Unit 5 if you need help with this.

Introduction

The three basic trigonometric functions, sine, cosine and tangent, are different from the other functions which we study in that they repeat themselves again and again. We call this behaviour **periodicity**. After one **period**, the graph repeats itself. The period of the basic sine function, $y = \sin \theta$, is 360°. The function values repeat themselves every 360° (see Figure 1).

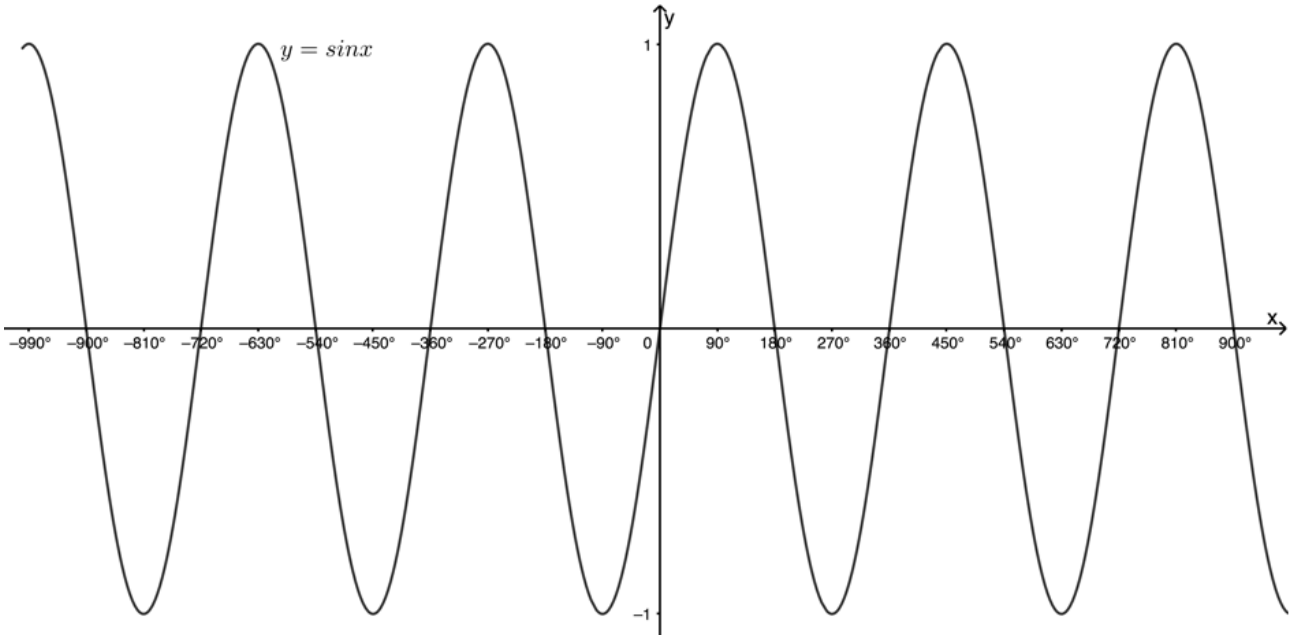


Figure 1: $y = \sin x$

Note

If you have an internet connection, spend some time playing with this online simulation to make sure that you understand why the period of the sine function is 360° (one full revolution of the unit circle).

The same is true for the cosine function. The period of $y = \cos\theta$, is 360° (see Figure 2).

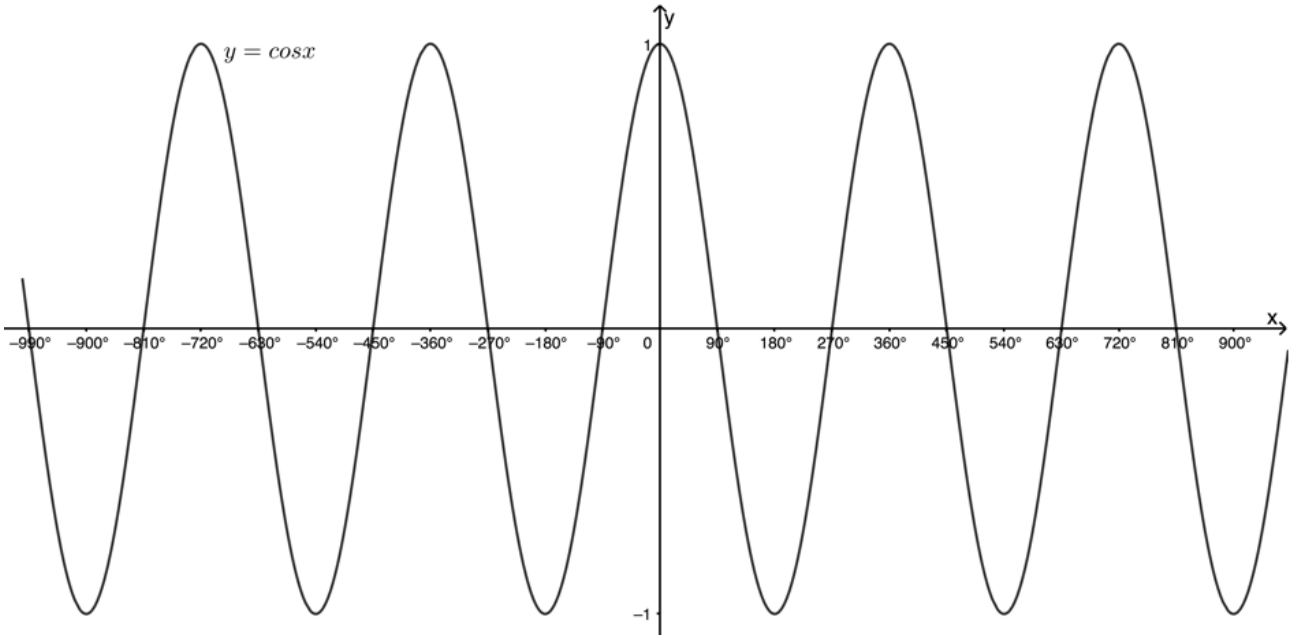


Figure 2: $y = \cos x$

The period of $y = \tan\theta$, is 180°.The function values repeat themselves every 180° (see Figure 3).

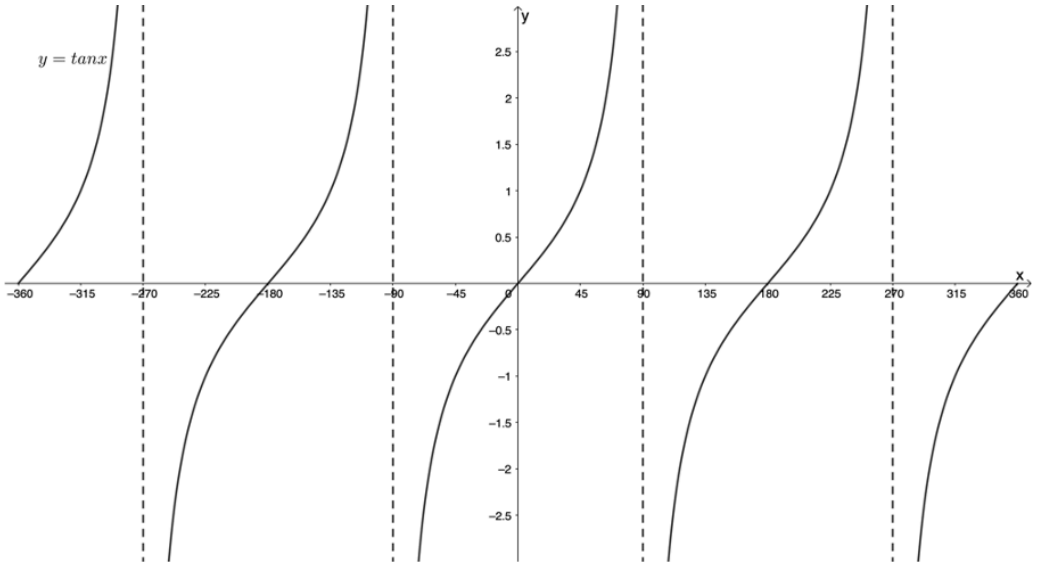


Figure 3: $y = \tan x$

As we learnt in level 2 subject outcome 2.1 unit 5, this periodicity makes the trigonometric functions, and especially sine and cosine, particularly useful in modelling things such as sound waves.

The other feature of the sine and cosine functions that is unique among the functions we have studied is the fact that they have a specific **amplitude**. Amplitude is the maximum distance between the function and its midline. The amplitude of $y = \sin\theta$ is 1 (see Figure 4) and the amplitude of $y = \cos\theta$ is also 1 (see Figure 5).

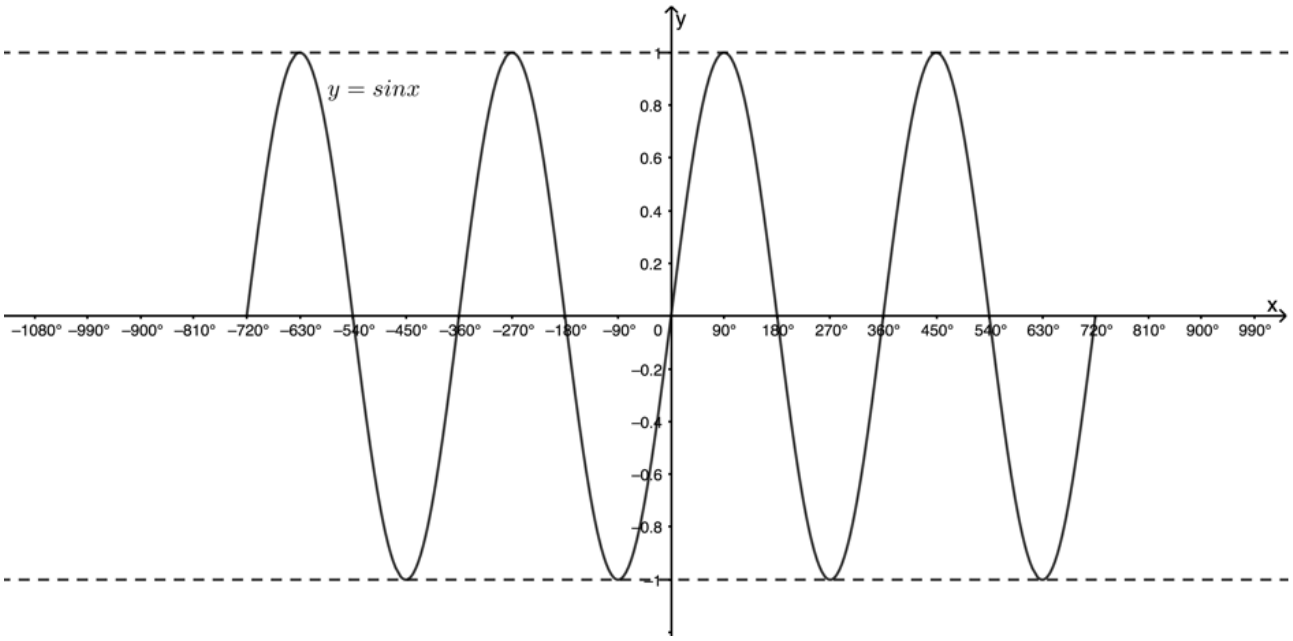


Figure 4: $y = \sin x, -720^\circ \leq x \leq 720^\circ$

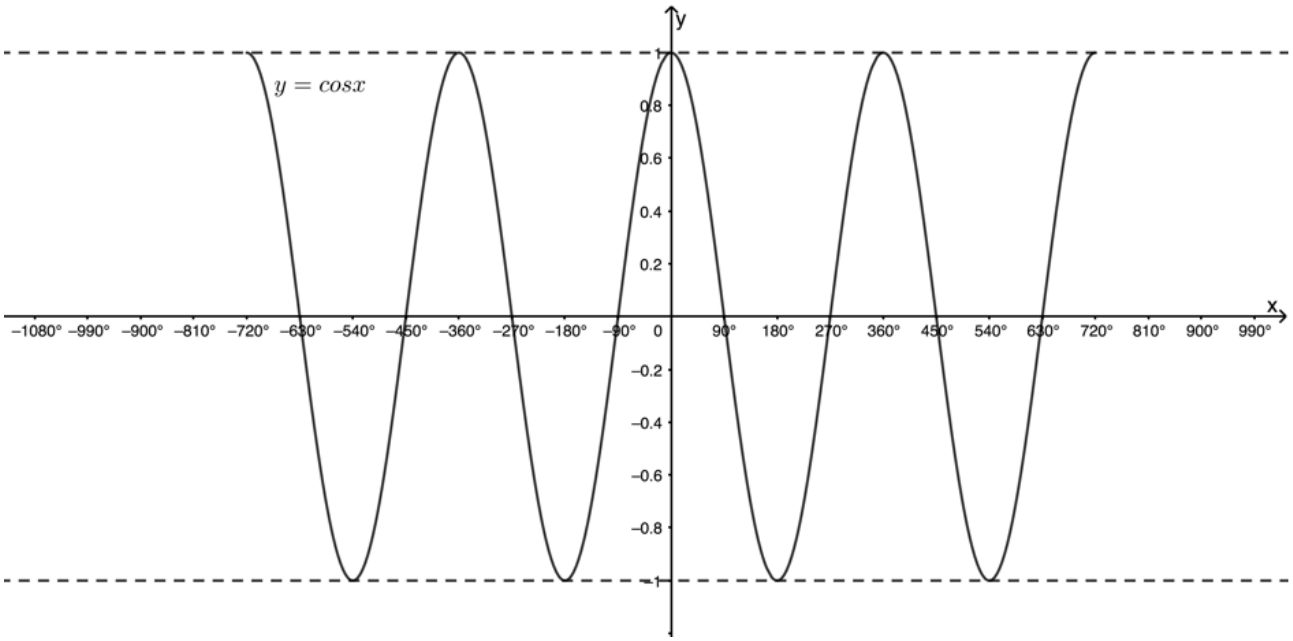


Figure 5: $y = \cos x, -720^\circ \leq x \leq 720^\circ$In $y = \sin\theta$, because the amplitude is 1 and the graph's midline is the x-axis, the maximum value of $y = \sin\theta$ is 1 and the minimum value is -1. In the interval $0^\circ \leq \theta \leq 360^\circ$, the graph of $y = \sin\theta$ has a **maximum turning point** at $(90^\circ, 1)$ and a minimum turning point at $(270^\circ, -1)$ (see Figure 6).

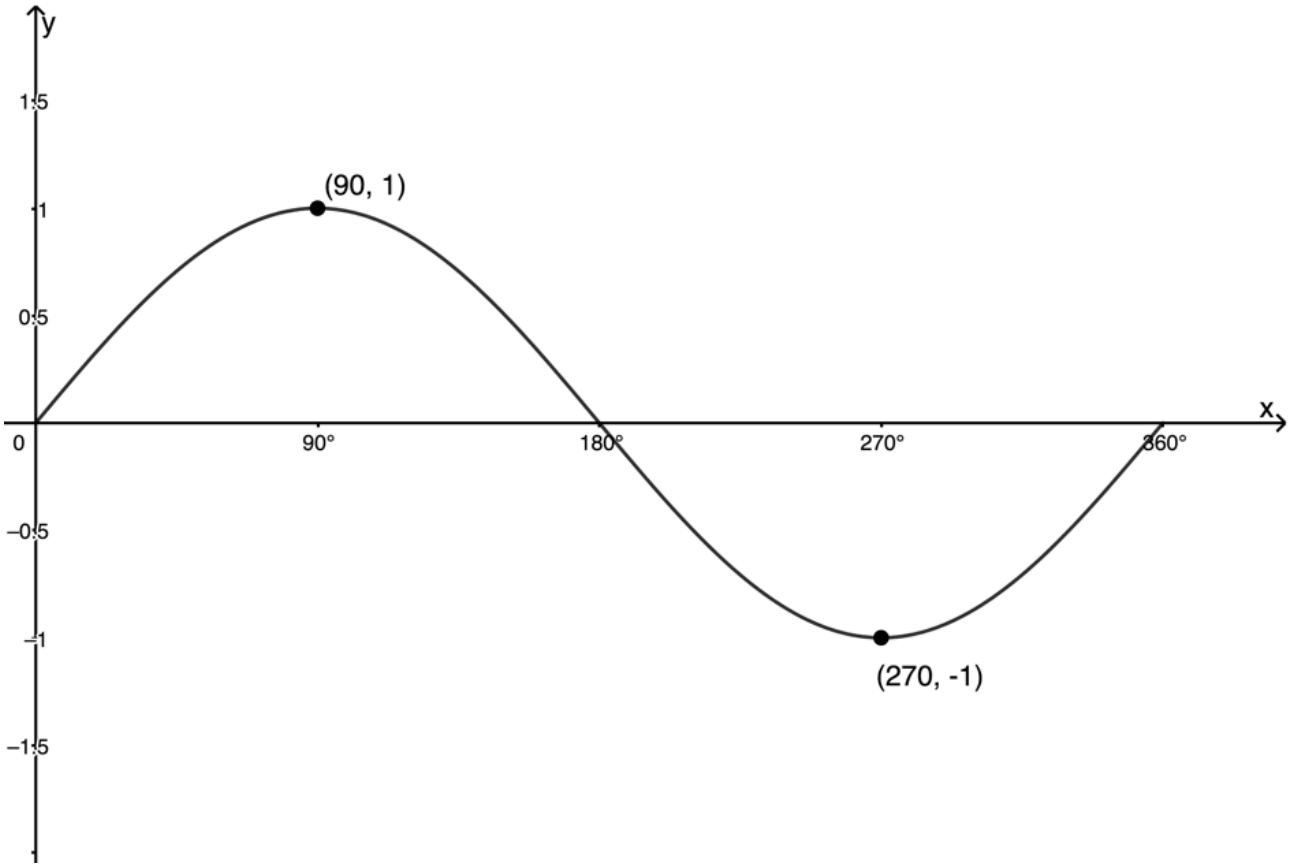


Figure 6: Maximum and minimum turning points of $y = \sin x$ *in the interval* $0^\circ \leq x \leq 360^\circ$

In level 2 we were introduced to the **sine function** of the form $y = a \sin \theta + q$. We discovered that the value of q shifted the graph vertically up and down while the value of a changed the amplitude of the graph.

Take note!

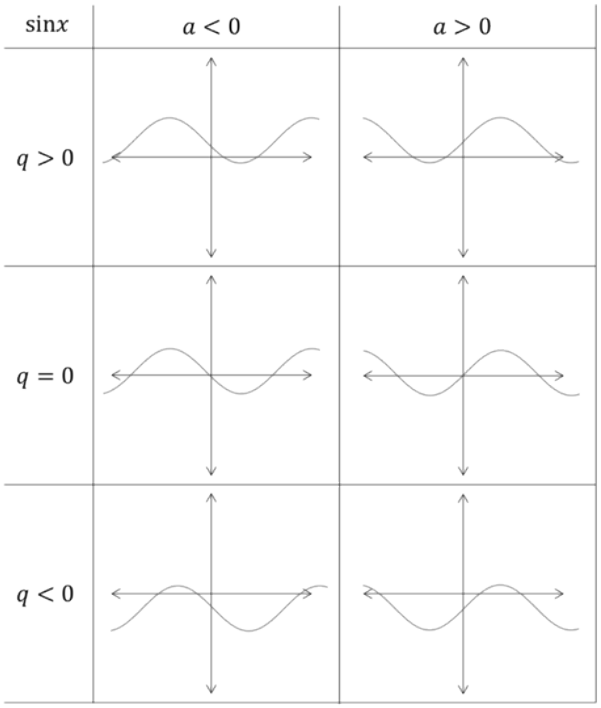
Period of $y = a \sin \theta + q$ is 360°.

The effect of q:

- if $q > 0$, the function is shifted **up** by q units
- if $q < 0$, the function is shifted **down** by q units.

The effect of a:

- if $a > 1$, the amplitude **increases** to a
- if $0 < a < 1$, the amplitude **decreases** to a
- if $a < 0$, the function is reflected about the x-axis
- if $a < -1$, the amplitude **increases** to a
- if $-1 < a < 0$, the amplitude **decreases** to a.



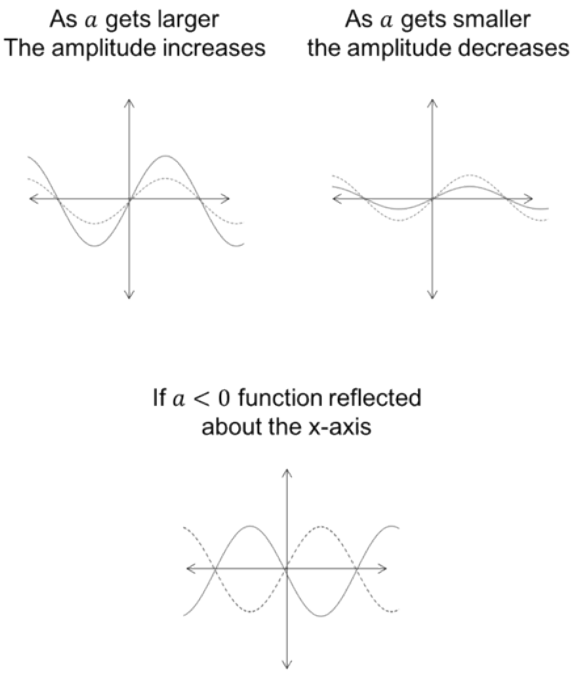


Figure 7: Summary of effects of a and q on $y = a\sin x + q$

Note

If you have access to the internet, visit this interactive simulation.

Here you will find a sine function of the form $y = a\sin x + q$ with sliders to change the values of a and q. Spend some time playing with it to make sure that you understand how changing the values of a and q affects the shape and location of the sine function of the form $y = a\sin x + q$.

The sine function of the form $y = \sin kx$

By now, you should have a good understanding of the effects of a and q on $y = a\sin x + q$. But what happens if we change the form of the function to $y = \sin kx$. We know that the period of the 'normal' sine function $y = \sin x$ is $360°$. How does the value of k in $y = \sin kx$ change the shape of the graph?

Activity 7.1: What is the period?

Time required: 40 minutes

What you need:

- a pen or pencil
- a calculator
- several pieces of paper

What to do:

1. Start by making a sketch of $y = \sin x$ for the interval $-360^\circ \leq x \leq 360^\circ$ on a piece of paper. If you need to, you can complete the following table of values first.

x	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
$\sin x$									

2. Now complete the following tables of values and sketch the additional functions on separate sets of axes for the interval $-360^\circ \leq x \leq 360^\circ$. Just make sure that you use the same scale for each set of axes so that you can compare the graphs.

x	-180°	-135°	-90°	-45°	0°	45°	90°	135°	180°
$\sin(2x)$									

x	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
$\sin\left(\frac{1}{2}x\right)$									
$\sin(-x)$									

3. Use your sketches to complete the following table.

	Domain	**Range**	**Amplitude**	**Period**
$y = \sin x$				
$y = \sin(2x)$				
$y = \sin\left(\frac{1}{2}x\right)$				
$y = \sin(-x)$				

4. Using the results of the table you completed in step 3, describe the effect of k on the graph of the function $y = \sin kx$.

What did you find?

1. Below is the completed table of values and the sketch of

$y = \sin x$, for the interval $-360^\circ \leq x \leq 360^\circ$ (see Figure 8).

x	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
$\sin x$	0	1	0	-1	0	1	0	-1	0

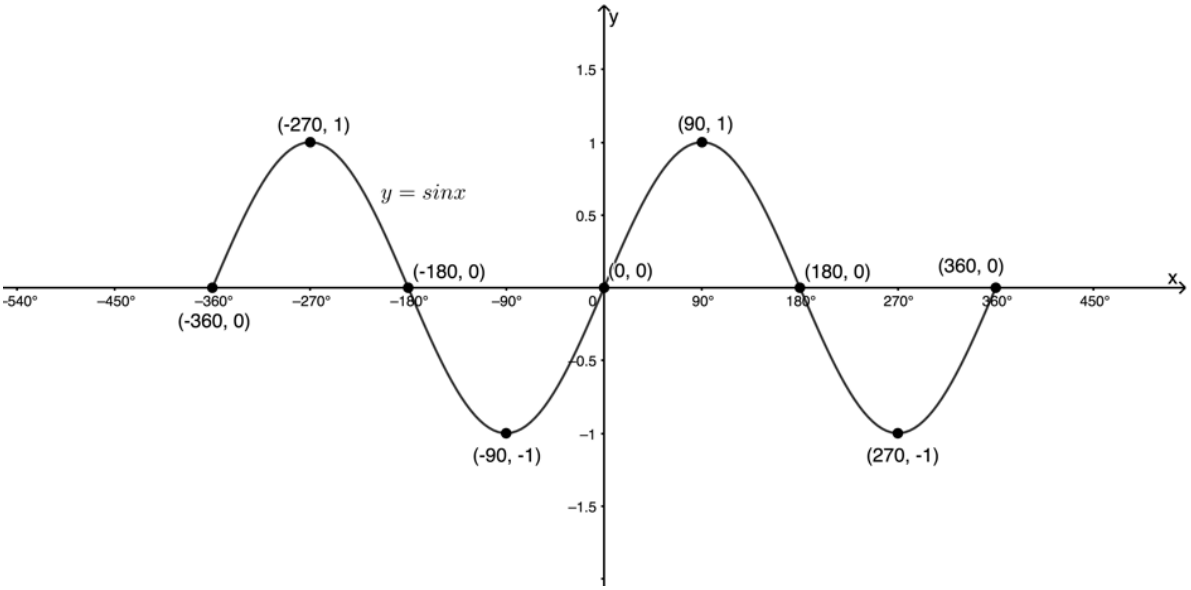


Figure 8: Graph of $y = \sin x, -360^\circ \leq x \leq 360^\circ$

2. Here are the completed tables of values for the other functions.

x	-180°	-135°	-90°	-45°	0°	45°	90°	135°	180°
$\sin(2x)$	0	1	0	-1	0	1	0	-1	0

x	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
$\sin\left(\frac{1}{2}x\right)$	0	-0.707	-1	-0.707	0	0.707	1	0.707	0
$\sin(-x)$	0	-1	0	1	0	-1	0	1	0

Below the graphs of $y = \sin(2x)$ (see Figure 9), $y = \sin\left(\frac{1}{2}x\right)$ (see Figure 10) and $y = \sin(-x)$ (see Figure 11) are drawn for the interval $-360^\circ \leq x \leq 360^\circ$. For ease of comparison with $y = \sin x$, the graph of $y = \sin x$ has also been included as the dotted line.

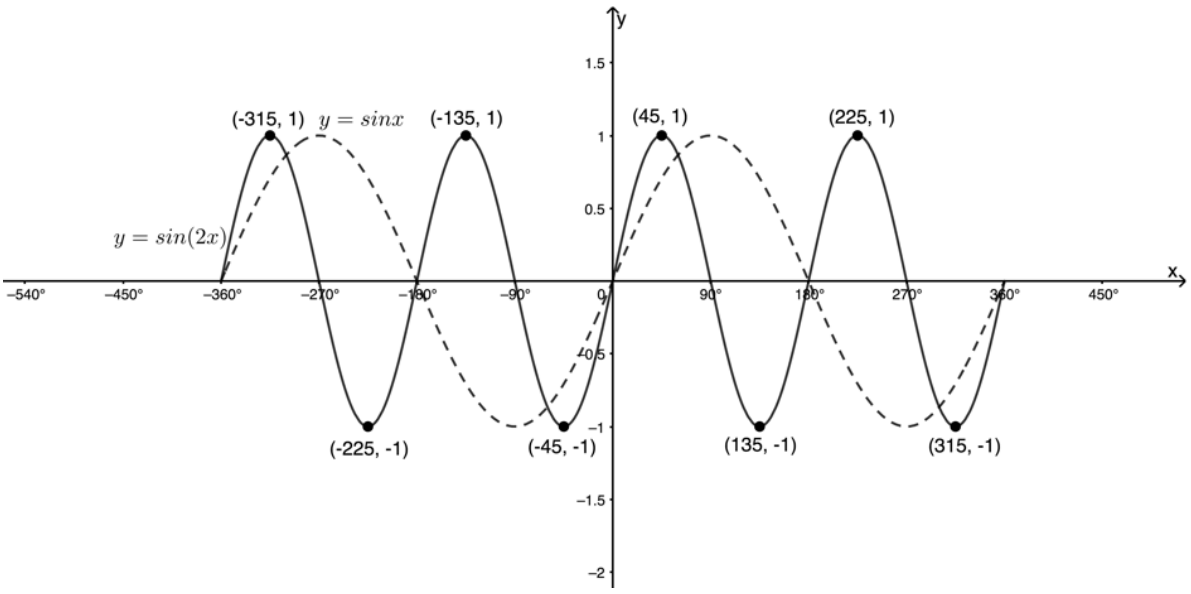


Figure 9: Graph of $y = \sin(2x), \quad -360^\circ \leq x \leq 360^\circ$

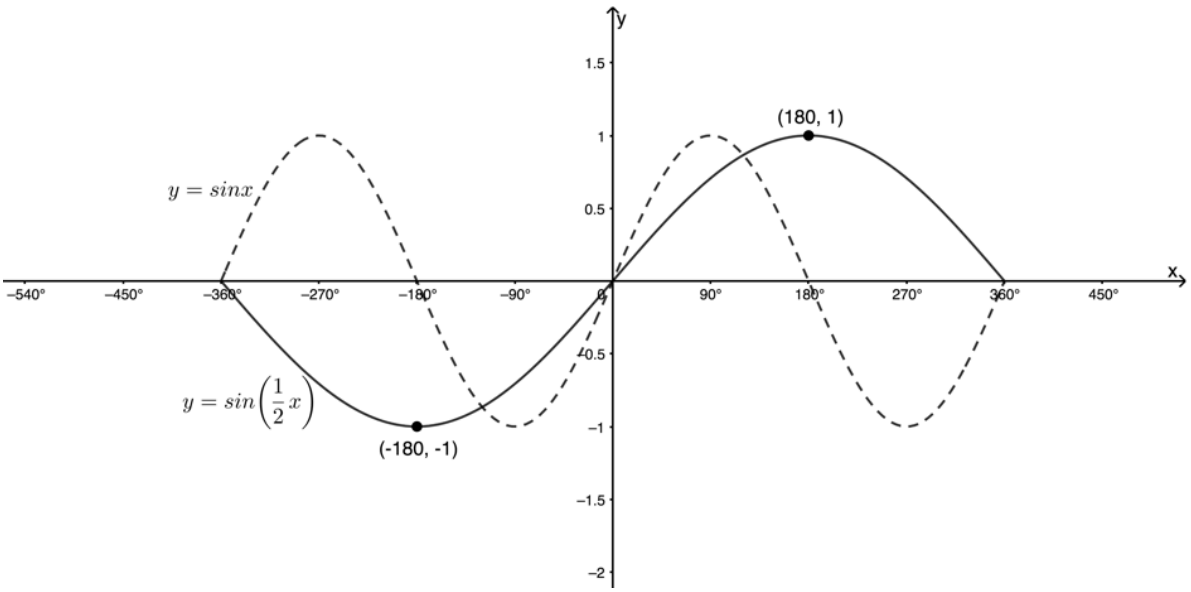


Figure 10: Graph of $y = \sin\left(\frac{1}{2}x\right), \quad -360^\circ \leq x \leq 360^\circ$

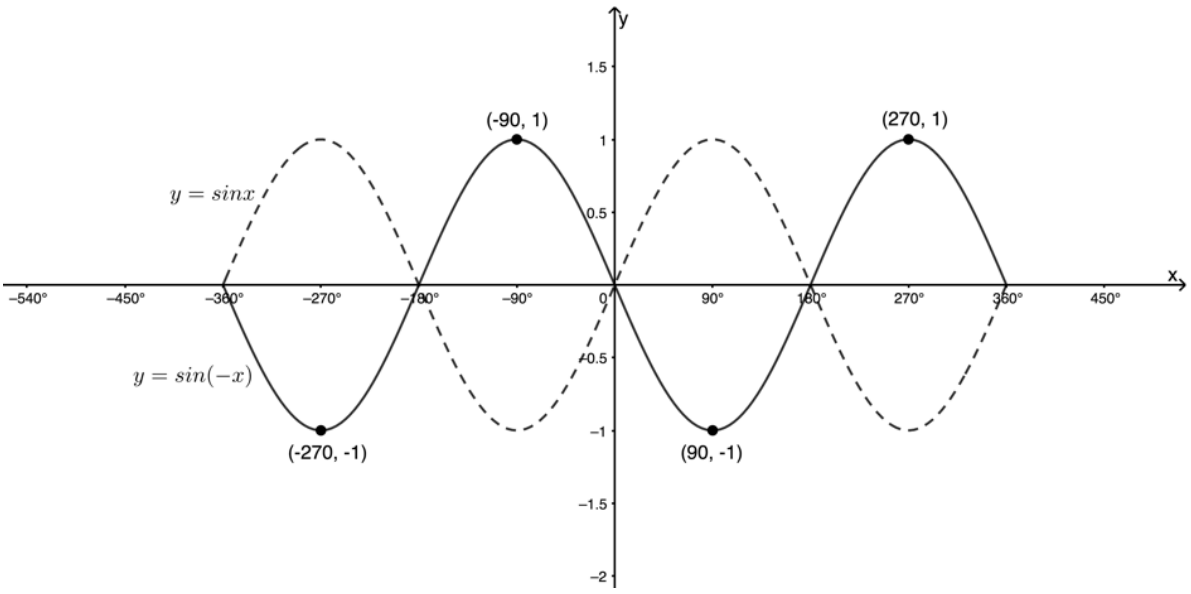


Figure 11: Graph of $y = \sin(-x), \quad -360^\circ \leq x \leq 360^\circ$

3. Here is the completed comparison table.

	Domain	Range	Amplitude	Period
$y = \sin x$	$\{x \mid x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$	$\{y \mid y \in \mathbb{R}, -1 \leq y \leq 1\}$	1	360°
$y = \sin(2x)$	$\{x \mid x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$	$\{y \mid y \in \mathbb{R}, -1 \leq y \leq 1\}$	1	180°
$y = \sin\left(\frac{1}{2}x\right)$	$\{x \mid x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$	$\{y \mid y \in \mathbb{R}, -1 \leq y \leq 1\}$	1	720°
$y = \sin(-x)$	$\{x \mid x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$	$\{y \mid y \in \mathbb{R}, -1 \leq y \leq 1\}$	1	360°

4. We can see from the sketches that the value of k in $y = \sin kx$ does NOT affect the domain, range or amplitude of the graph. It does, however, affect the period of the graph. The period of the graph when:

   - $k = 1$ is 360°
   - $k = 2$ is $\dfrac{360^\circ}{2} = 180^\circ$
   - $k = \dfrac{1}{2}$ is $\dfrac{360^\circ}{\frac{1}{2}} = 360^\circ \times 2 = 720^\circ$
   - $k = -1$ is 360° but the graph is reflected about the x-axis. In other words $\sin(-x) = -\sin(x)$.

From Activity 7.1 we can see that the value of k in $y = \sin kx$ affects the period of the function and that we can calculate the period as $\text{period} = \dfrac{360^\circ}{|k|}$ where $|k|$ is the absolute value of k, in other words the positive value of k. This means that if $k > 1$, the period **decreases** and if $0 < k < 1$, the period **increases**.

If $k < 0$, it changes the period AND reflects the graph about the x-axis because $\sin(-x) = -\sin(x)$.

Take note!

In $y = \sin kx$ the $\text{Period} = \dfrac{360^\circ}{|k|}$.

$\sin(-x) = -\sin(x)$. Therefore, if $k < 0$, it changes the period AND reflects the graph about the x-axis.

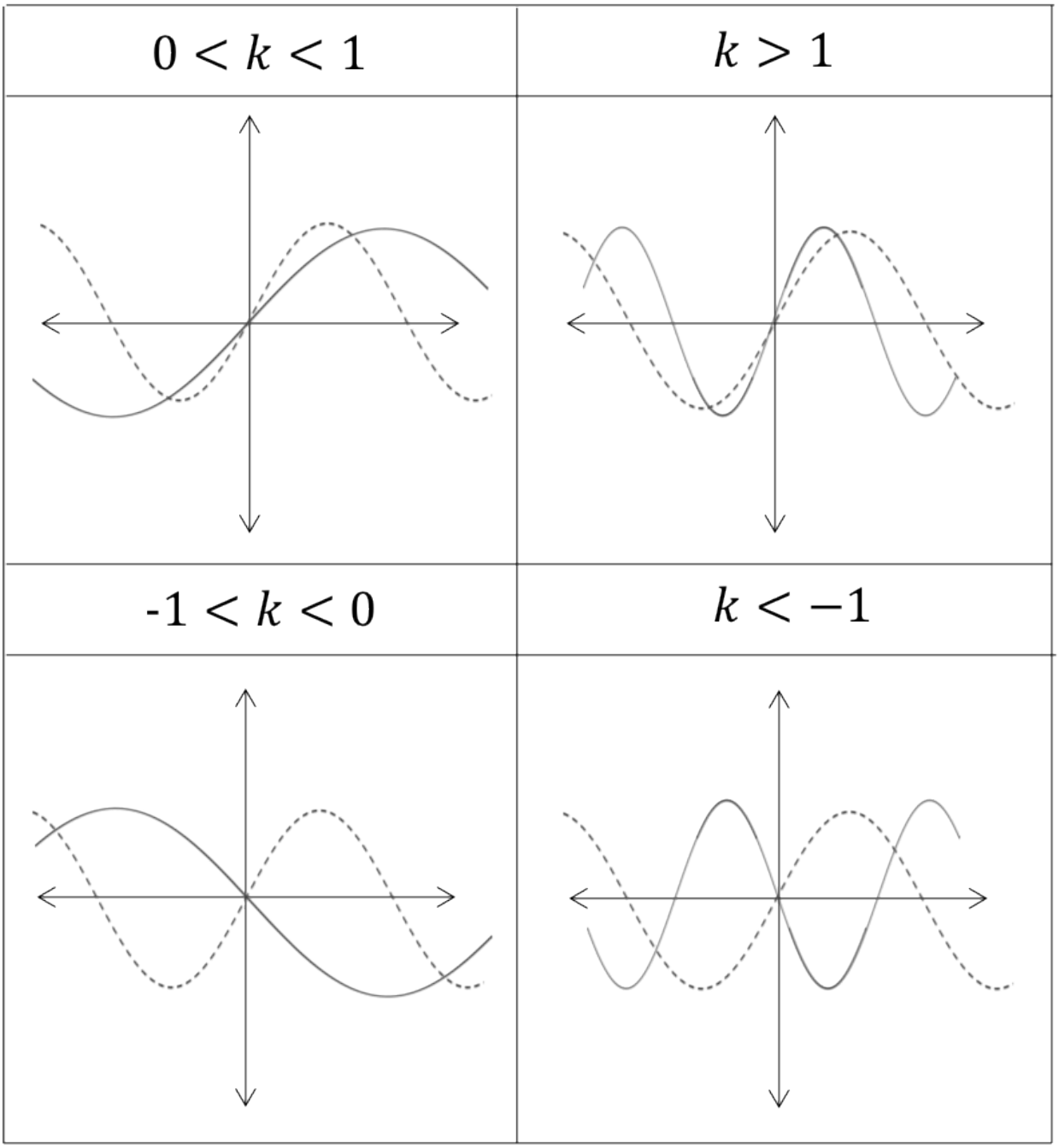


Figure 12: The effect of the value of k on $y = \sin kx$

Example 7.1

Given the function $y = \sin\left(\frac{3x}{2}\right)$ for the interval $0^\circ \leq x \leq 360^\circ$, determine the:

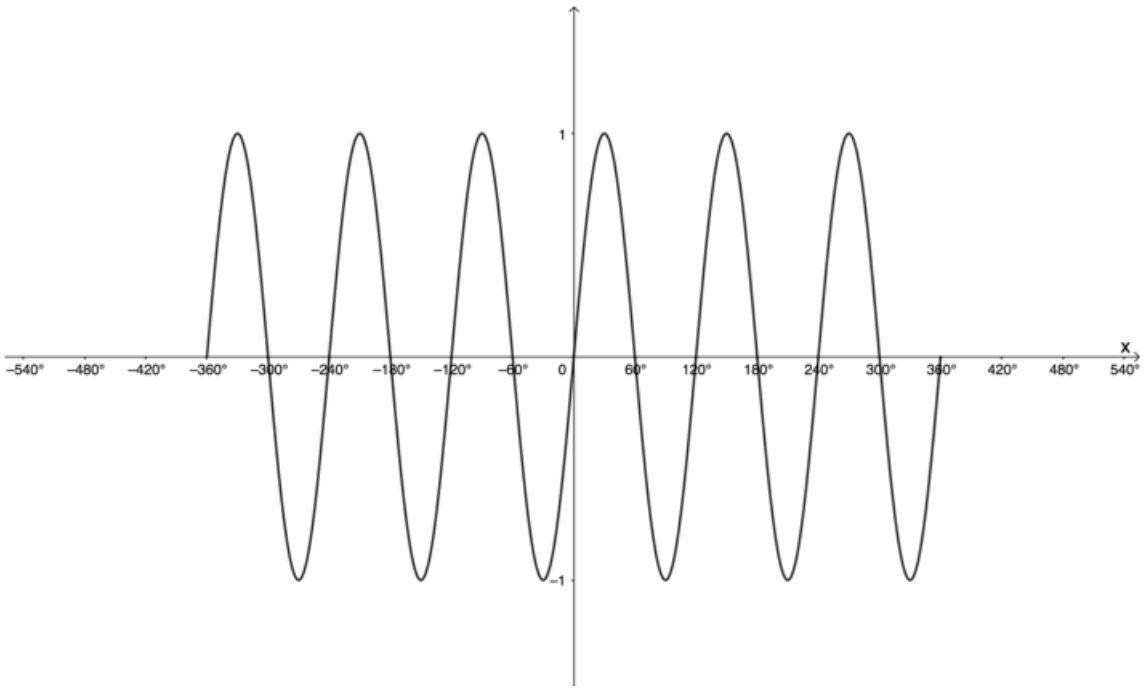
1. domain and range
2. period.

Solution

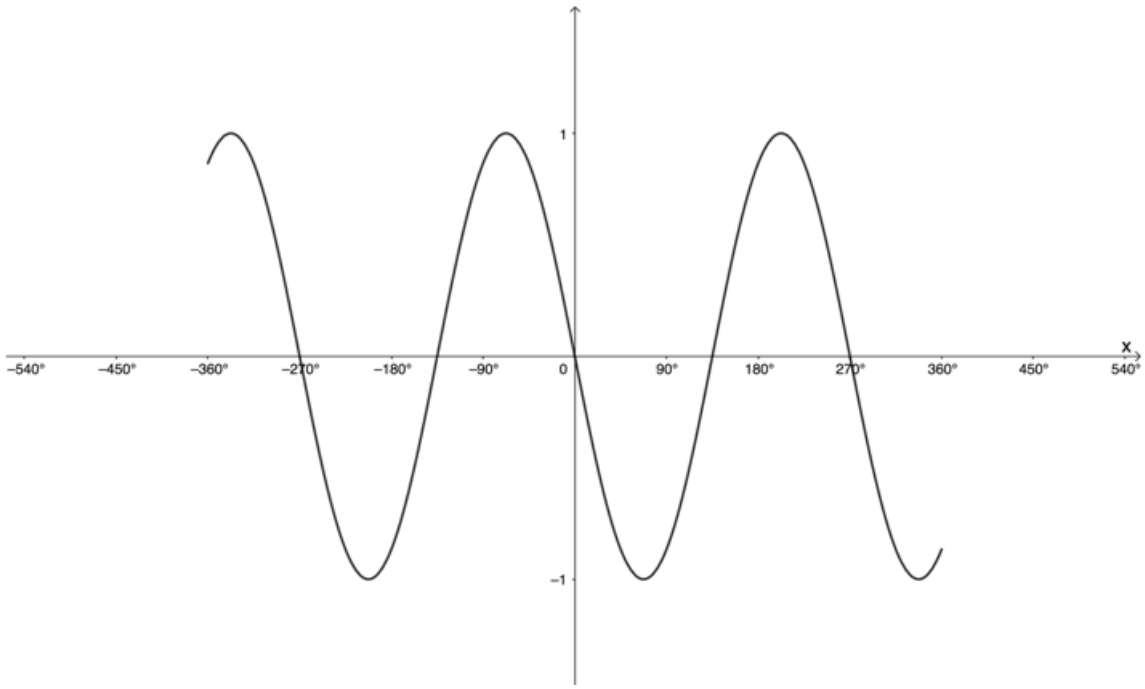
1. Domain: $\{x|x \in \mathbb{R}, 0^\circ \leq x \leq 360^\circ\}$
   Range: $\{y|y \in \mathbb{R}, -1 \leq y \leq 1\}$
2. $k = \frac{3}{2}$. Therefore, the period of the graph is $\frac{360^\circ}{\left|\frac{3}{2}\right|} = \frac{2 \times 360^\circ}{3} = 240^\circ$.

Exercise 7.1

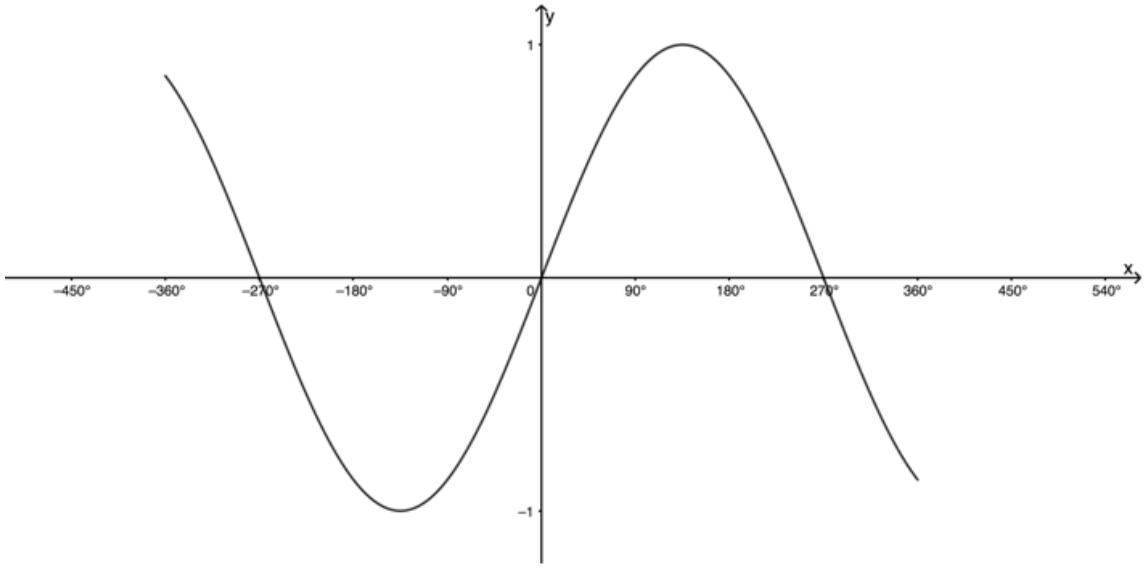
1. Determine the periods of the following functions:
   a. $y = \sin(3x)$
   b. $y = \sin\left(-\frac{1}{2}x\right)$
   c. $y = \sin\left(\frac{7}{3}x\right)$
2. For each function of the form $y = \sin kx$ below, determine the value of k.
   a.



   b.



c.



The full solutions are at the end of the unit.

Sketch functions of the form $y = a \sin kx$

Now that we know that the value of k affects the period of the function $y = \sin kx$, we can combine this with our knowledge of the effects of a and examine functions of the form $y = a \sin kx$.

The effects of a on $y = a \sin kx$ are exactly the same as they were on $y = a \sin x + q$. Before we learn how to sketch functions of the form $y = a \sin kx$, spend some time playing with this online simulation.

Here you will find a graph of the form $y = a \sin kx$ with sliders to change the values of a and k. As you change these values look at the effect this has on the amplitude and/or the period.

Now open this interactive simulation.

Here you will find a graph of the function $y = a \sin kx + q$ for the interval $-360^\circ \leq x \leq 360^\circ$. Pay particular attention to how the amplitude, period and the values of the turning points and intercepts with the axes change as you change the values of a k and q.

The best way to sketch functions of the form $y = a \sin kx$ is to transform the basic function of $y = \sin x$ depending on the values of a and k. To do this, you need to know the set of 'anchor points' of $y = \sin x$, as transformation of these points will help you to sketch functions of the form $y = a \sin kx$.

Take note!

The 'anchor points' for $y = \sin x$ are illustrated in Figure 13 for the interval $-360^\circ \leq x \leq 360^\circ$.

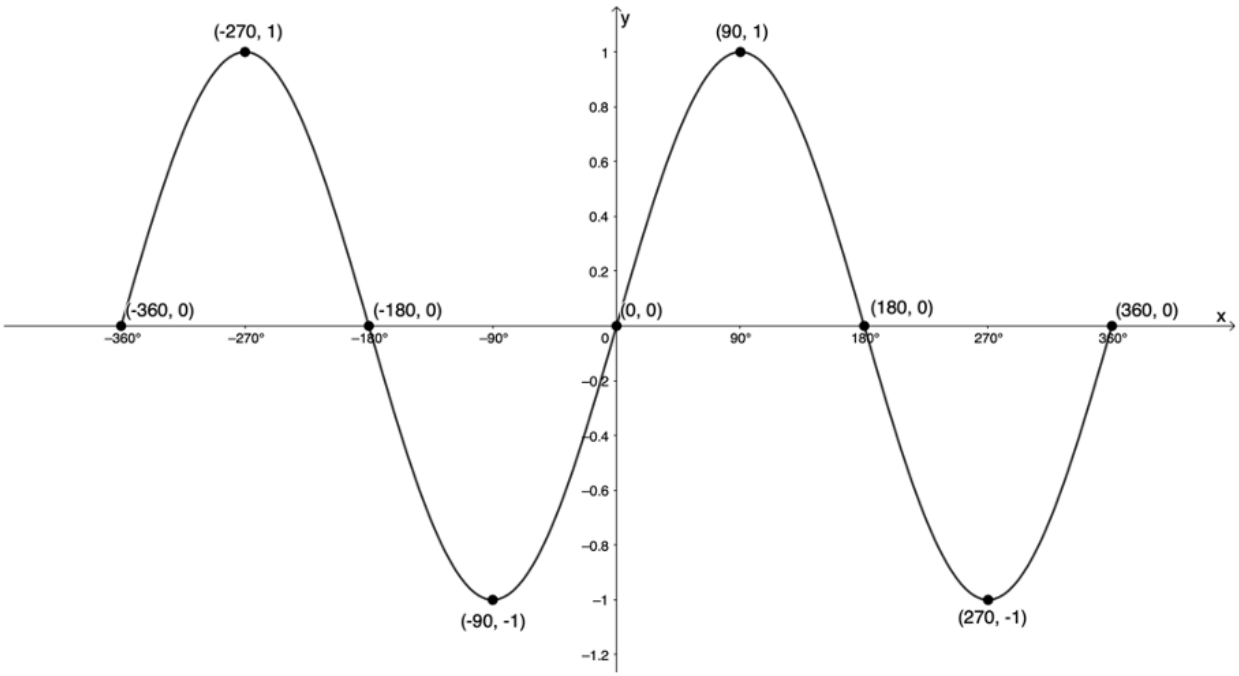


Figure 13: The 'anchor points' of $y = \sin x$

Example 7.2

Sketch the function $y = 2\sin 2x$ for the interval $-180^\circ \leq x \leq 180^\circ$.

Solution

The function is of the form $y = a\sin kx$ with $a = 2$ and $k = 2$.

Because $k = 2$, we know that the period of the function will be $\frac{360^\circ}{|2|} = 180^\circ$. We need to transform one period of 'anchor points' as follows:

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin 2x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$(90^\circ, 0)$	$(135^\circ, -1)$	$(180^\circ, 0)$

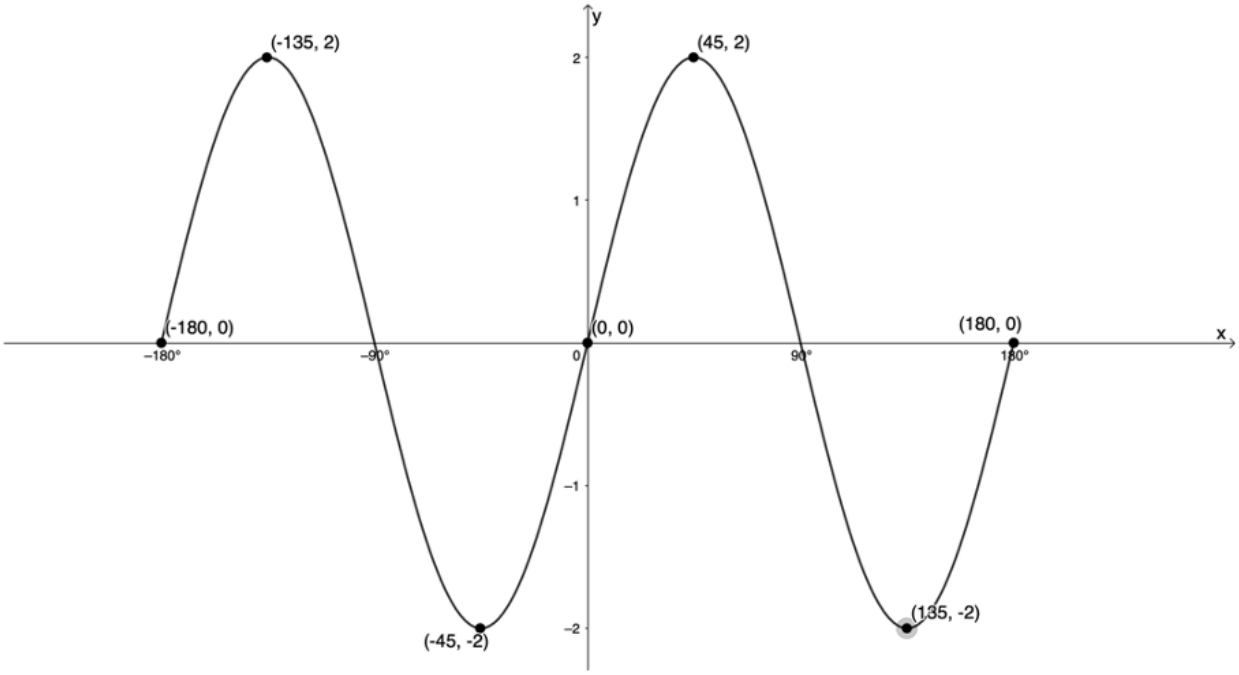
Because the period has been halved, the value of each of the x-coordinates is also halved. We can see that the function will start at zero, rise to its maximum turning point $(45^\circ, 1)$, fall back to zero, fall to its minimum turning point $(135^\circ, -1)$ and rise back to zero in 180°, the new period.

Because $a = 2$, we know that each function value (or y-value) is going to be multiplied by 2 thereby making the amplitude of the function 2. We need to further transform one period of 'anchor points' as follows:

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin 2x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$(90^\circ, 0)$	$(135^\circ, -1)$	$(180^\circ, 0)$
$2\sin 2x$	$(0^\circ, 0)$	$(45^\circ, 2)$	$(90^\circ, 0)$	$(135^\circ, -2)$	$(180^\circ, 0)$

We can see that the function will start at zero, rise to its new maximum turning point $(45^\circ, 2)$, fall back to zero, fall to its new minimum turning point $(135^\circ, -2)$ and rise back to zero in 180°, the new period.

We can now plot our transformed 'anchor points' and draw the graph. We first draw it for the interval $0^\circ \leq x \leq 180^\circ$ and then continue to pattern for the full interval of $-180^\circ \leq x \leq 180^\circ$.



Example 7.3

Given the function $f(x) = 3\sin\left(-\frac{1}{2}x\right)$.

1. Sketch the graph of $f(x)$ for the interval $-360^\circ \leq x \leq 360^\circ$.
2. State the domain and range of $f(x)$.
3. List the turning points of $f(x)$.
4. What is the amplitude of $f(x)$?
5. What is the period of $f(x)$?

Solutions

1. The function is of the form $y = a\sin kx$ Therefore, $a = 3$ and $k = -\frac{1}{2}$.

Because $k = -\frac{1}{2}$, we know that the period of the function will be $\frac{360^\circ}{\left|-\frac{1}{2}\right|} = \frac{360^\circ}{\frac{1}{2}} = 720^\circ$ .But $k < 0$ and we know that $\sin(-x) = -\sin x$. Therefore, we need to multiply the x-values of the 'anchor points' by 2 for the new period and multiply the y-values of the 'anchor points' by -1 to reflect the fact that $k < 0$.

We need to transform one period of 'anchor points' as follows:

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin\left(-\frac{1}{2}x\right)$	$(0^\circ, 0)$	$(180^\circ, -1)$	$(360^\circ, 0)$	$(540^\circ, 1)$	$(720^\circ, 0)$

Because the period has been doubled, the value of each of the x-coordinates is also doubled. We can see that the function will start at zero, drop to its minimum turning point $(180^\circ, -1)$, rise back to zero, rise to its maximum turning point $(540^\circ, -1)$ and fall back to zero in 720°, the new period.

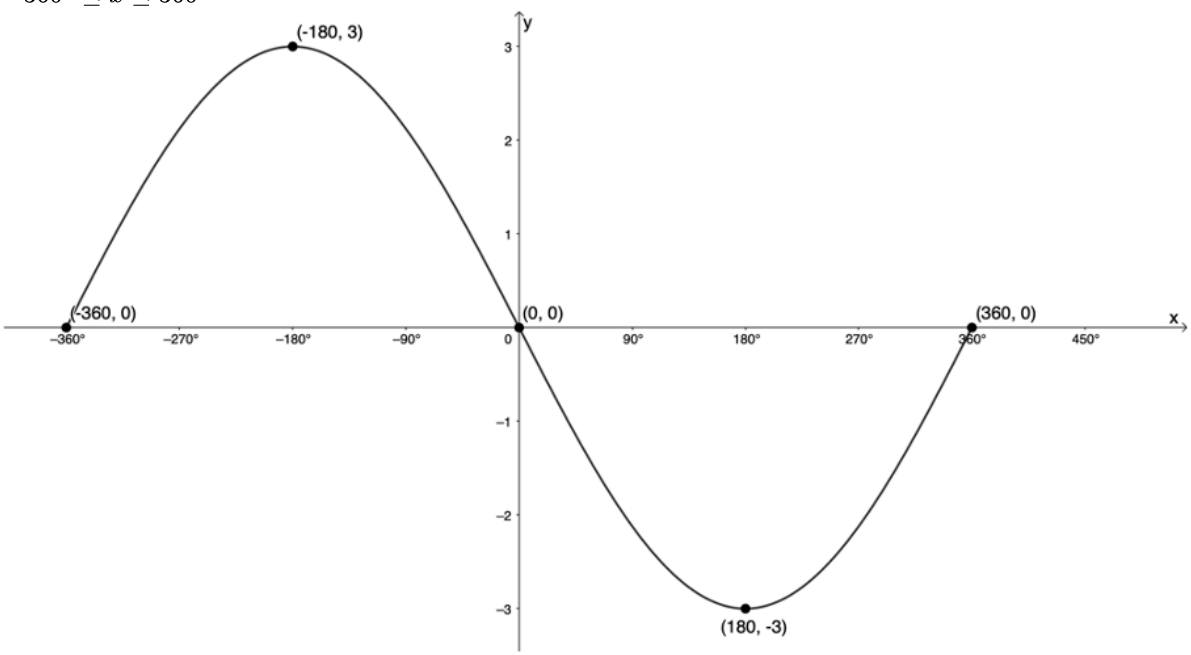
Because $a = 3$, we know that each function value (or y-value) is going to be multiplied by 3 thereby making the amplitude of the function 3. We need to further transform one period of 'anchor points' as follows:

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin\left(-\frac{1}{2}x\right)$	$(0^\circ, 0)$	$(180^\circ, -1)$	$(360^\circ, 0)$	$(540^\circ, 1)$	$(720^\circ, 0)$
$3\sin\left(-\frac{1}{2}x\right)$	$(0^\circ, 0)$	$(180^\circ, -3)$	$(360^\circ, 0)$	$(540^\circ, 3)$	$(720^\circ, 0)$

We can see that the function will start at zero, fall to its new minimum turning point $(180^\circ, -3)$, rise back to zero, rise to its new maximum turning point $(540^\circ, 3)$and fall back to zero in 720°, the new period.

We can now plot our transformed 'anchor points' and draw the graph for the interval

$-360^\circ \leq x \leq 360^\circ$.



2. Domain: $\{x | x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$
   Range: $\{f(x) | f(x) \in \mathbb{R}, -3 \leq f(x) \leq 3\}$
3. Turning points: $(-180^\circ, 3)$ and $(180^\circ, -3)$
4. Amplitude is 3
5. Period is 720°



Exercise 7.2

Sketch the following functions for the indicated intervals:

1. $y = -2 \sin \frac{\theta}{2}$ for $-360^\circ \leq \theta \leq 360^\circ$
2. $y = \sin\left(-\frac{3x}{2}\right)$ for $-180^\circ \leq x \leq 180^\circ$
3. $y = \frac{1}{2} \sin 3\theta$ for $-180^\circ \leq \theta \leq 180^\circ$

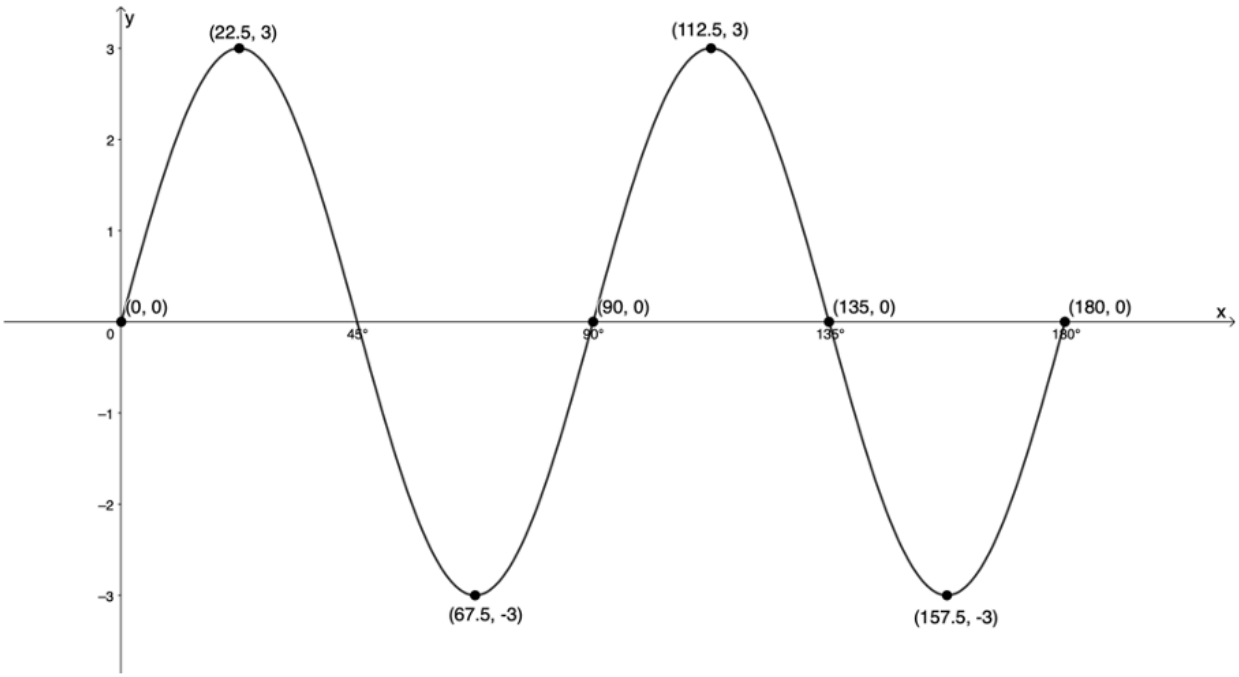
The full solutions are at the end of the unit.

Find the equation of a sine function of the form $y = a \sin kx$

By examining the amplitude and period of a graph of the form $y = a \sin kx$, we can determine the values of a and k.

Example 7.4

Given the following graph of the form $y = a \sin kx$, determine the values of a and k.



Solution

From the graph we can see that the amplitude is 3. We can also see that the graph is not reflected about the x-axis. From zero, it rises to its maximum turning point in the same way as does the graph of $y = \sin x$. Therefore, $a = 3$ and $y = 3 \sin kx$.

The period is 90°.

$$\text{Period} = \frac{360^\circ}{|k|}$$

$$90^\circ = \frac{360^\circ}{|k|}$$

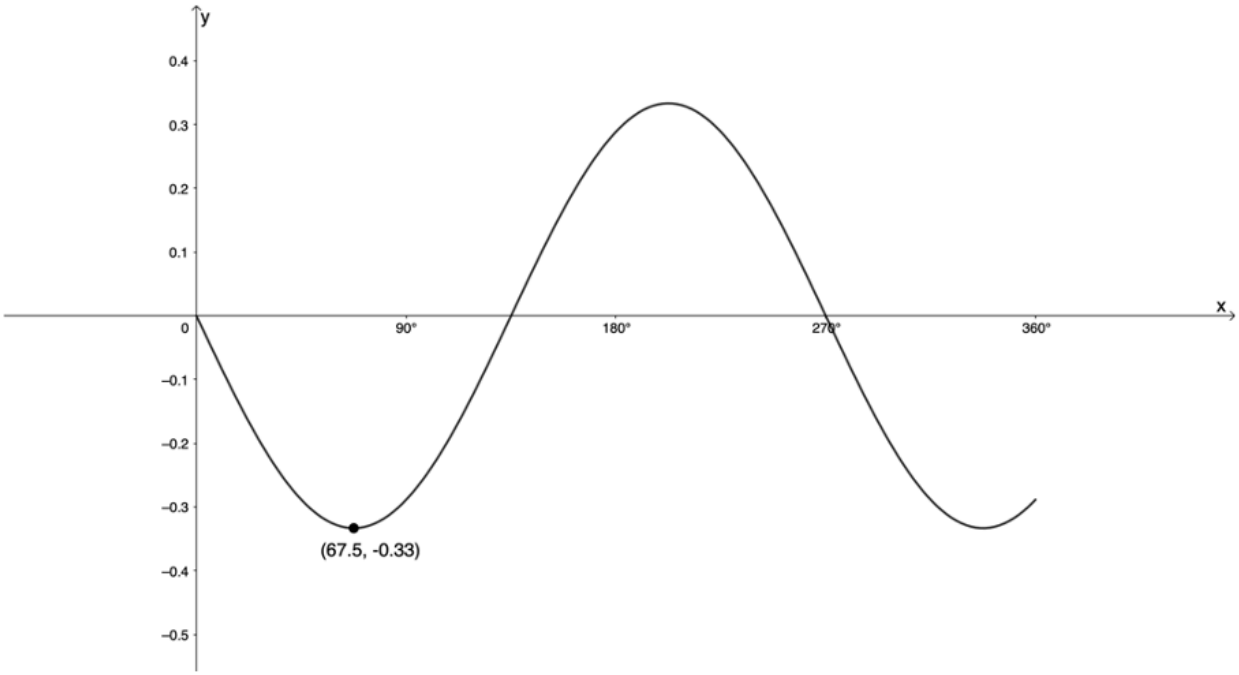
$$\therefore |k| = 4$$

Because the graph is not reflected about the x-axis, $k = 4$.

$y = 3 \sin 4x$

Example 7.5

Given the following graph of the form $y = a \sin kx$, determine the values of a and k.



Solution

From the graph we can see that the amplitude is $\frac{1}{3}$. We can also see that the graph is reflected about the x-axis. From zero, it falls to its minimum turning point unlike the graph of $y = \sin x$, which rises to its maximum turning point. Therefore, $a = -\frac{1}{3}$ and $y = -\frac{1}{3}\sin kx$.

$$\text{Period} = \frac{360^\circ}{|k|}$$

$$270^\circ = \frac{360^\circ}{|k|}$$

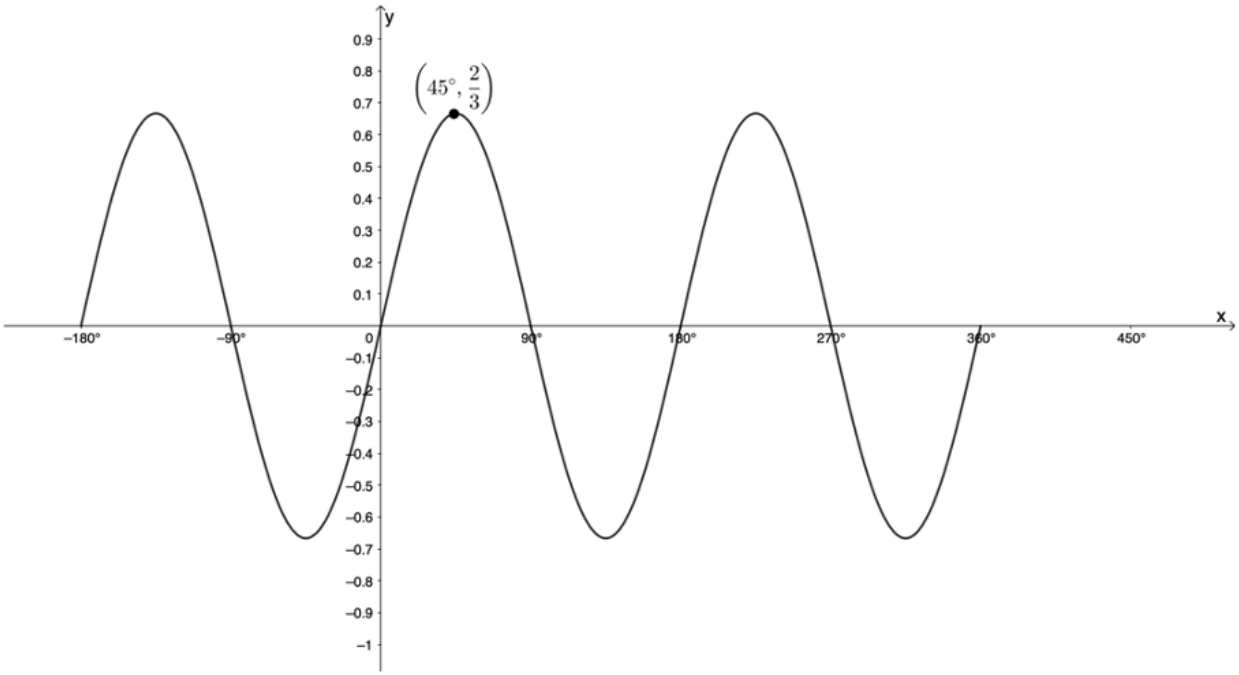
$$\therefore |k| = \frac{360^\circ}{270^\circ} = \frac{4}{3}$$

$$y = -\frac{1}{3}\sin\left(\frac{4}{3}x\right)$$

Note: that we could also assign the negative to the value of k and state that the equation of the function is $y = \frac{1}{3}\sin\left(-\frac{4}{3}x\right)$. Because $-\sin x = \sin(-x)$, $-\frac{1}{3}\sin\left(\frac{4}{3}x\right) = \frac{1}{3}\sin\left(-\frac{4}{3}x\right)$.

Exercise 7.3

Given the graph of the form $y = a\sin kx$:



1. Determine the values of a and k.
2. State the domain and range of the function.

The full solutions are at the end of the unit.

Summary

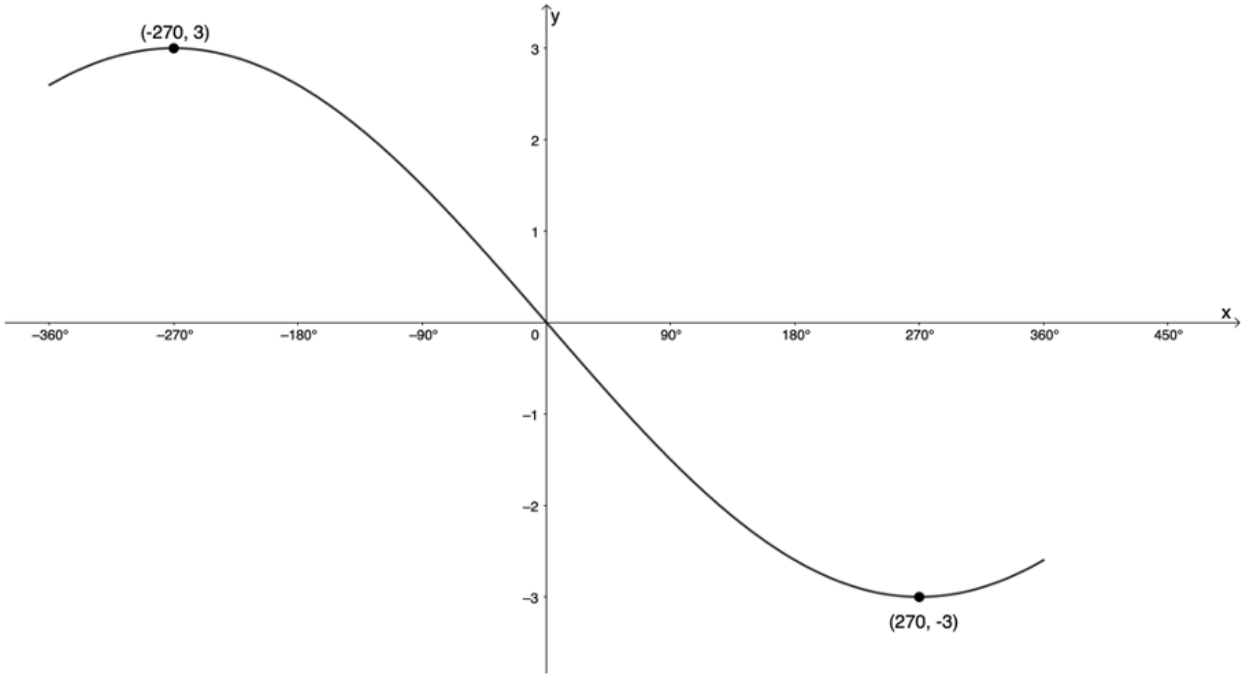
In this unit you have learnt the following:

- The effects of a and k on the sine graph of the form $y = a \sin k\theta$.
- How to sketch functions of the form $y = a \sin k\theta$.
- How to find the values of a and k from a given sine graph of the form $y = a \sin k\theta$.

Unit 7: Assessment

Suggested time to complete: 45 minutes

1. Sketch the following functions for the given intervals:
   a. $3y = \sin\left(-\frac{1}{3}x\right)$ for $-720° \leq x \leq 360°$
   b. $g(x) = -4 \sin 2x$ for $0° \leq x \leq 270°$
2. From the graph below of the form $y = a \sin kx$, determine the values of a and k.



The full solutions are at the end of the unit.

Unit 7: Solutions

Exercise 7.1

1.

a. $y = \sin(3x)$

$k = 3$

$\text{Period} = \dfrac{360^\circ}{|3|} = \dfrac{360^\circ}{3} = 120^\circ$

b. $y = \sin\left(-\dfrac{1}{2}x\right)$

$k = -\dfrac{1}{2}$

$\text{Period} = \dfrac{360^\circ}{\left|-\dfrac{1}{2}\right|} = \dfrac{360^\circ}{\dfrac{1}{2}} = 720^\circ$

c. $y = \sin\left(\dfrac{7}{3}x\right)$

$k = \dfrac{7}{3}$

$\text{Period} = \dfrac{360^\circ}{\left|\dfrac{7}{3}\right|} = \dfrac{3 \times 360^\circ}{7} = 154.29^\circ$

2.

a. The period of the graph is 120°.

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$120^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = 3$$

The graph is not reflected about the x-axis. Therefore, $k > 0$. $k = 3$.

b. The period of the graph is 270°.

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$270^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = \frac{360^\circ}{270^\circ} = \frac{4}{3}$$

The graph is reflected about the x-axis. Therefore, $k < 0$. $k = -\frac{4}{3}$.

c. The period of the graph is 540°.

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$540^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = \frac{360^\circ}{540^\circ} = \frac{2}{3}$$

The graph is not reflected about the x-axis. Therefore, $k > 0$. $k = \frac{2}{3}$.

Back to Exercise 7.1

Exercise 7.2

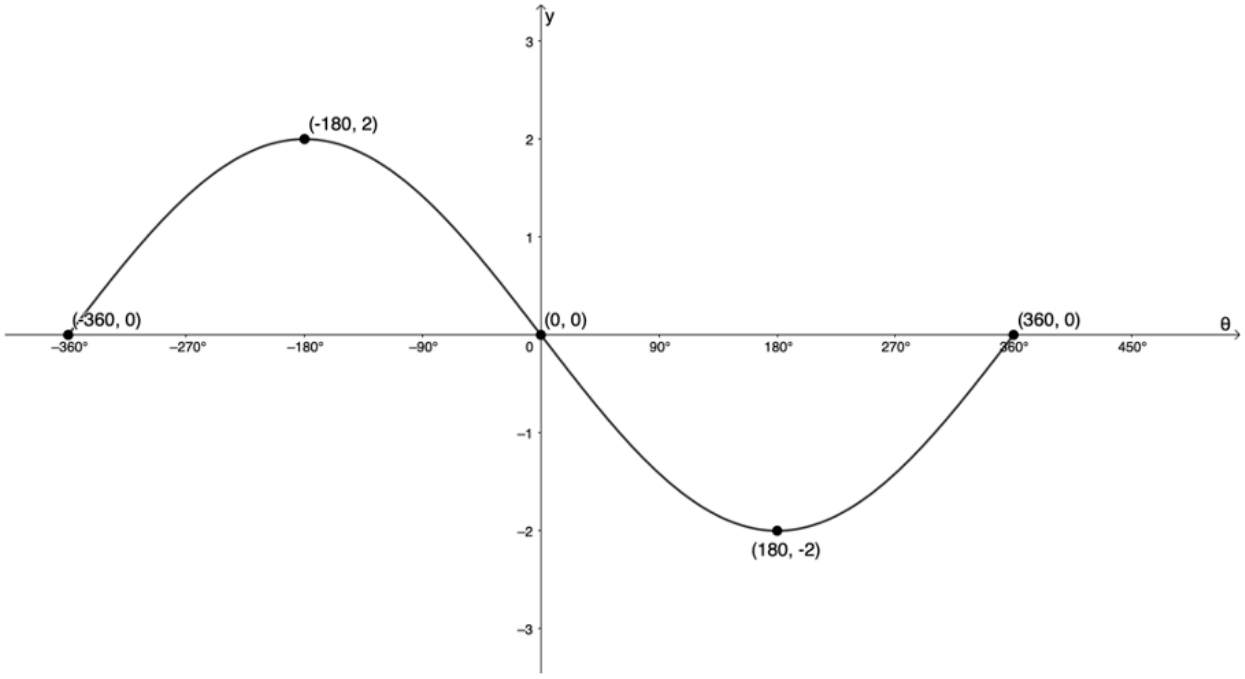
1. $y = -2\sin\frac{\theta}{2}$ for $-360^\circ \leq \theta \leq 360^\circ$

$k = \frac{1}{2}$

$\text{Period} = \frac{360^\circ}{|k|} = \frac{360^\circ}{\left|\frac{1}{2}\right|} = 720^\circ$

$a = -2$. Amplitude is 2 and graph is reflected about the x-axis.

$\sin\theta$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$-2\sin\left(\frac{\theta}{2}\right)$	$(0^\circ, 0)$	$(180^\circ, -2)$	$(360^\circ, 0)$	$(540^\circ, 2)$	$(720^\circ, 0)$



2. $y = \sin\left(-\frac{3x}{2}\right)$ for $-180^\circ \leq x \leq 180^\circ$

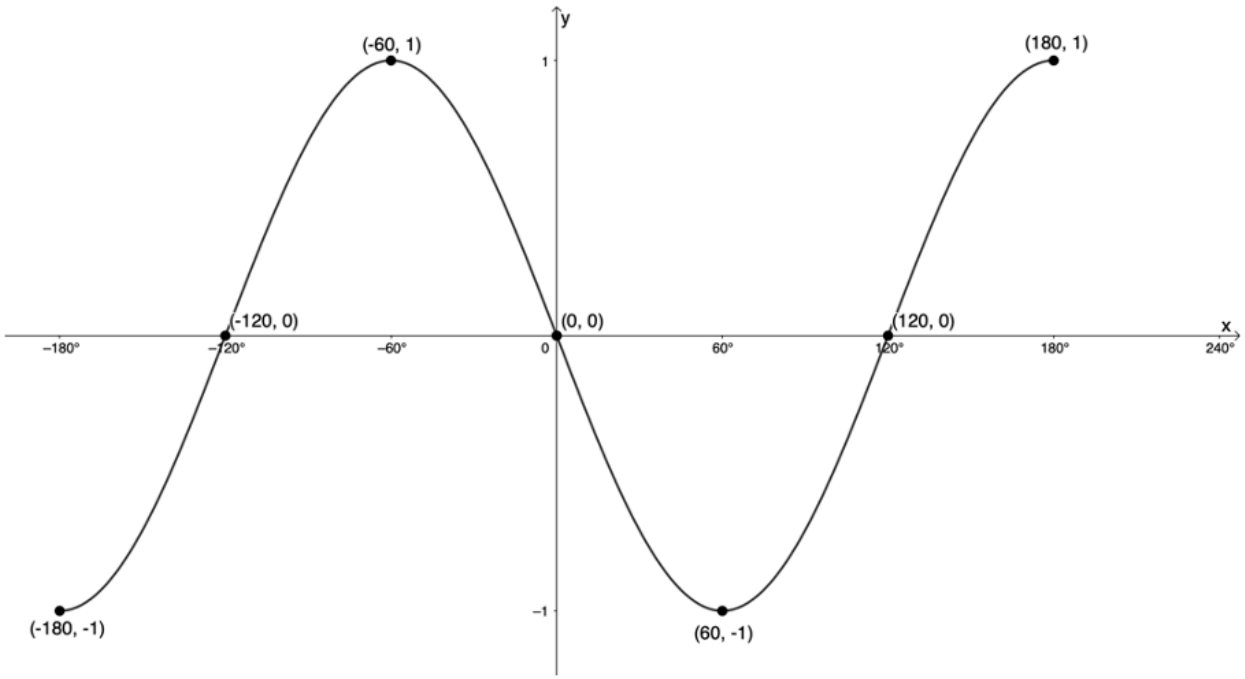
   $k = -\frac{3}{2}$

   $\text{Period} = \frac{360^\circ}{|k|} = \frac{360^\circ}{\left|-\frac{3}{2}\right|} = \frac{360^\circ}{\frac{3}{2}} = 240^\circ$

   $k < 0$. Therefore, graph is reflected about the x-axis.

   $a = 1$. Amplitude is 1.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin\left(-\frac{3}{2}x\right)$	$(0^\circ, 0)$	$(60^\circ, -1)$	$(120^\circ, 0)$	$(180^\circ, 1)$	$(240^\circ, 0)$



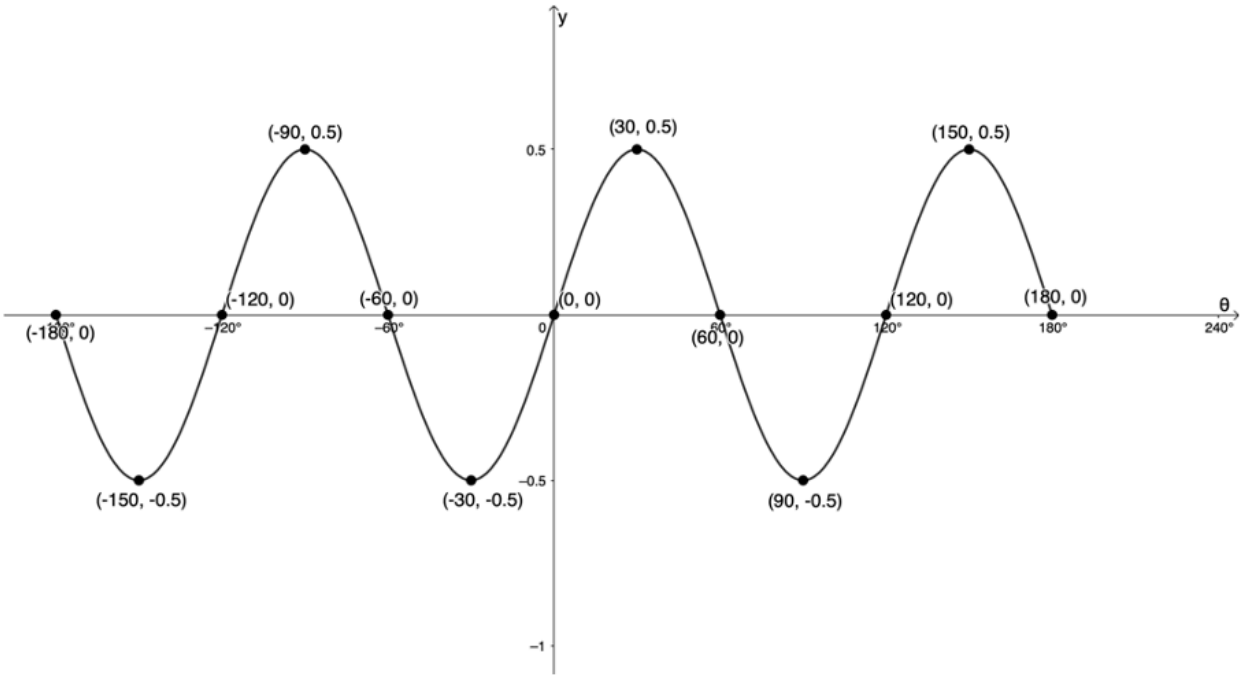
3. $y = \frac{1}{2}\sin 3\theta$ for $-180^\circ \leq x \leq 180^\circ$

   $k = 3$

   $\text{Period} = \frac{360^\circ}{|k|} = \frac{360^\circ}{|3|} = 120^\circ$

   $a = \frac{1}{2}$. Amplitude is $\frac{1}{2}$.

$\sin\theta$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\frac{1}{2}\sin(3\theta)$	$(0^\circ, 0)$	$(30^\circ, \frac{1}{2})$	$(60^\circ, 0)$	$(90^\circ, -\frac{1}{2})$	$(120^\circ, 0)$



Back to Exercise 7.2

Exercise 7.3

1. The amplitude of the function is $\frac{2}{3}$ and graph is not reflected about the x-axis. Therefore, $a = \frac{2}{3}$ and $y = \frac{2}{3}\sin kx$.

   The period is 180°. Therefore $k = 2$.

   $y = \frac{2}{3}\sin 2x$

2. Domain: $\{x | x \in \mathbb{R}, \ -180^\circ \leq x \leq 360^\circ\}$

   Range: $\{y | y \in \mathbb{R}, \ -\frac{2}{3} \leq x \leq \frac{2}{3}\}$

Back to Exercise 7.3

Unit 7: Assessment

1.

a. $3y = \sin\left(-\frac{1}{3}x\right)$ for $-720^\circ \leq x \leq 360^\circ$

$3y = \sin\left(-\frac{1}{3}x\right)$

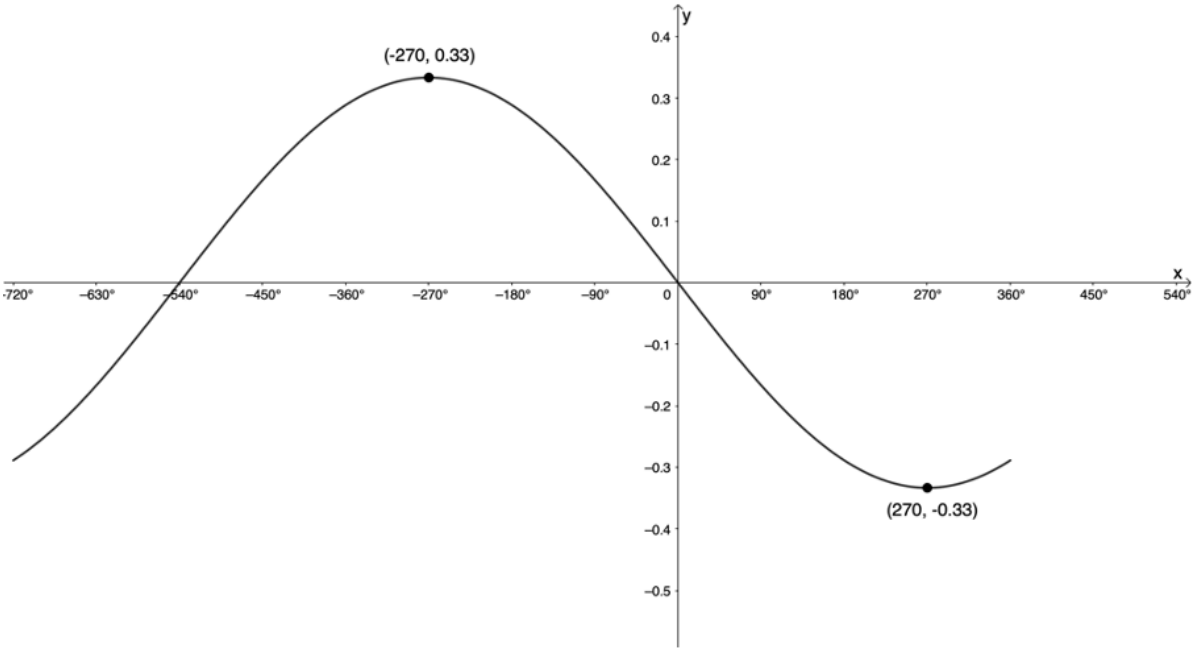
$\therefore y = \frac{1}{3}\sin\left(-\frac{1}{3}x\right)$

$k = -\frac{1}{3}$. Therefore, $\text{period} = \frac{360^\circ}{\left|\frac{1}{3}\right|} = 1\ 080^\circ$.

$k < 0$. Therefore, the graph is reflected about the x-axis.

$a = \frac{1}{3}$. Therefore, amplitude is $\frac{1}{3}$.

$\sin\theta$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\frac{1}{3}\sin\left(-\frac{1}{3}x\right)$	$(0^\circ, 0)$	$(270^\circ, -\frac{1}{3})$	$(540^\circ, 0)$	$(810^\circ, \frac{1}{3})$	$(1\ 080^\circ, 0)$

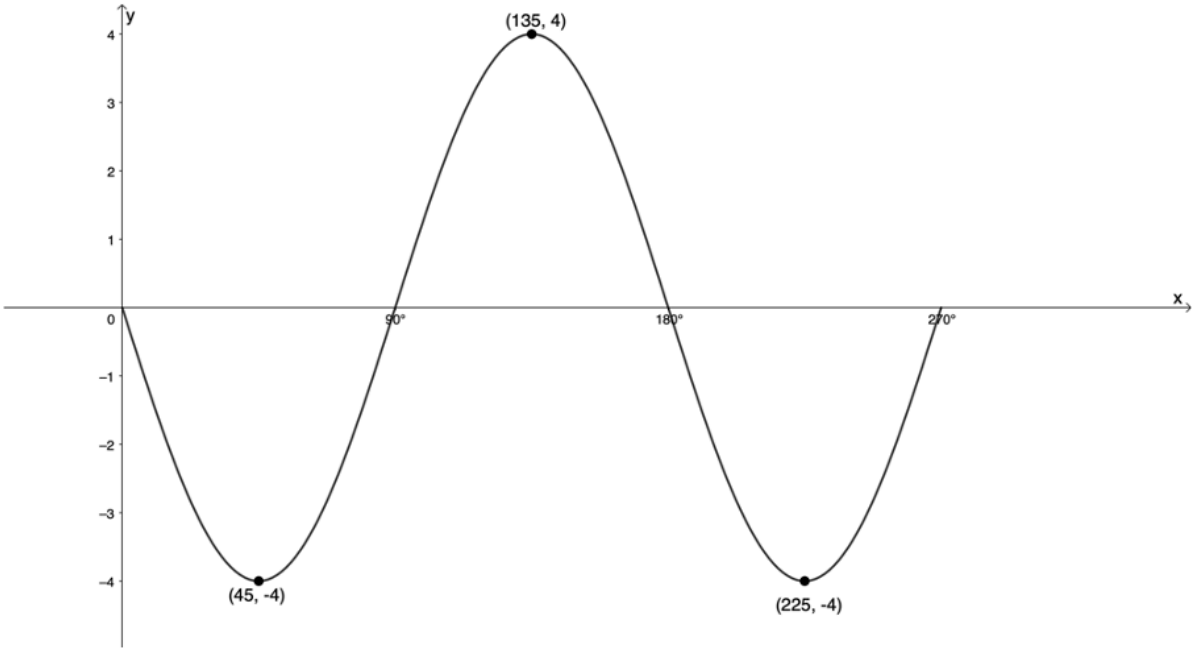


b. $g(x) = -4\sin 2x$ for $0^\circ \leq x \leq 270^\circ$

$k = 2$. Therefore, $\text{period} = \frac{360^\circ}{|2|} = 180^\circ$.

$a = -4$. Therefore, amplitude is 4 and the graph is reflected about the x-axis.

$\sin\theta$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$-4\sin 2x$	$(0^\circ, 0)$	$(45^\circ, -4)$	$(90^\circ, 0)$	$(135^\circ, 4)$	$(180^\circ, 0)$



2. The amplitude is 3 and the graph is reflected about the x-axis. Therefore, $a = -3$ and $y = -3 \sin kx$.
The graph completes half a period between -270° and 270°. Therefore, the period is $2 \times 540^\circ = 1\ 080^\circ$.

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$1\ 080^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = \frac{360^\circ}{1\ 080^\circ} = \frac{1}{3}$$
$$y = -3 \sin\left(\frac{1}{3}x\right)$$

Back to Unit 7: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure6 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure7 © DHET is licensed under a CC BY (Attribution) license
- figure8 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure9 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure10 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure11 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure12 © DHET is licensed under a CC BY (Attribution) license
- exercise7.1Q2a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.1Q2b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.1Q2c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure13 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example7.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example7.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example7.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example7.5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.2A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.2A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 8: Horizontal transformation of the sine graph

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = \sin(\theta + p)$.
- Determine the effects of positive and negative values of p on the sine graph $y = \sin(\theta + p)$.
- Find the value of p from a given sine graph of the form $y = \sin(\theta + p)$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \sin x$ and $y = \sin \theta$ are the same function.

What you should know

Before you start this unit, make sure you can:

- Sketch sine functions of the form $y = a \sin \theta + q$. Refer to level 2 subject outcome 2.1 Unit 5 if you need help with this.

Introduction

So far in this subject outcome, we have been investigating the effects of the values of a, p and q on various functions. For example, for the functions $y = a(x + p)^2 + q$, $y = \frac{a}{(x + p)} + q$ and $y = a.b^{x+p} + q$ we have discovered that q is responsible for a vertical shift of the graph of the function while p is responsible for a horizontal shift such that:

- if $q > 0$, the graph shifts q units **up**
- if $q < 0$, the graph shifts q units **down**
- if $p > 0$, the graph shifts q units to the **left**
- if $p < 0$, the graph shifts q units to the **right**.

Why q results in a vertical shift

If we consider the basic quadratic function, we can see why this is the case. Consider $f(x) = x^2$. If we add a constant of q to the equation such that it becomes $f(x) = x^2 + q$, we can see that we will be adding this constant to the value of $f(x)$ (the y-value) for every value of x we feed into the function. In other words, if $f(x) = x^2$, then $x^2 + q = f(x) + q$. If $q = 2$, for example, we will add this same positive constant to each function and shift the whole graph 2 units up (see Figure 1).

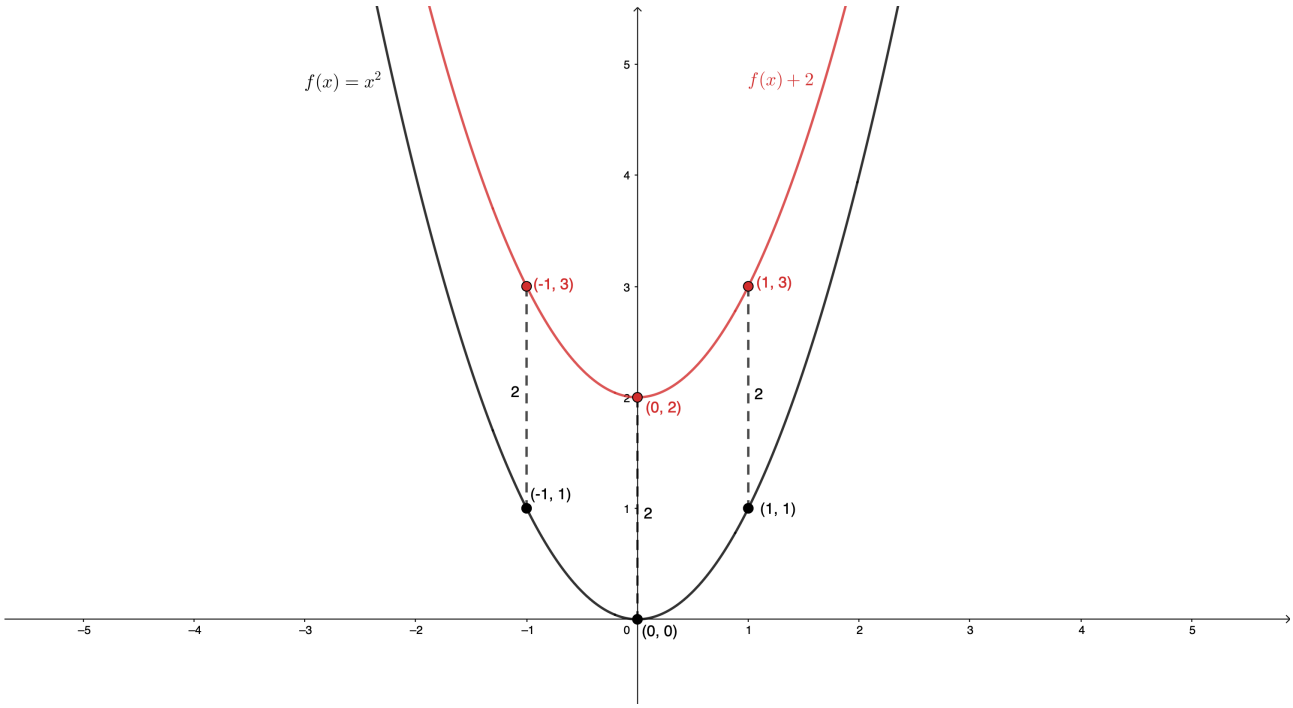


Figure 1: The effect of q on the function $f(x) = x^2 + q$

Why p results in a horizontal shift

Now let's consider the role of p. Again, let $f(x) = x^2$. When $x = 2$ the function value is $f(2) = 2^2 = 4$. When $x = 4$, the function value is $f(4) = 4^2 = 16$. But if we replaced x with $(x - 2)$ in the function, then the function value when $x = 4$ would be $f(4 - 2) = f(2) = 2^2 = 4$. In other words, the function value that used to occur when $x = 2$ now occurs when $x = 4$. The graph has been shifted two units to the right.

Therefore, replacing x with $(x + p)$ shifts the graph horizontally by $-p$ units. If $p < 0$, the graph shifts p units to the right. If $p > 0$, the graph shifts p units to the left.

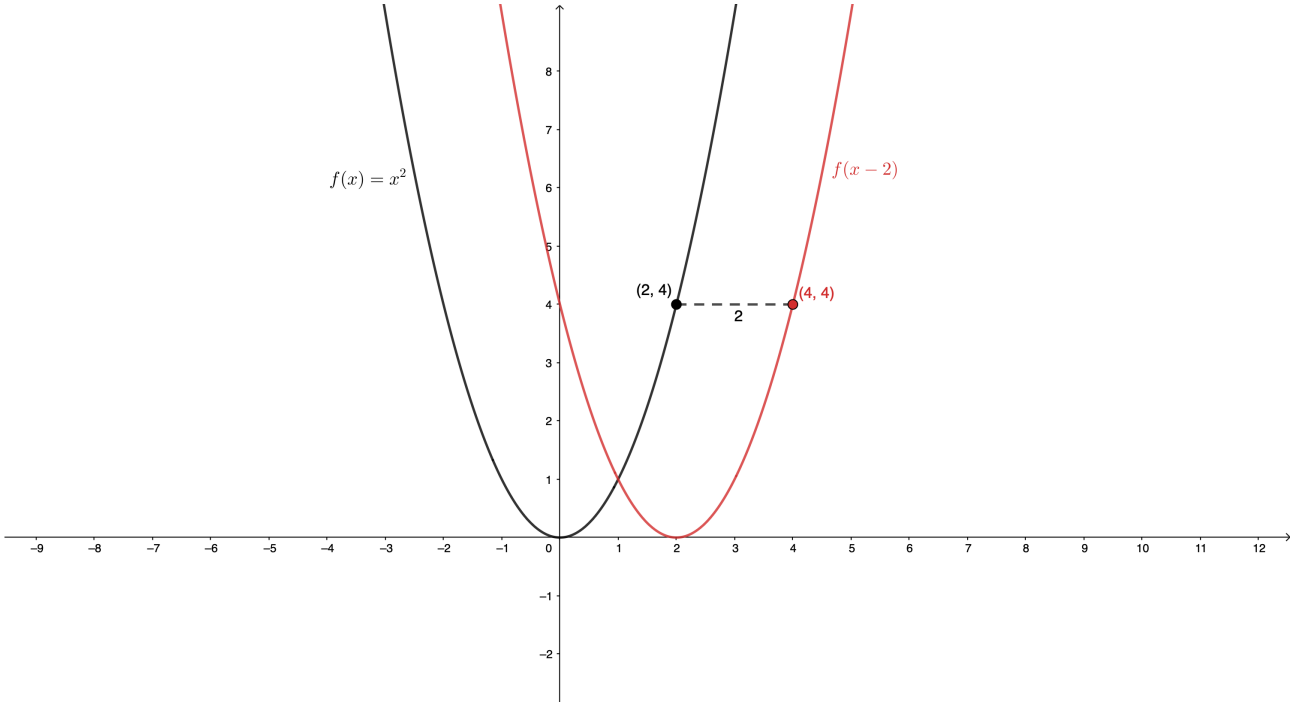


Figure 2: The effect of p on the function $f(x + p)^2$

The sine function of the form $y = \sin(x + p)$

By looking at the effects of p on the quadratic function above, you should be able to predict the effect that p will have on the graph of $y = \sin(x + p)$. But let's investigate.

Activity 8.1: Shifts?

Time required: 40 minutes

What you need:

- a pen or pencil
- a calculator
- several pieces of paper

What to do:

1. Start by making a sketch of $y = \sin x$ for the interval $0° \leq x \leq 360°$ on your piece of paper. If you need to, you can complete the following table of values first.

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin x$					

2. Now complete the following tables of values and sketch the additional functions on the same set of axes for the interval $0° \leq x \leq 360°$.

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin(x + 30°)$					

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin(x - 60°)$					

3. Using your sketches, describe the effect of p on the graph of the function $y = \sin(x + p)$.

What did you find?

1. Below is the completed table of values and the sketch of $y = \sin x,\ 0° \leq x \leq 360°$ (see Figure 3).

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin x$	0	1	0	-1	0

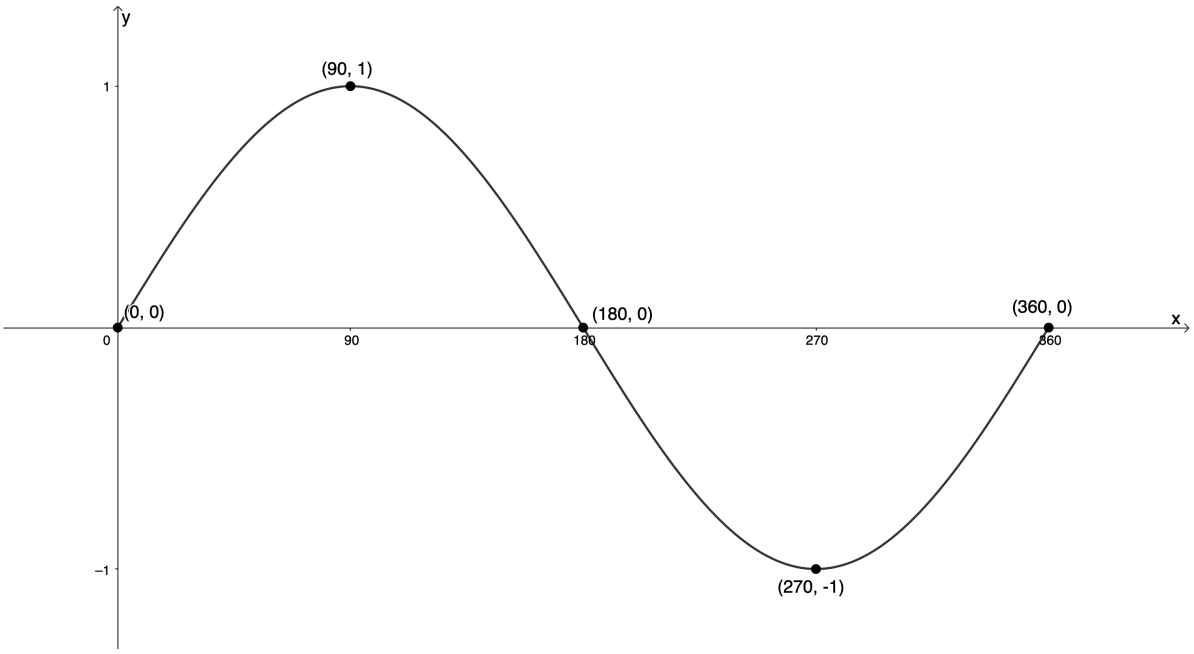


Figure 3: Graph of $y = \sin x,\ 0^\circ \leq x \leq 360^\circ$

2. Here are the completed tables of values for the other functions.

x	0°	90°	180°	270°	360°
$\sin(x + 30^\circ)$	$\frac{1}{2}$	0.866	$-\frac{1}{2}$	-0.866	$\frac{1}{2}$

x	0°	90°	180°	270°	360°
$\sin(x - 60^\circ)$	-0.866	$\frac{1}{2}$	0.866	$-\frac{1}{2}$	-0.866

Below are the graphs of $y = \sin x$, $y = \sin(x + 30^\circ)$ and $y = \sin(x - 60^\circ)$ (see Figure 4).

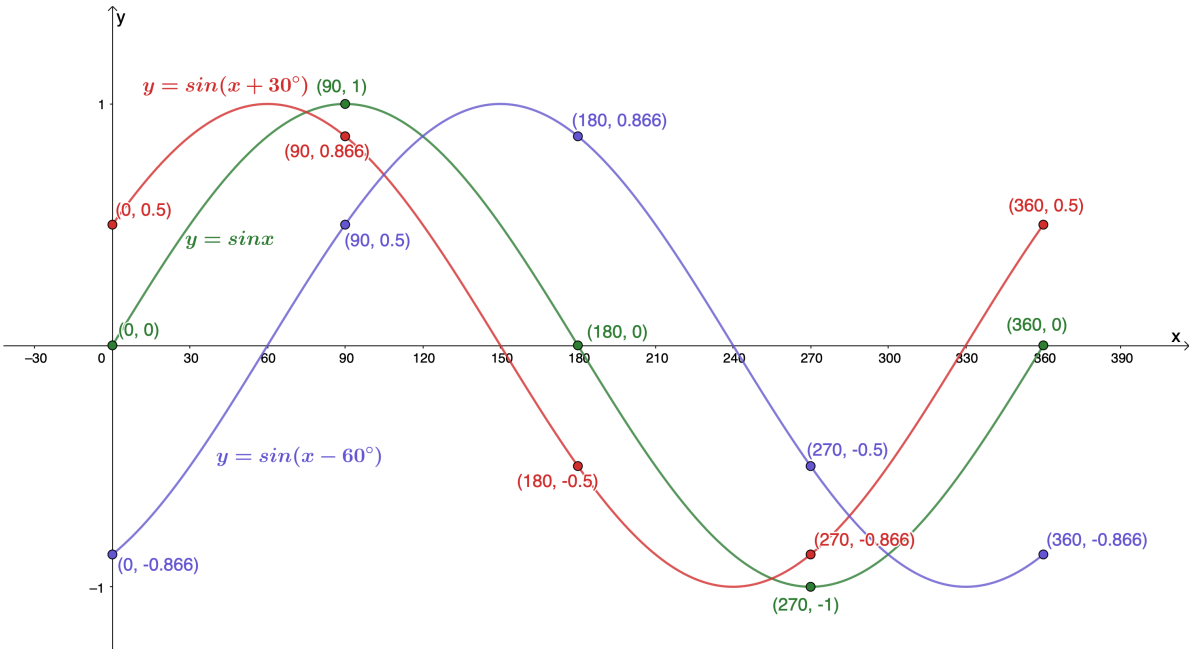


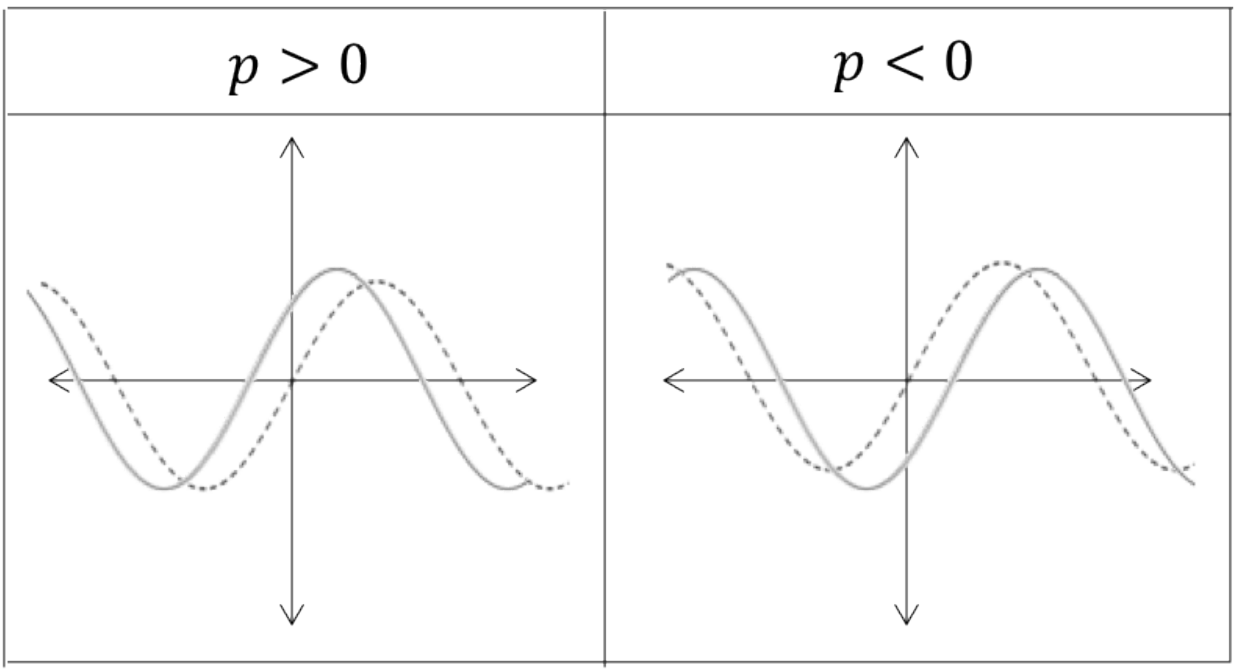
Figure 4: Graphs of $y = \sin x,\ y = \sin(x + 30°),\ y = \sin(x - 60°),\ 0° \leq x \leq 360°$

3. We can see from the sketches that the value of p in $y = \sin(x + p)$ results in a horizontal shift of the graph left or right by p units. In the case of $y = \sin(x + 30°)$ the graph is shifted $30°$ **to the left**. In the case of $y = \sin(x - 60°)$, the graph is shifted $60°$ **to the right**. We can see this by looking at the maximum and minimum turning points of the graphs as well as the x-intercepts

Take note!

In $y = \sin(x + p)$:

- if $p > 0$, the graph is shifted p units **to the left**
- if $p < 0$, the graph is shifted p units **to the right**.



Note

If you have an internet connection, spend some time playing with this interactive simulation.

Here you will find a graph of the function $y = \sin(x + p)$ with a slider to change the value of p. Pay particular attention to how changing the value of p affects the location of the turning points and the intercepts with the x-axis.

Sketch functions of the form $y = \sin(x + p)$

The best way to sketch functions of the form $y = \sin(x + p)$ is to transform the basic function of $y = \sin x$ depending on the value of p. To do this, you need to know the set of 'anchor points' of $y = \sin x$, as transformation of these points will help you to sketch functions of the form$y = \sin(x + p)$.

Example 8.1

Given the function $y = \sin(x - 45^\circ)$:

1. Sketch the function for the interval $0^\circ \leq x \leq 360^\circ$.
2. State the domain and range.
3. State the period.
4. State the amplitude.

Solution

1. The function is of the form $y = \sin(x + p)$ with, $p = -45^\circ$.
   We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 45° to the right.

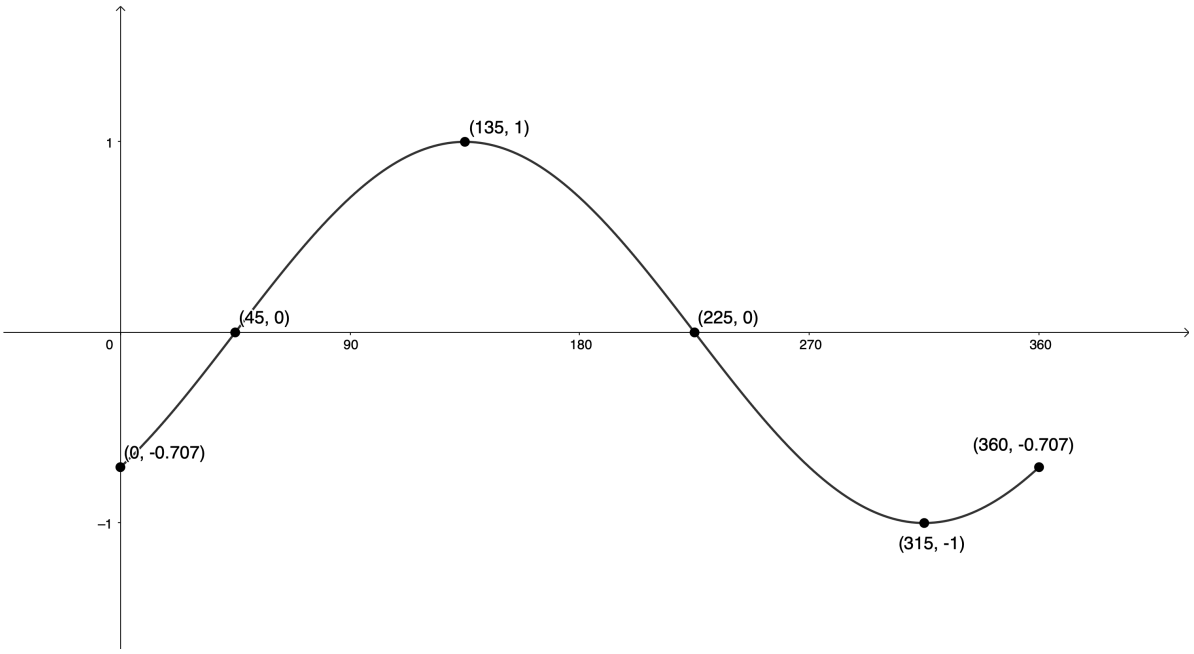
$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x - 45^\circ)$	$(45^\circ, 0)$	$(135^\circ, 1)$	$(225^\circ, 0)$	$(315^\circ, -1)$	$(405^\circ, 0)$

   To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.

   y-intercept (let $x = 0$):
   $y = \sin(0^\circ - 45^\circ)$
   $\therefore y = -0.707$

   Because the period of the function is 360° we know that the function value at 360° will also be -0.707.

   We can now plot our transformed 'anchor points' and draw the graph for the interval $0^\circ \leq x \leq 360^\circ$.



2. Domain: $\{x | x \in \mathbb{R}, 0^\circ \leq x \leq 360^\circ\}$

Range: $\{y|y \in \mathbb{R}, -1 \leq x \leq 1\}$

3. The period is 360°.
4. The amplitude is 1.

Take note!

In general, the domain of the function $y = \sin(x + p)$ is $x \in \mathbb{R}$. We had to restrict the domain in Example 8.1, because the interval in which we were working was restricted.

Exercise 8.1

Sketch the following functions for the indicated intervals on separate sets of axes:

1. $f(x) = \sin(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
2. $g(x) = \sin(x - 45^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
3. $h(x) = \sin(x + 20^\circ)$ for $-360^\circ \leq x \leq 360^\circ$

The full solutions are at the end of the unit.

Sketch functions of the form $y = a\sin(x + p)$

Now that we know that the value of p shifts the function of the form $y = \sin(x + p)$ horizontally by p degrees, we can combine our knowledge of the effects of a with this and examine functions of the form $y = a\sin(x + p)$.

Example 8.2

Sketch the graph of $f(x) = 2\sin(x + 25^\circ)$ for the interval $-360^\circ \leq x \leq 360^\circ$.

Solution

The function is of the form $y = a\sin(x + p)$ where $a = 2$ and $p = 25^\circ$.

We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 25° to the left and each of the y-values is going to be multiplied by 2.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$2\sin(x + 25^\circ)$	$(-25^\circ, 0)$	$(65^\circ, 2)$	$(155^\circ, 0)$	$(245^\circ, -2)$	$(335^\circ, 0)$

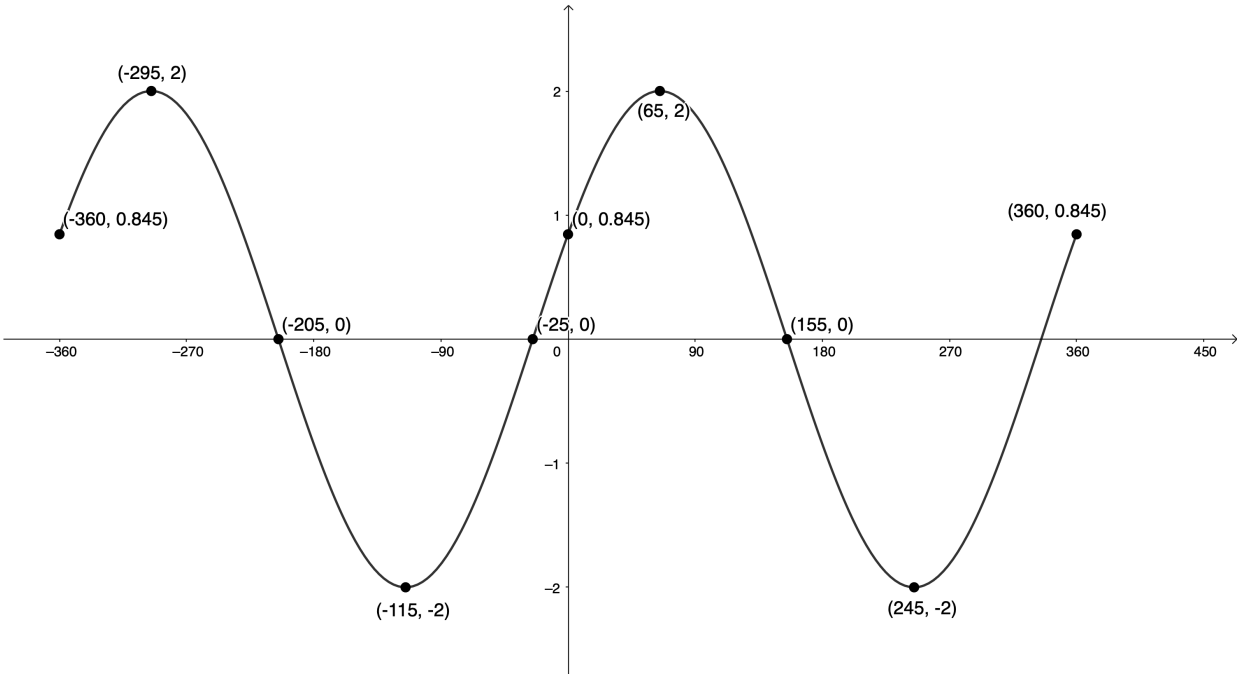
To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.

y-intercept (let $x = 0$):

$$y = 2\sin(0 + 25^\circ)$$
$$\therefore y = 0.845$$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 0.845.

We can now plot our transformed 'anchor points' and draw the graph for the interval $-360^\circ \leq x \leq 360^\circ$.



Example 8.3

Given the function $f(x) = \sin(30^\circ - x)$:

1. Sketch the graph for the interval $0^\circ \leq x \leq 360^\circ$.
2. State the intercepts with the axes.
3. State the domain and range.
4. State the amplitude and period.

Solutions

1. The function is not in the form $y = \sin(x + p)$. We first need to get it into this form.

$$\begin{aligned} f(x) &= \sin(30^\circ - x) \\ &= \sin(-x + 30^\circ) \\ &= \sin(-(x - 30^\circ)) \\ &= -\sin(x - 30^\circ) \end{aligned}$$

We know from the previous unit that $\sin(-x) = -\sin(x)$. In other words, the graph of $f(x) = \sin(30^\circ - x)$ is going to be a reflection of the standard sine graph about the x-axis because $a = -1$.

We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 30° to the right and each of the y-values of these points needs to be multiplied by -1 .

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x - 30^\circ)$	$(30^\circ, 0)$	$(120^\circ, -1)$	$(210^\circ, 0)$	$(300^\circ, 1)$	$(390^\circ, 0)$

To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.

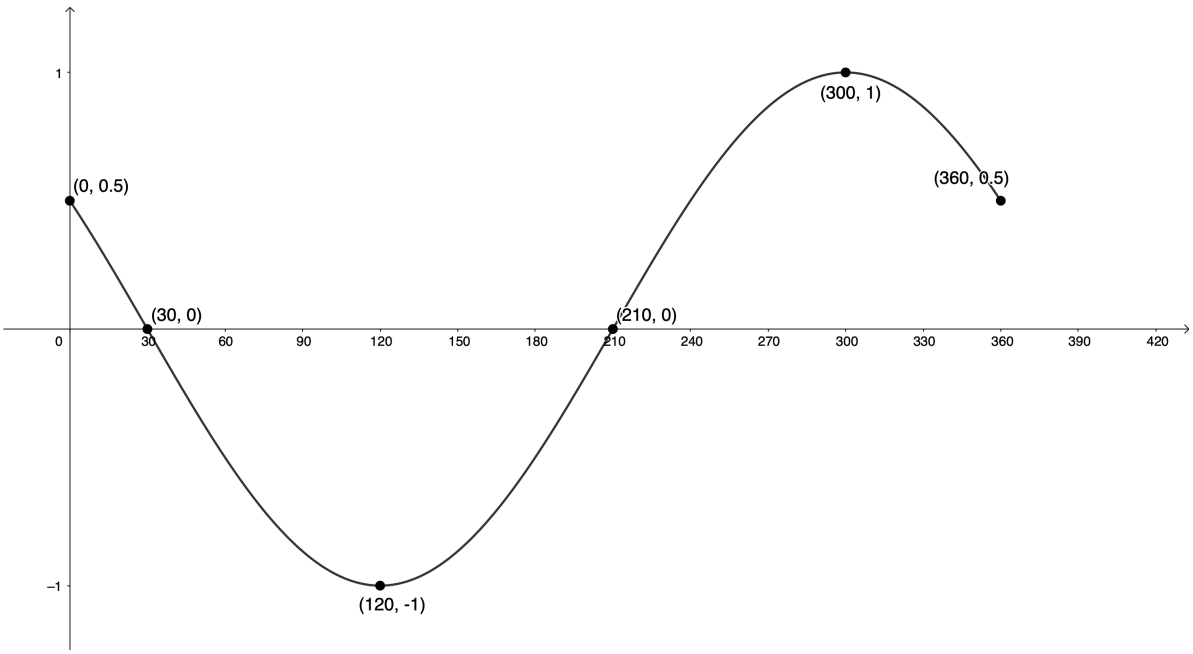
y-intercept (let $x = 0$):

$$f(0) = \sin(30^\circ - 0)$$

$$\therefore f(0) = \frac{1}{2}$$

Because the period of the function is 360° we know that the function value at 360° will also be $\frac{1}{2}$.

We can now plot our transformed 'anchor points' and draw the graph for the interval $0^\circ \leq x \leq 360^\circ$.



2. y-intercept: $(0^\circ, \frac{1}{2})$
   x-intercepts: $(30^\circ, 0)$ and $(210^\circ, 0)$
3. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$

Range: $\{y|y \in \mathbb{R}, \ -1 \leq x \leq 1\}$

4. The period is 360°.
   The amplitude is 1.



Exercise 8.2

For each of the following functions, sketch the graph for the indicated interval and state the domain and range:

1. $y = -2\sin(x + 30^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
2. $2y = \sin(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
3. $\frac{1}{3}y = \sin(60^\circ - x)$ for $-360^\circ \leq x \leq 360^\circ$

The full solutions are at the end of the unit.

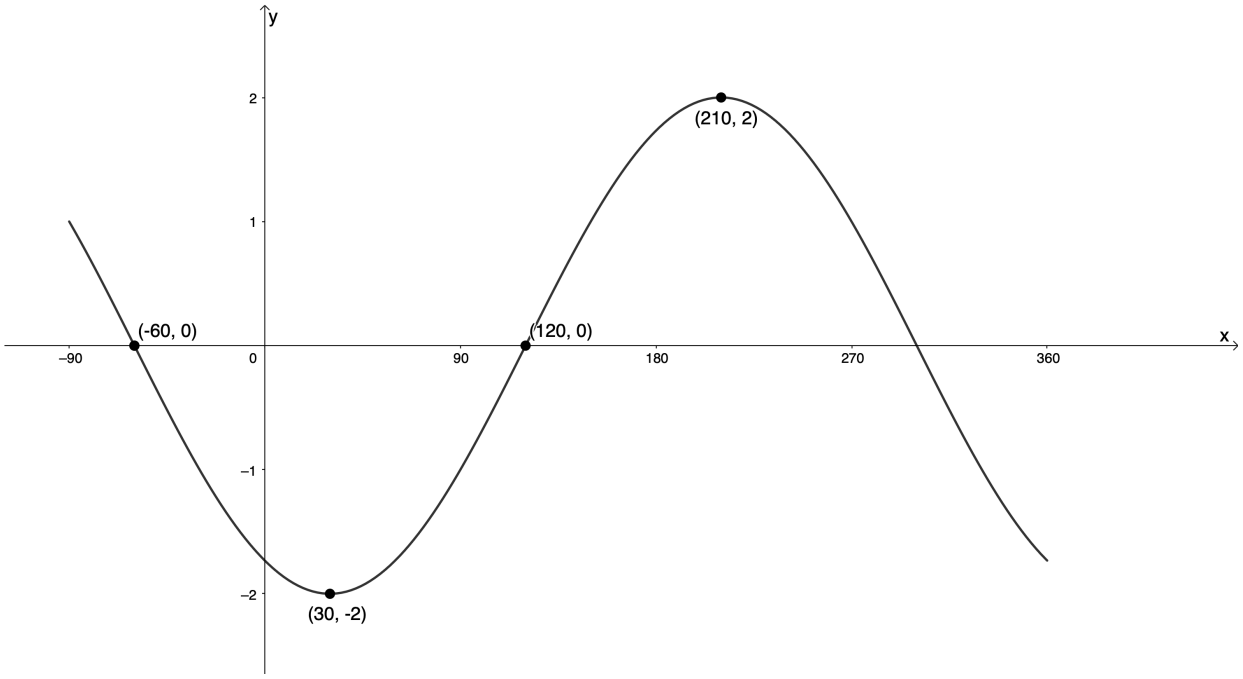
Find the equation of a sine function of the form $y = a\sin(x + p)$

If we examine the amplitude and the horizontal shift of $y = a\sin(x + p)$, we can determine the values of a and p.



Example 8.4

The graph below is a function of the form $y = a\sin(x + p)$. Determine the values of a and p.



Solution

We are told that the function is of the form $y = a\sin(x + p)$.

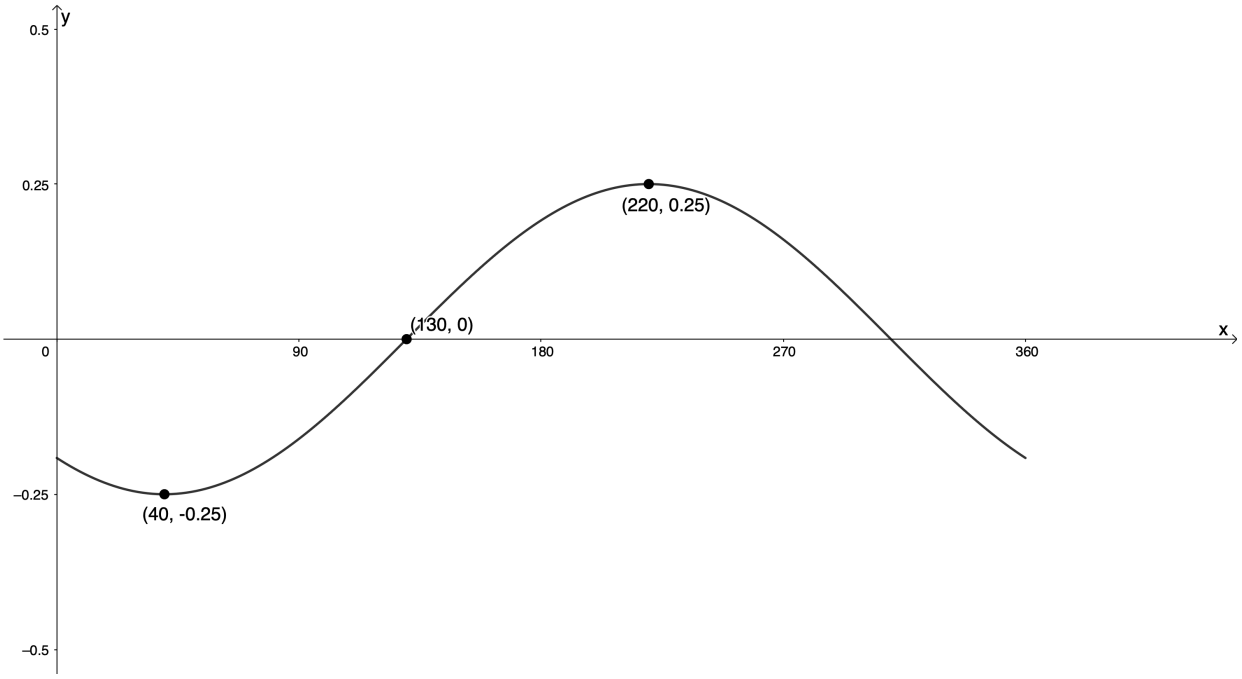
Firstly, we can see that the amplitude of the graph is 2 units. We do not yet know whether $a = 2$ or $a = -2$.

We know that the function $y = \sin x$ passes through the origin and then reaches a maximum turning point at $(90^\circ, 1)$. This function instead reaches a minimum turning point at $(30^\circ, -2)$. Therefore, we can say that the point $(90^\circ, 1)$ has been transformed into the point $(30^\circ, -2)$.

This means that the point has been shifted horizontally 60° to the left and the y-value has been multiplied by -2. Therefore, $a = -2$ and $p = 60^\circ$, and the function is $y = -2\sin(x + 60^\circ)$.

Exercise 8.3

Given the graph of the form $y = a\sin(x + p)$:



1. Determine the values of a and p.
2. State the domain and range of the function.

The full solutions are at the end of the unit.

Summary

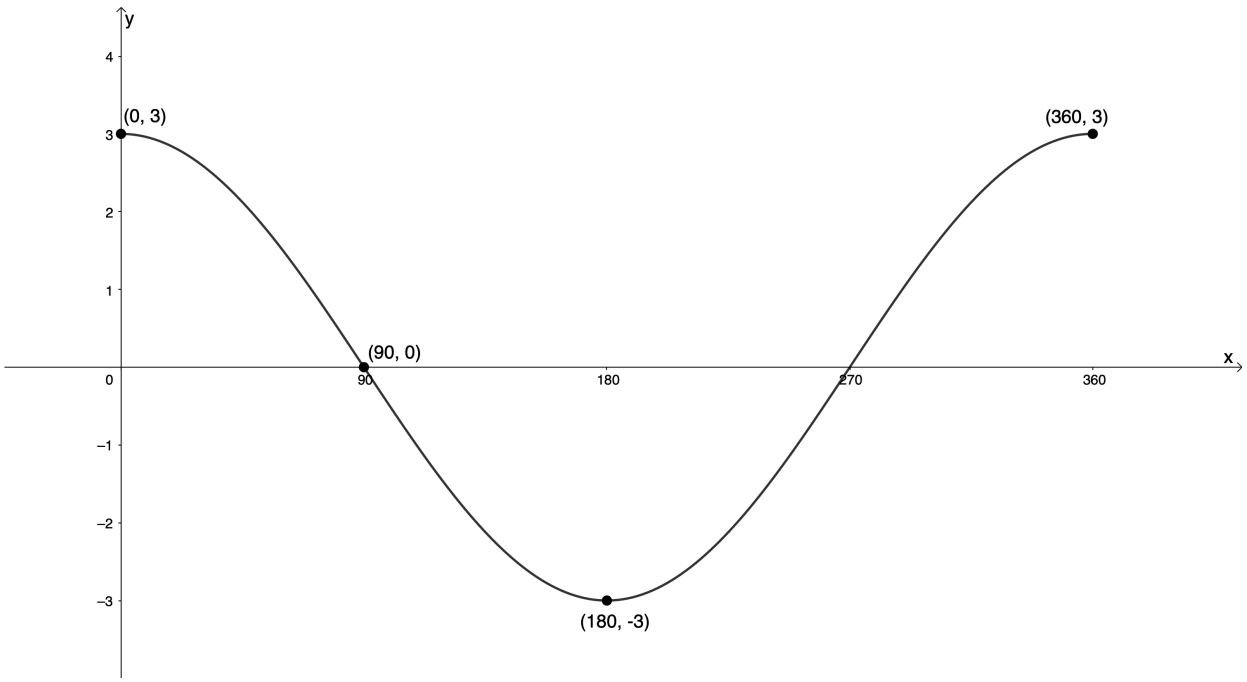
In this unit you have learnt the following:

- The effects of a and p on the sine graph of the form $y = a\sin(x + p)$.
- How to sketch functions of the form$y = a\sin(x + p)$.
- How to find the values of a and p from a given sine graph of the form$y = a\sin(x + p)$.

Unit 8: Assessment

Suggested time to complete: 45 minutes

1. Sketch the following functions for the given intervals:
   a. $3y = \sin(x - 45^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
   b. $g(x) = -4\sin(60^\circ - x)$ for $0^\circ \leq x \leq 360^\circ$
2. From the graph below of the form $y = a\sin(x + p)$, determine the values of a and p.



The full solutions are at the end of the unit.

Unit 8: Solutions

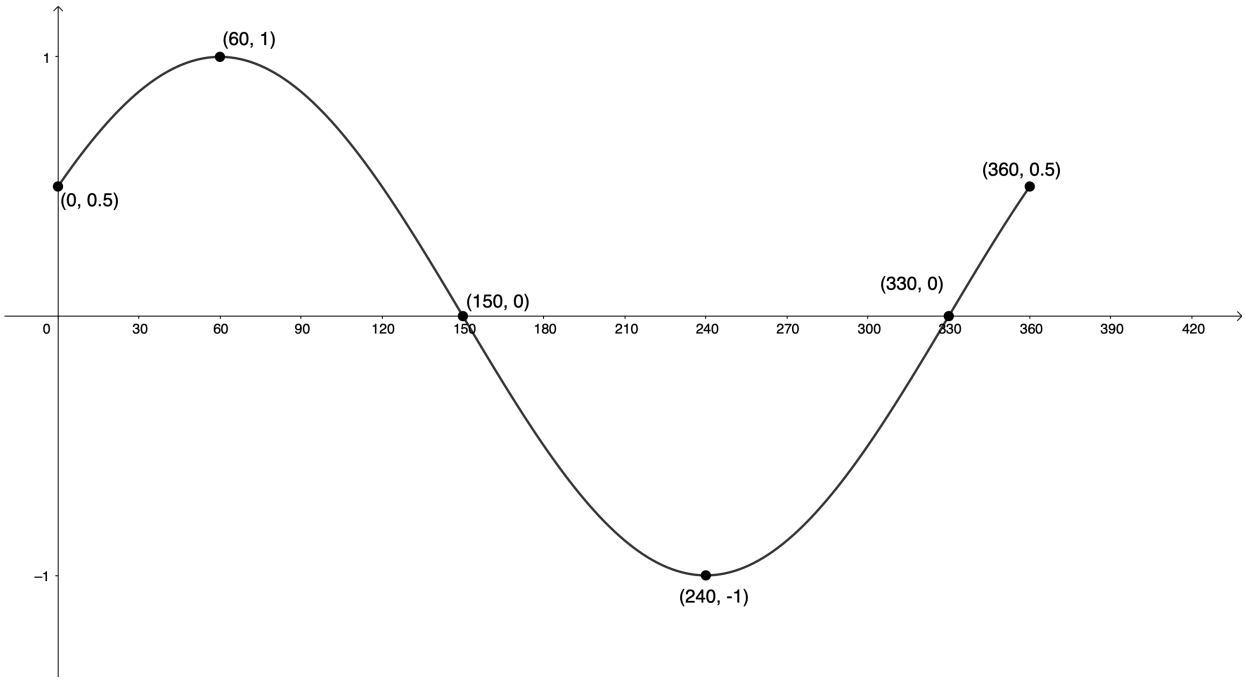
Exercise 8.1

1. $f(x) = \sin(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
   $p = 30^\circ$. Therefore, the graph will be shifted 30° to the left.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x + 30^\circ)$	$(-30^\circ, 0)$	$(60^\circ, 1)$	$(150^\circ, 0)$	$(240^\circ, -1)$	$(330^\circ, 0)$

   y-intercept (let $x = 0$):
   $f(0) = \sin(0^\circ + 30^\circ)$
   $\therefore f(0) = \frac{1}{2}$

   Because the period of the function is 360° we know that the function value at 360° will also be $\frac{1}{2}$.



2. $g(x) = \sin(x - 45^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
   $p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

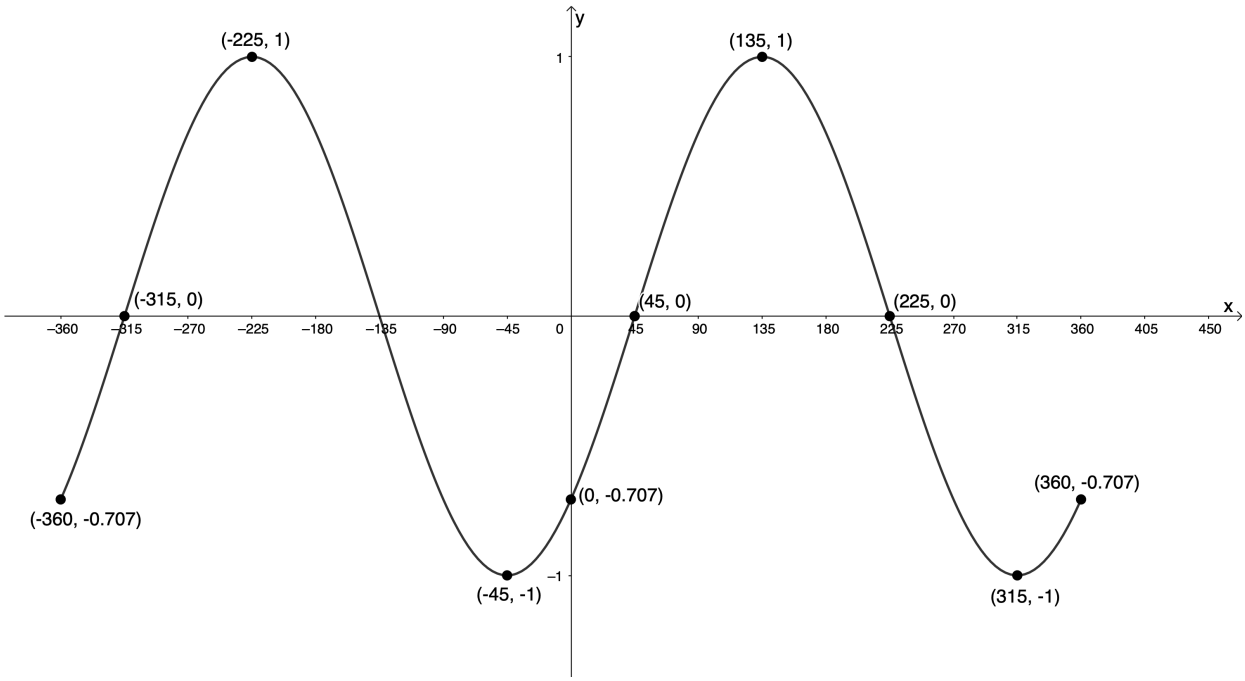
$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x - 45^\circ)$	$(45^\circ, 0)$	$(135^\circ, 1)$	$(225^\circ, 0)$	$(315^\circ, -1)$	$(405^\circ, 0)$

y-intercept (let $x = 0$):

$$g(0) = \sin(0^\circ - 45^\circ)$$

$$\therefore g(0) = -\frac{1}{\sqrt{2}} = -0.707$$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be -0.707.

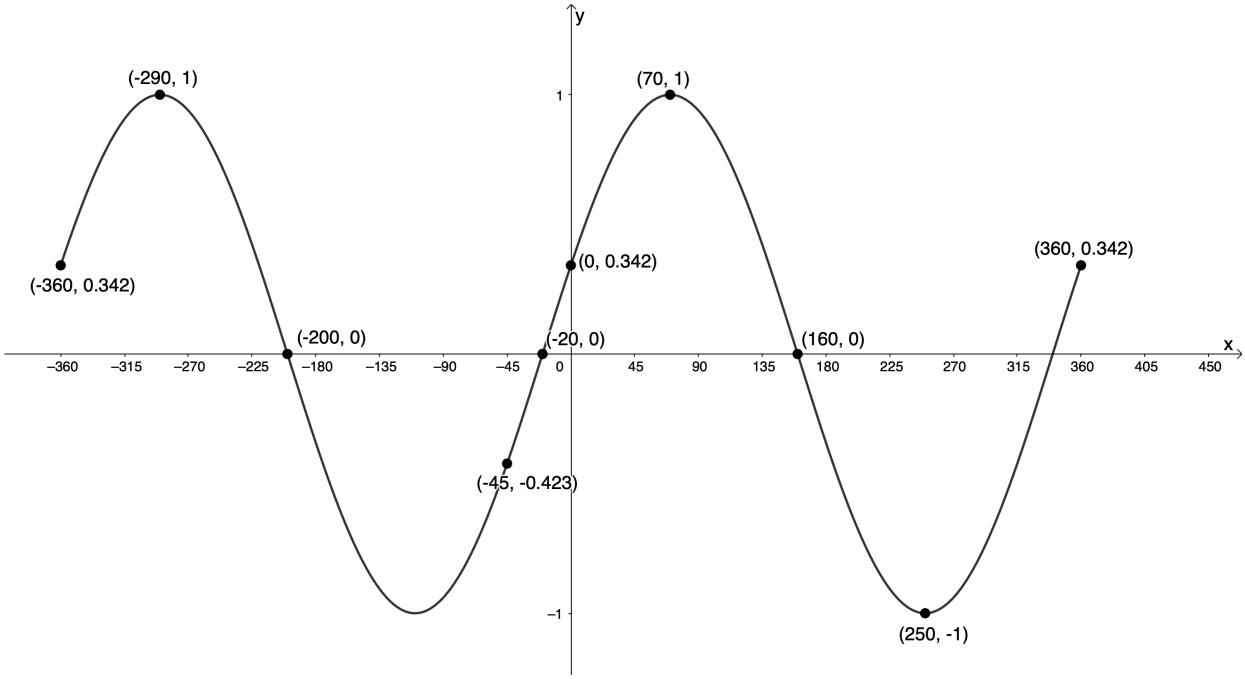


3. $h(x) = \sin(x + 20^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
   $p = 20^\circ$. Therefore, the graph will be shifted 20° to the left.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x + 20^\circ)$	$(-20^\circ, 0)$	$(70^\circ, 1)$	$(160^\circ, 0)$	$(250^\circ, -1)$	$(340^\circ, 0)$

y-intercept (let $x = 0$):
$h(0) = \sin(0^\circ + 20^\circ)$
$\therefore h(0) = 0.342$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 0.342.



Back to Exercise 8.1

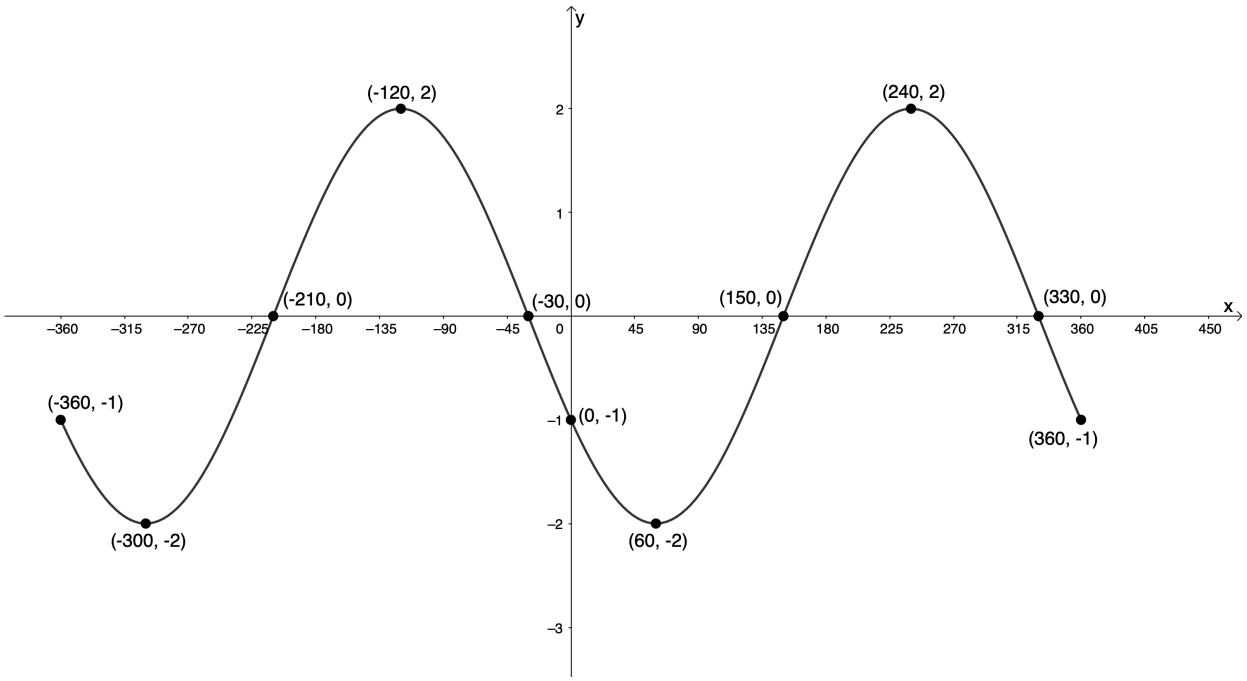
Exercise 8.2

1. $y = -2\sin(x + 30^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
   $a = -2$. Therefore, the amplitude will be 2 and the graph will be reflected about the x-axis.
   $p = 30^\circ$. Therefore, the graph will be shifted 30° to the left.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$-2\sin(x + 30^\circ)$	$(-30^\circ, 0)$	$(60^\circ, -2)$	$(150^\circ, 0)$	$(240^\circ, 2)$	$(330^\circ, 0)$

   y-intercept (let $x = 0$):
   $y = -2\sin(0^\circ + 30^\circ)$
   $\therefore y = -1$

   Because the period of the function is 360° we know that the function values at -360° and 360° will also be -1.



   Domain: $\{x | x \in \mathbb{R}, \ -360^\circ \leq x \leq 360^\circ\}$
   Range: $\{y | y \in \mathbb{R}, \ -2 \leq y \leq 2\}$

2. $2y = \sin(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
   $2y = \sin(x + 30^\circ)$
   $\therefore y = \frac{1}{2}\sin(x + 30^\circ)$
   $a = \frac{1}{2}$. Therefore, the amplitude will be $\frac{1}{2}$.
   $p = 30^\circ$. Therefore, the graph will be shifted 30° to the left.

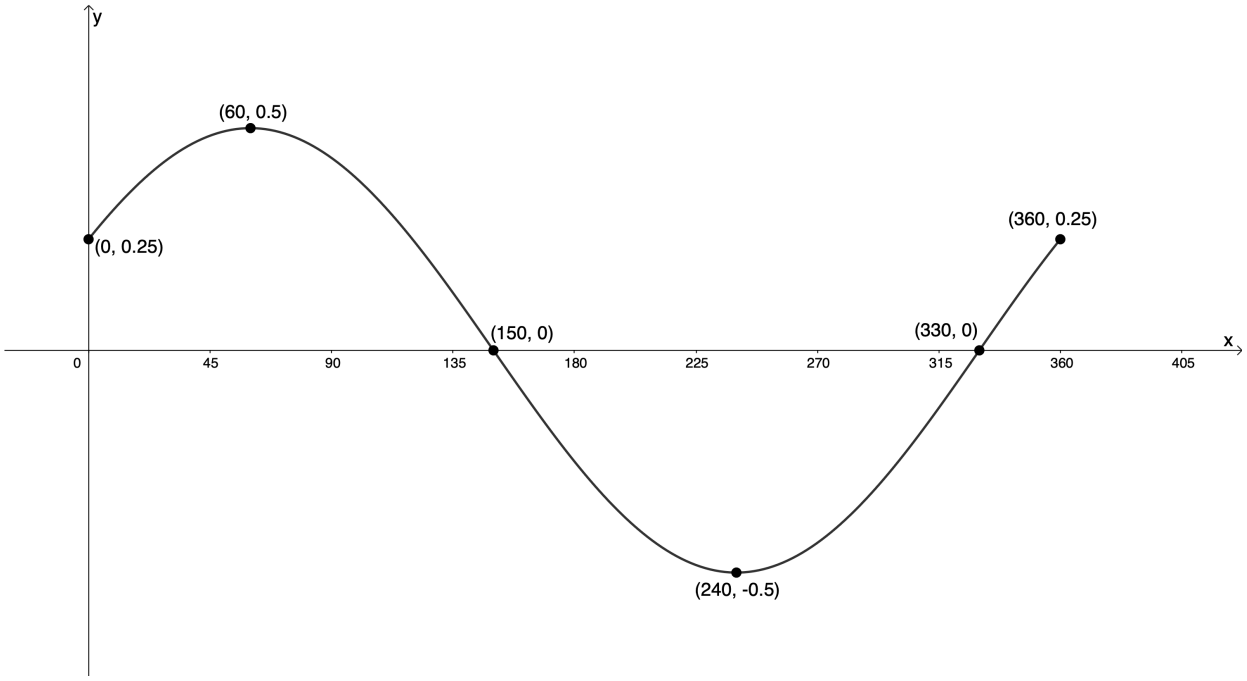
$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\frac{1}{2}\sin(x + 30^\circ)$	$(-30^\circ, 0)$	$(60^\circ, \frac{1}{2})$	$(150^\circ, 0)$	$(240^\circ, -\frac{1}{2})$	$(330^\circ, 0)$

y-intercept (let $x = 0$):

$$y = \frac{1}{2}\sin(0^\circ + 30^\circ)$$

$$\therefore y = \frac{1}{4}$$

Because the period of the function is 360° we know that the function value at 360° will also be $\frac{1}{4}$.



Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$

Range: $\{y | y \in \mathbb{R},\ -\frac{1}{2} \leq y \leq \frac{1}{2}\}$

3. $\frac{1}{3}y = \sin(60^\circ - x)$ for $-360^\circ \leq x \leq 360^\circ$

$$\frac{1}{3}y = \sin(60^\circ - x)$$

$$\therefore y = 3\sin(60^\circ - x)$$

$$= 3\sin(-x + 60^\circ)$$

$$= 3\sin(-(x - 60^\circ))$$

$$= -3\sin(x - 60^\circ)$$

$a = -3$. Therefore, the amplitude will be 3 and the graph will be reflected about the x-axis.

$p = -60^\circ$. Therefore, the graph will be shifted 60° to the right.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$-3\sin(x - 60^\circ)$	$(60^\circ, 0)$	$(150^\circ, -3)$	$(240^\circ, 0)$	$(330^\circ, 3)$	$(420^\circ, 0)$

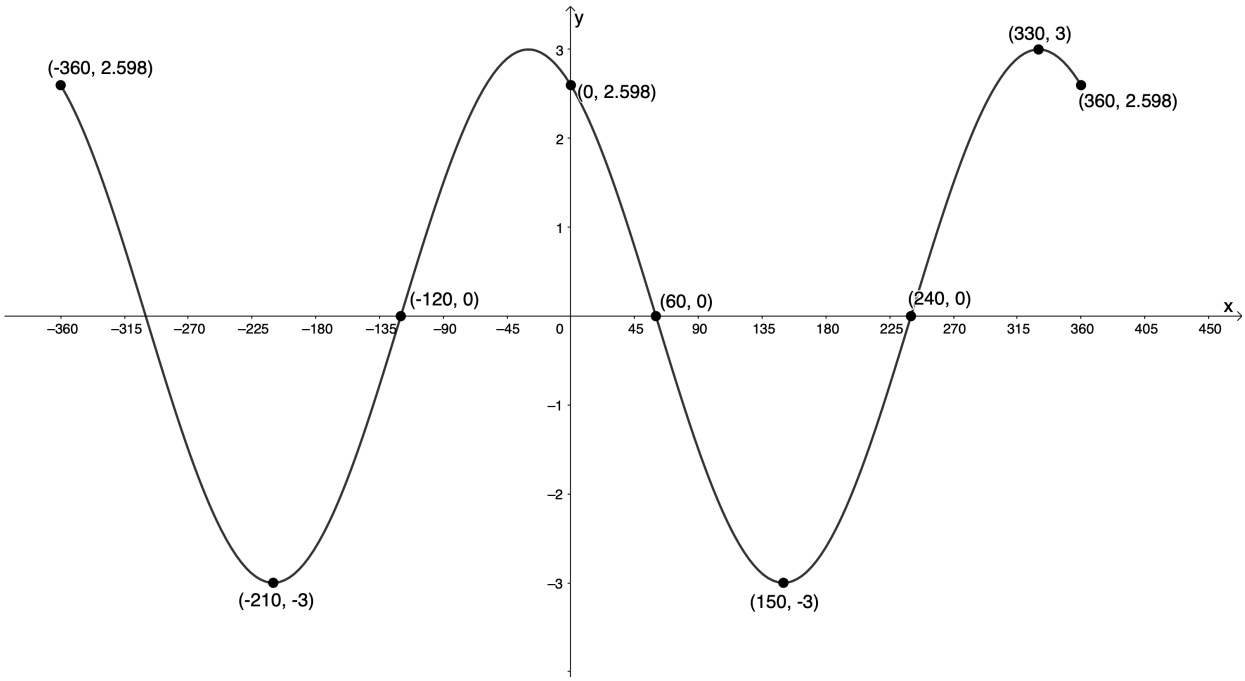
y-intercept (let $x = 0$):

$$y = -3\sin(0^\circ - 60^\circ)$$

$$\therefore y = -3 \times -\frac{\sqrt{3}}{2} = 2,598$$

Because the period of the function is 360° we know that the function values at -360° and 360° will also

be 2.598.



Domain: $\{x|x \in \mathbb{R},\ -360^\circ \leq x \leq 360^\circ\}$
Range: $\{y|y \in \mathbb{R},\ -3 \leq y \leq 3\}$

Back to Exercise 8.2

Exercise 8.3

1. Amplitude is $0.25 = \frac{1}{4}$

   $(90^\circ, 1)$ has been transformed to $(40^\circ, -\frac{1}{4})$. Therefore, $a = -\frac{1}{4}$ and $p = 50^\circ$.

   $y = -\frac{1}{4}\sin(x + 50^\circ)$
2. Domain: $\{x|x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$

   Range: $\{y|y \in \mathbb{R},\ -\frac{1}{4} \leq y \leq \frac{1}{4}\}$

Back to Exercise 8.3

Unit 8: Assessment

1.

   a. $3y = \sin(x - 45^\circ)$ for $-360^\circ \leq x \leq 360^\circ$

   $3y = \sin(x - 45^\circ)$

   $\therefore y = \frac{1}{3}\sin(x - 45^\circ)$

   $a = \frac{1}{3}$. Therefore, the amplitude will be $\frac{1}{3}$.

   $p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

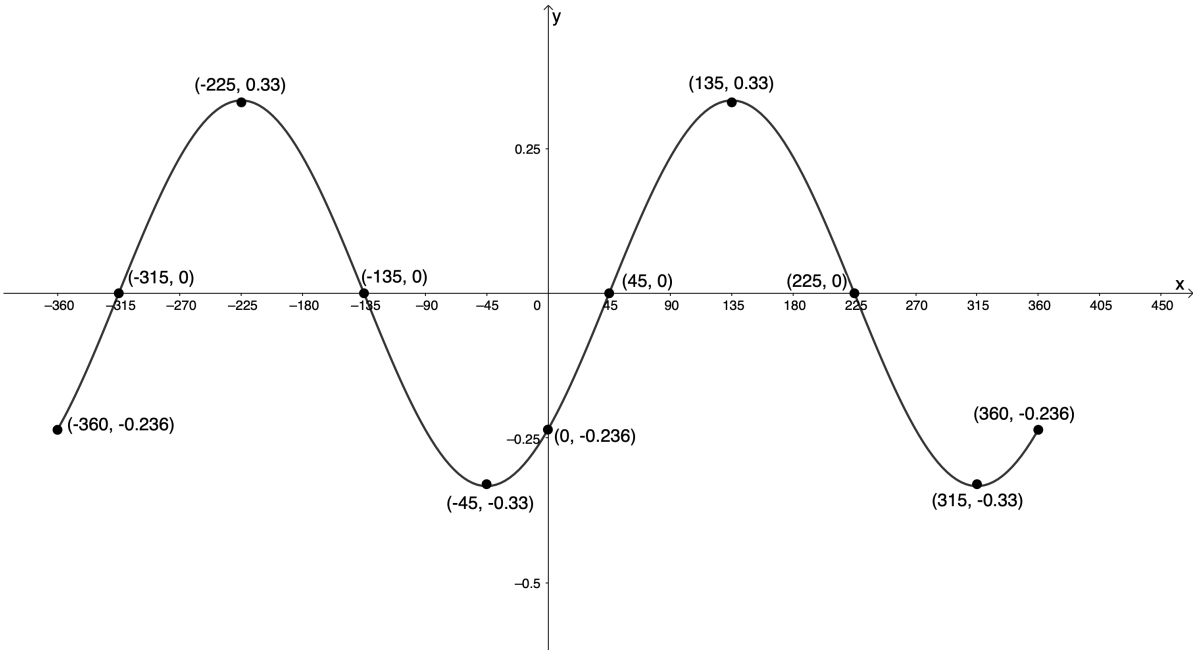
$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\frac{1}{3}\sin(x - 45^\circ)$	$(45^\circ, 0)$	$(135^\circ, \frac{1}{3})$	$(225^\circ, 0)$	$(315^\circ, -\frac{1}{3})$	$(405^\circ, 0)$

y-intercept (let $x = 0$):

$$y = \frac{1}{3}\sin(0^\circ - 45^\circ)$$

$$\therefore y = \frac{1}{3} \times -\frac{1}{\sqrt{2}} = -0.236$$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 0.236.



b. $g(x) = -4\sin(60^\circ - x)$ for $0^\circ \leq x \leq 360^\circ$

$$\begin{aligned} g(x) &= -4\sin(60^\circ - x) \\ &= -4\sin(-x + 60^\circ) \\ &= -4\sin(-(x - 60^\circ)) \\ &= 4\sin(x - 60^\circ) \end{aligned}$$

$a = 4$. Therefore, the amplitude will be 4.

$p = -60^\circ$. Therefore, the graph will be shifted 60° to the right.

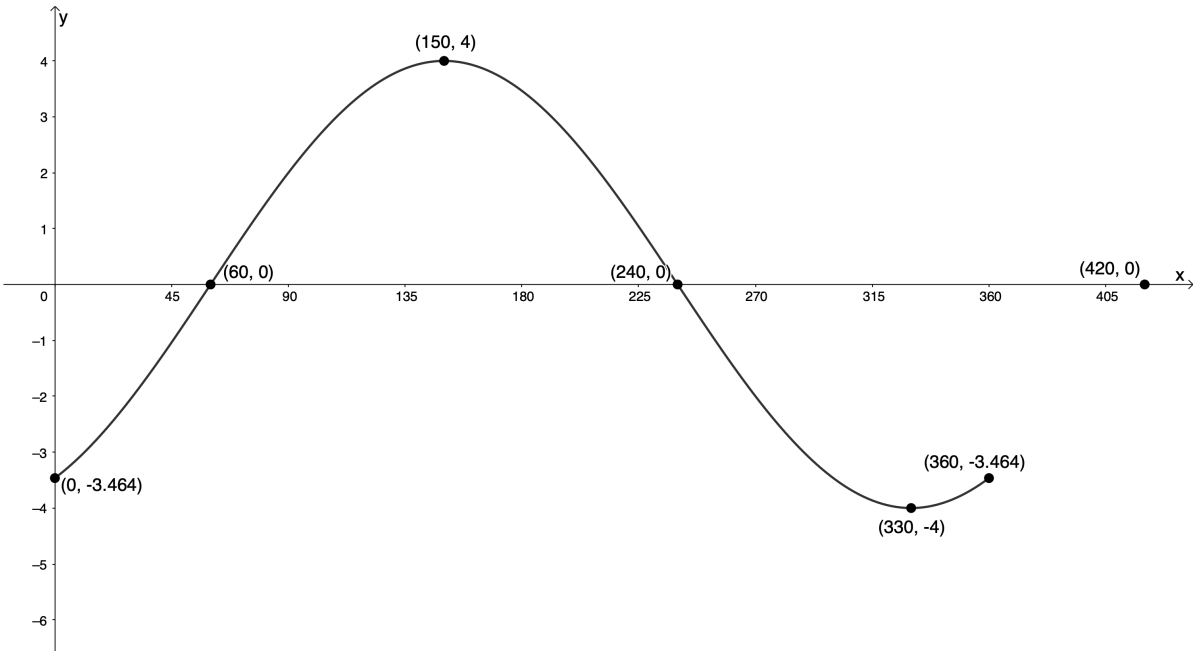
$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$4\sin(x + 60^\circ)$	$(60^\circ, 0)$	$(150^\circ, 4)$	$(240^\circ, 0)$	$(330^\circ, -4)$	$(420^\circ, 0)$

y-intercept (let $x = 0$):

$$g(0) = 4\sin(0 - 60^\circ)$$

$$\therefore y = 4 \times -\frac{\sqrt{3}}{2} = -3.464$$

Because the period of the function is 360° we know that the function value at 360° will also be -3.464.



2. The function is of the form $y = a\sin(x + p)$. The amplitude is 3. The point $(90^\circ, 1)$ has been transformed to $(0^\circ, 3)$. Therefore, the graph has been shifted $90^\circ$to the left. Hence $a = 3$ and $p = 90^\circ$ and the function is $y = 3\sin(x + 90^\circ)$.

Back to Unit 8: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- takenote © DHET is licensed under a CC BY (Attribution) license
- example8.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example8.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example8.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example8.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.1A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.1A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.1A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.2A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.2A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 9: The cosine function

DYLAN BUSA

Unit 9 outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = a\cos k\theta$.
- Determine the effects of a and k on the cosine graph of the form $y = a\cos k\theta$.
- Find the values of a and k from a given cosine graph of the form $y = a\cos k\theta$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \cos x$ and $y = \cos\theta$ are the same function.

What you should know

Before you start this unit, make sure you can:

- Sketch cosine functions of the form $y = a\cos\theta + q$. Refer to level 2 subject outcome 2.1 Unit 5 if you need help with this.

Introduction

We know that the **period** of the cosine function is $360°$ (same as the sine function) (see Figure 1).

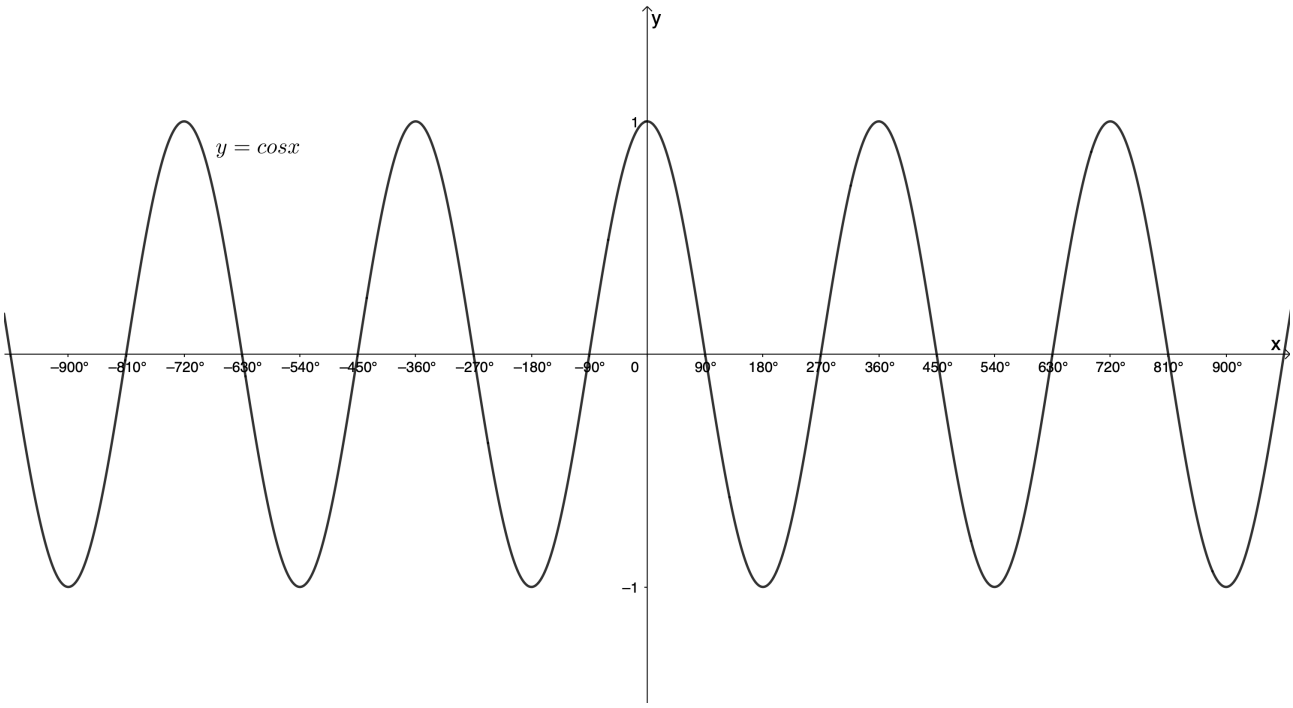


Figure 1: $y = \cos x$

We also know that the **amplitude** is the maximum distance of the function from its midpoint and that the amplitude of $y = \cos x$ is 1 (see Figure 2).

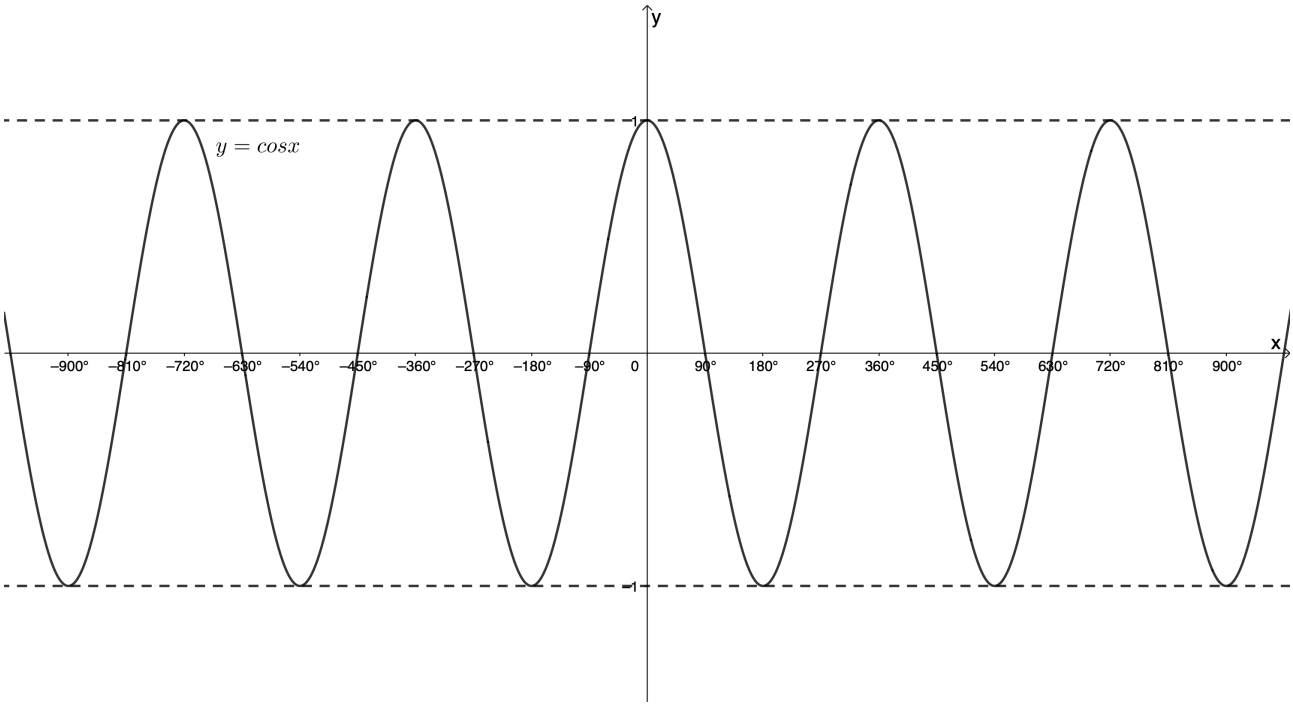


Figure 2: Graph of $y = \cos x$ showing amplitude

In $y = \cos x$, because the amplitude is 1 and the graph's midpoint is the x-axis, the maximum value of $y = \cos\theta$ is 1 and the minimum value is -1. In the interval $0^\circ \leq x \leq 360^\circ$, the graph of $y = \cos x$ has a **maximum turning point** at $(0^\circ, 1)$ and $(360^\circ, 1)$, and a minimum turning point at $(180^\circ, -1)$ (see Figure 3).

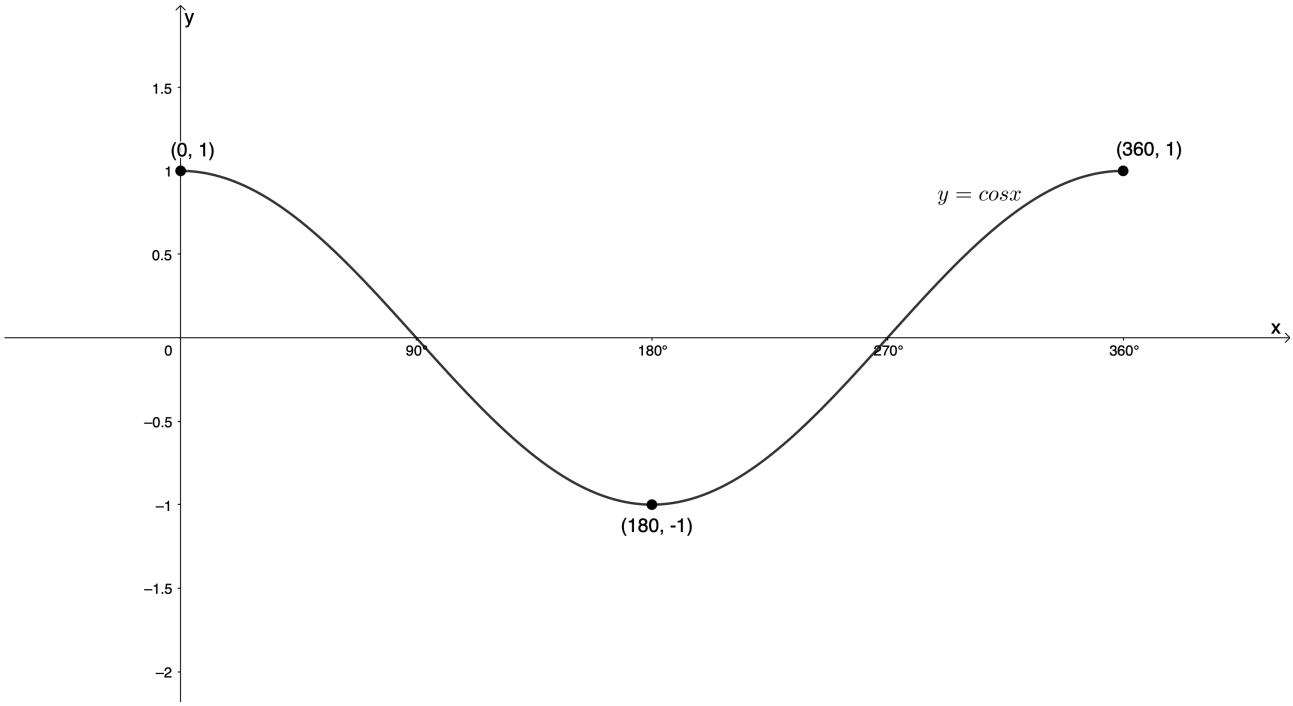


Figure 3: Maximum and minimum turning points of $y = \cos x$ in the interval $0^\circ \leq x \leq 360^\circ$

In level 2 we were introduced to the cosine function of the form $y = a\sin\theta + q$. We discovered that the value of q shifted the graph vertically up and down while the value of a changed the amplitude of the graph.

Take note!

The period of $y = a\cos\theta + q$ is $360°$.

The effect of q:

- if $q > 0$, the function is shifted **up** by q units
- if $q < 0$, the function is shifted **down** by q units.

The effect of a:

- if $a > 1$, the amplitude **increases** to a
- if $0 < a < 1$, the amplitude **decreases** to a
- if $a < 0$, the function is reflected about the x-axis
- if $a < -1$, the amplitude **increases** to a
- if $-1 < a < 0$, the amplitude **decreases** to a.

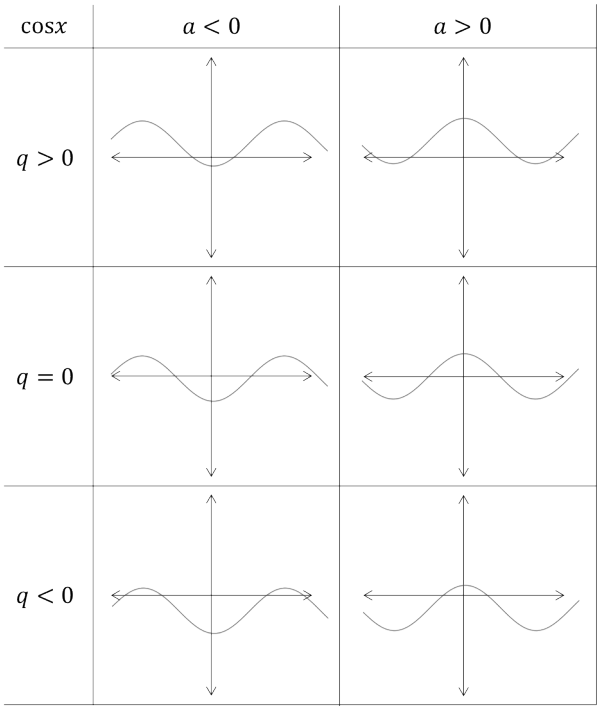
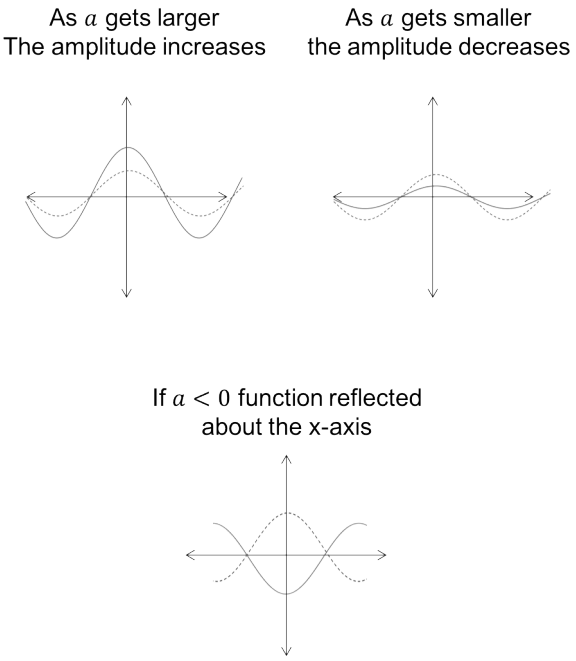


Figure 4: Summary of effects of a and q on $y = a\cos x + q$



Note

If you have access to the internet, visit this interactive simulation.

Here you will find a cosine function of the form $y = a\cos x + q$ with sliders to change the values of a and q. Spend some time playing with it to make sure that you understand how changing the values of a and q affect the shape and location of the cosine function of the form $y = a\cos x + q$.

The cosine function of the form $y = \cos kx$

In Activity 7.1 in unit 7, we saw that the value of k in $y = \sin kx$ affects the period of the function and that we can calculate the period as:

$\text{Period} = \dfrac{360^\circ}{|k|}$ where $|k|$ is the absolute value of k, in other words the positive value of k.

This means that if $k > 1$, the period **decreases** and if $0 < k < 1$, the period **increases**.

The same is true for the cosine function $y = \cos kx$. The period can be calculated as follows:

$\text{Period} = \dfrac{360^\circ}{|k|}$ where $|k|$ is the absolute value of k, in other words the positive value of k.

This means that if $k > 1$, the period **decreases** and if $0 < k < 1$, the period **increases**.

If $k < 0$, it changes the period, but the graph is not reflected about the x-axis. This is because the graph of $y = \cos x$ is symmetrical about the y-axis. Therefore $\cos(-x) = \cos x$ (see Figure 5).

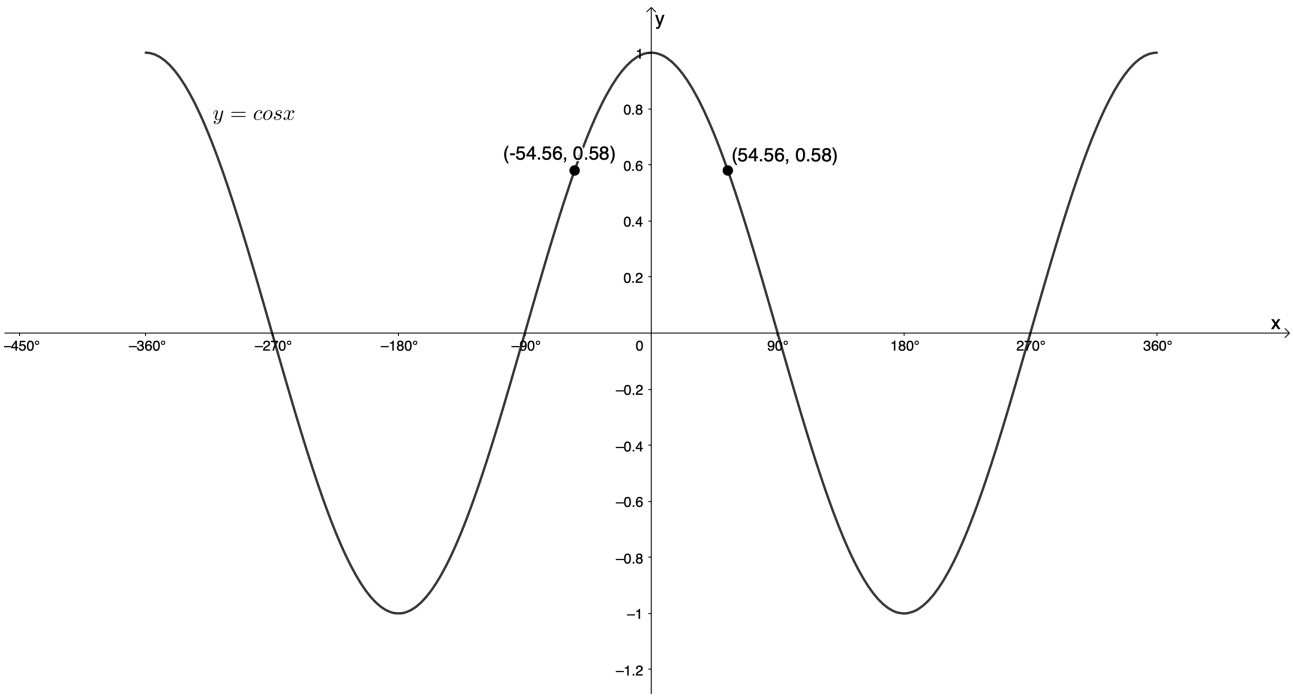


Figure 5: $\cos(-x) = \cos x$

Another way of looking at this is that angles of $-\theta$ lie in the fourth quadrant where the cosine function is positive, as it is in the first quadrant. Therefore, there is no change of sign for the function values.

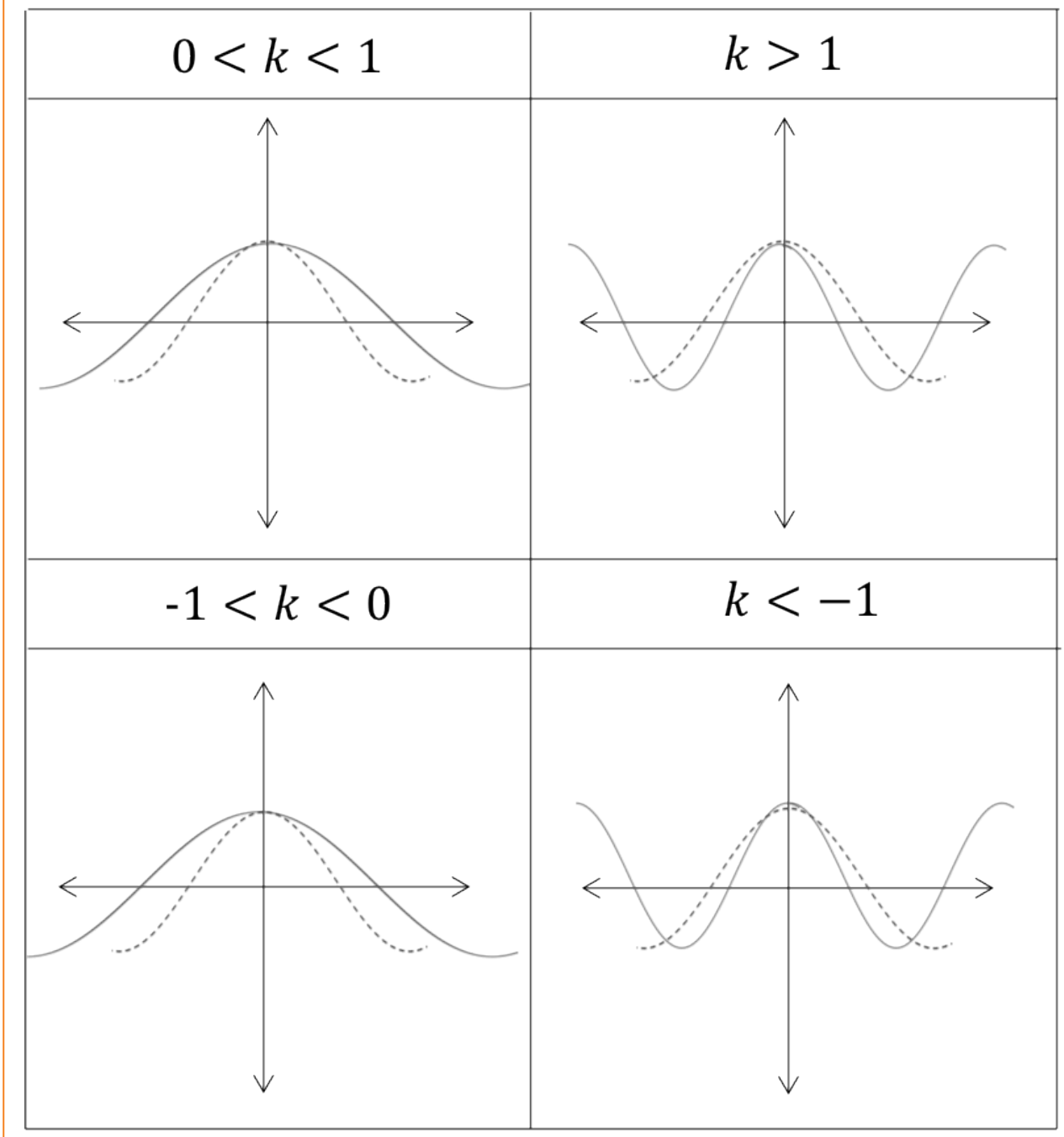
Note

If you have an internet connection, visit this online simulation to compare the effect values of $k < 0$ have on the sine and cosine functions.

Take note!

In $y = \cos kx$ the $\text{period} = \dfrac{360^\circ}{|k|}$.

$\cos(-x) = \cos(x)$, so the graph is not reflected about the x-axis for $k < 0$.



Example 9.1

Given the function $y = \cos\left(\frac{2x}{3}\right)$ for the interval $0° \leq x \leq 360°$, determine the:

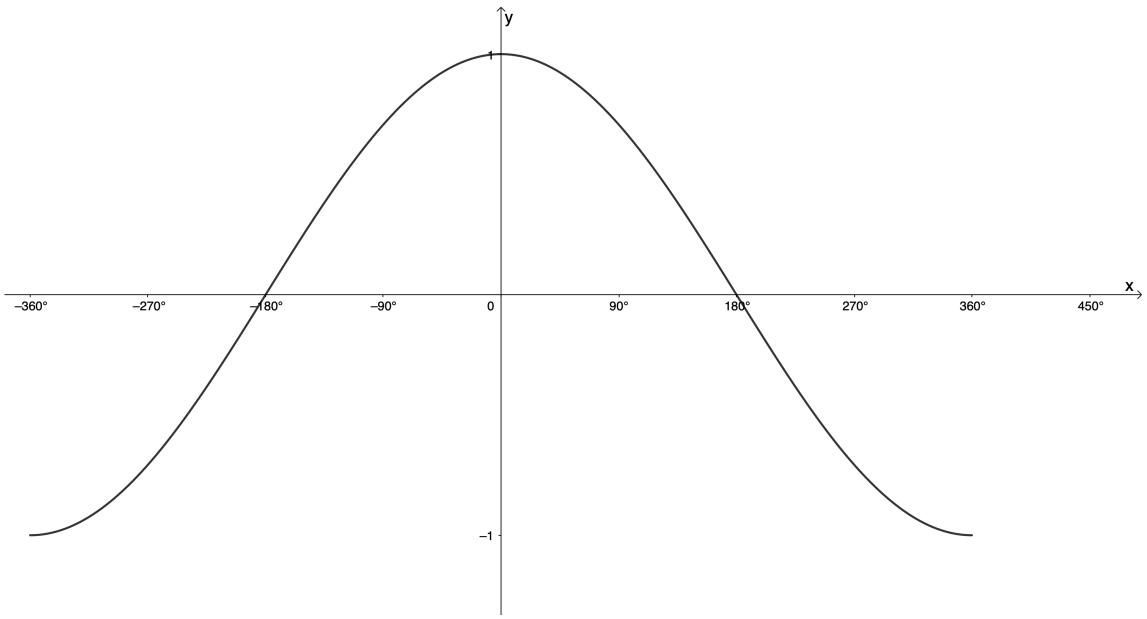
1. domain and range
2. period.

Solutions

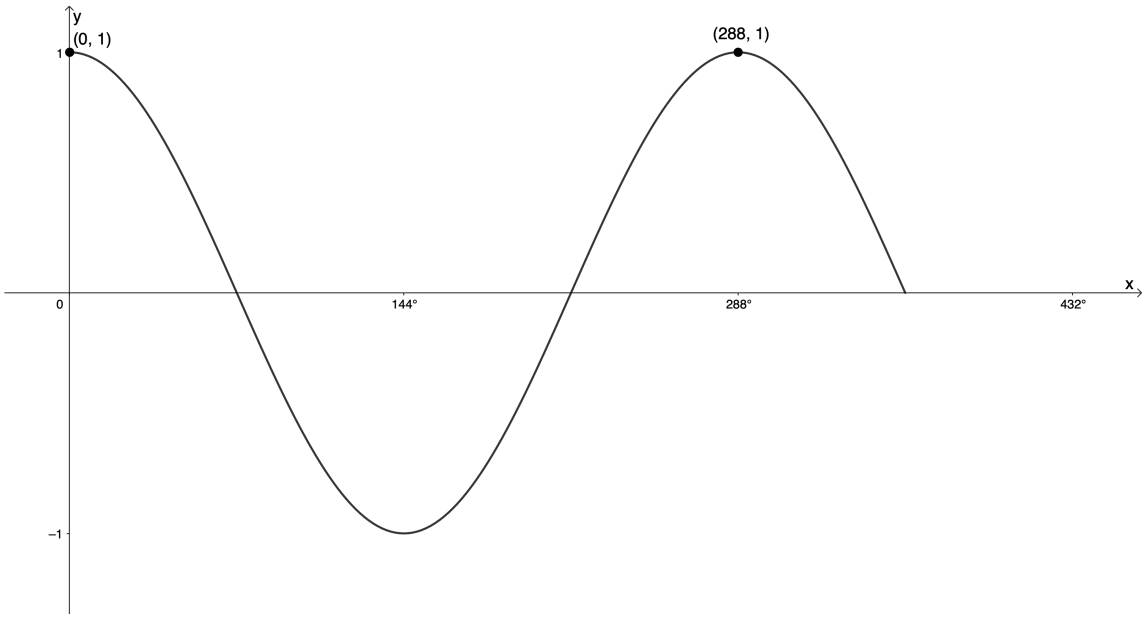
1. Domain: $\{x|x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$
   Range: $\{y|y \in \mathbb{R},\ -1 \leq y \leq 1\}$
2. $k = \frac{2}{3}$. Therefore, the period of the graph is $\frac{360^\circ}{\left|\frac{2}{3}\right|} = \frac{3 \times 360^\circ}{2} = 540^\circ$.

Exercise 9.1

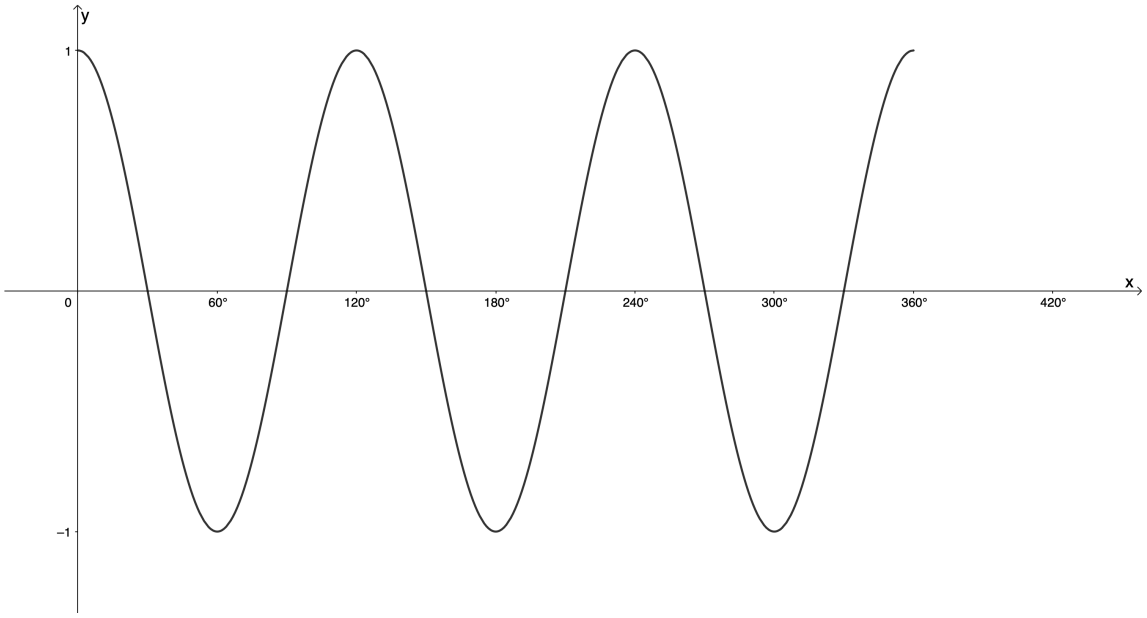
1. Determine the periods of the following functions:
   a. $y = \cos(2x)$
   b. $y = \cos\left(-\frac{3}{5}x\right)$
   c. $y = \cos\left(-\frac{3}{2}x\right)$
2. For each function of the form $y = \cos kx$, determine the value of k:
   a.



   b.



c.



The full solutions are at the end of the unit.

Sketch functions of the form $y = a\cos kx$

Now that we know that the value of k affects the period of the function $y = \cos kx$, we can combine this with our knowledge of the effects of a and examine functions of the form $y = a\cos kx$.

The effects of a on $y = a\cos kx$ are exactly the same as they were on $y = a\sin x + q$. Before we learn how to sketch functions of the form $y = a\cos kx$, spend some time playing with this online simulation.

In this simulation you will find a graph of the function $y = a\cos kx + q$ for the interval $-360^\circ \leq x \leq 360^\circ$. Pay particular attention to how the amplitude, period and the values of the turning points and intercepts with the axes change as you change the values of a, k and q.

The best way to sketch functions of the form $y = a\cos kx$ is to transform the basic function of $y = \cos x$ depending on the values of a and k. To do this, you need to know the set of 'anchor points' of $y = \cos x$, as transformation of these points will help you to sketch functions of the form $y = a\cos kx$.

Take note!

The 'anchor points' for $y = \cos x$ are illustrated in Figure 6 for the interval $-360^\circ \leq x \leq 360^\circ$.

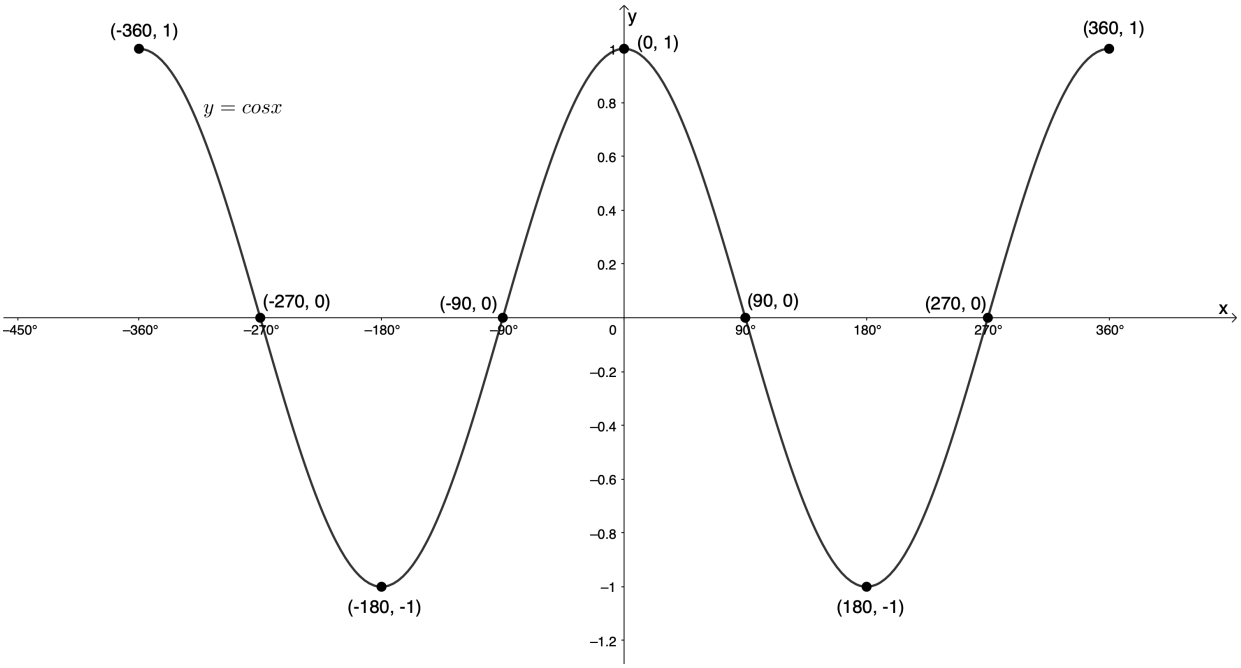


Figure 6: The 'anchor points' of $y = \cos x$

Example 9.2

Sketch the function $y = 2\cos 2x$ for the interval $-180^\circ \leq x \leq 180^\circ$.

Solution

The function is of the form $y = a\cos kx$, with$a = 2$ and $k = 2$.

Because $k = 2$, we know that the period of the function will be $\frac{360^\circ}{|2|} = 180^\circ$. We need to transform one period of 'anchor points' as follows:

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos 2x$	$(0^\circ, 1)$	$(45^\circ, 0)$	$(90^\circ, -1)$	$(135^\circ, 0)$	$(180^\circ, 1)$

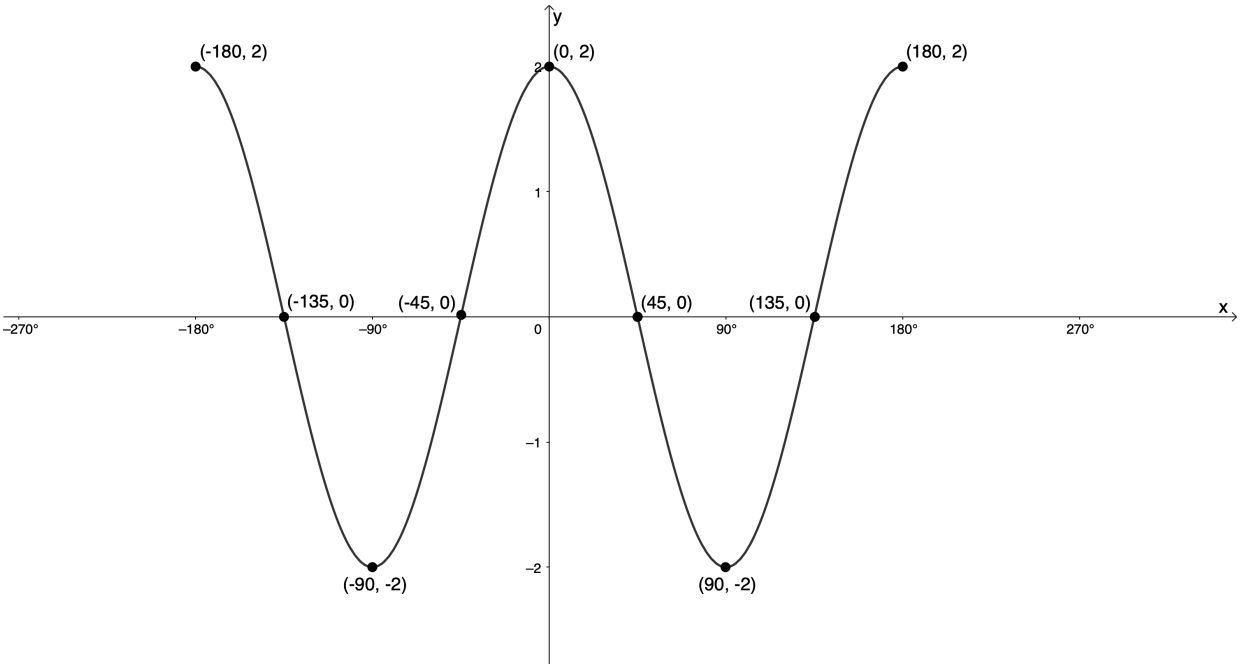
Because the period has been halved, the value of each of the x-coordinates is also halved. We can see that the function will start at1, which is its maximum turning point $(0^\circ, 1)$, fall to zero , fall to its minimum turning point $(90^\circ, -1)$, rise back to zero and rise back to its maximum turning point $(180^\circ, 1)$ in 180°, the new period.

Because $a = 2$, we know that each function value (or y-value) is going to be multiplied by 2 thereby making the amplitude of the function 2. We need to further transform one period of 'anchor points' as follows:

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos 2x$	$(0^\circ, 1)$	$(45^\circ, 0)$	$(90^\circ, -1)$	$(135^\circ, 0)$	$(180^\circ, 1)$
$2\cos 2x$	$(0^\circ, 2)$	$(45^\circ, 0)$	$(90^\circ, -2)$	$(135^\circ, 0)$	$(180^\circ, 2)$

We can see that the function will start at 2, which is its new maximum turning point $(0^\circ, 2)$, fall to zero, fall to its new minimum turning point $(90^\circ, -2)$, rise back to zero and rise back to its new maximum turning point $(180^\circ, 2)$ in 180°, the new period.

We can now plot our transformed 'anchor points' and draw the graph. We first draw it for the interval $0^\circ \leq x \leq 180^\circ$ and then continue the pattern for the full interval of $-180^\circ \leq x \leq 180^\circ$.



Example 9.3

Given the function $f(x) = 3\cos\left(-\frac{1}{2}x\right)$:

1. Sketch the graph of $f(x)$ for the interval $-360^\circ \leq x \leq 360^\circ$.
2. State the domain and range of $f(x)$.
3. List the turning points of $f(x)$.
4. What is the amplitude of $f(x)$?
5. What is the period of $f(x)$?

Solutions

1. The function is of the form $y = a\cos kx$ with $a = 3$ and $k = -\frac{1}{2}$.

   Because $k = -\frac{1}{2}$, we know that the period of the function will be $\frac{360^\circ}{\left|-\frac{1}{2}\right|} = \frac{360^\circ}{\frac{1}{2}} = 720^\circ$.

   We need to transform one period of 'anchor points' as follows:

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos\left(-\frac{1}{2}x\right)$	$(0^\circ, 1)$	$(180^\circ, 0)$	$(360^\circ, -1)$	$(540^\circ, 0)$	$(720^\circ, 1)$

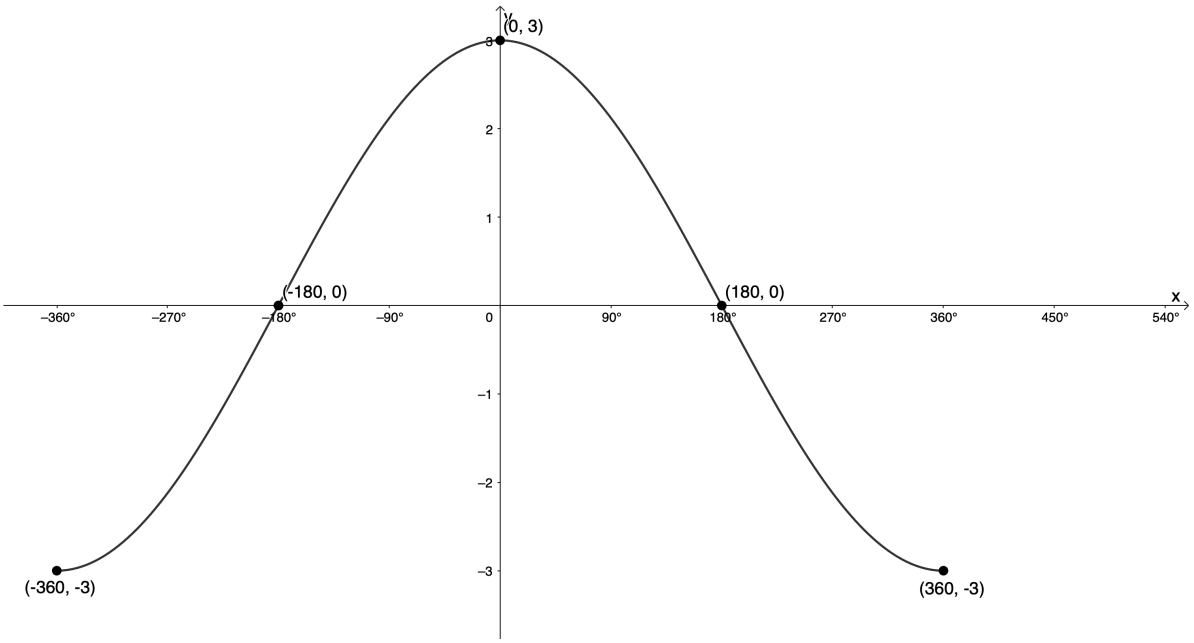
   Because the period has been doubled, the value of each of the x-coordinates is also doubled. We can see that the function will start at 1, which is its maximum turning point $(0^\circ, 1)$, fall to zero , fall to its minimum turning point $(360^\circ, -1)$, rise back to zero and rise back to its maximum turning point $(720^\circ, 1)$ in 720°, the new period.

   Because $a = 3$, we know that each function value (or y-value) is going to be multiplied by 3 thereby making the amplitude of the function 3. We need to further transform one period of 'anchor points' as follows:

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos\left(-\frac{1}{2}x\right)$	$(0^\circ, 1)$	$(180^\circ, 0)$	$(360^\circ, -1)$	$(540^\circ, 0)$	$(720^\circ, 1)$
$3\cos\left(-\frac{1}{2}x\right)$	$(0^\circ, 3)$	$(180^\circ, 0)$	$(360^\circ, -3)$	$(540^\circ, 0)$	$(720^\circ, 3)$

   We can see that the function will start at , which is its new maximum turning point $(0^\circ, 3)$, fall to zero , fall to its new minimum turning point $(360^\circ, -3)$, rise back to zero and rise back to its new maximum turning point $(720^\circ, 3)$ in 720°, the new period.

We can now plot our transformed 'anchor points' and draw the graph for the interval $-360^\circ \leq x \leq 360^\circ$.



2. Domain: $\{x|x \in \mathbb{R}, \ -360^\circ \leq x \leq 360^\circ\}$
   Range$\{f(x)|f(x) \in \mathbb{R}, \ -3 \leq f(x) \leq 3\}$
3. Turning points: $(-360^\circ, 3)$, $(0^\circ, 3)$ and $(360^\circ, -3)$
4. Amplitude is 3
5. Period is 720°

Exercise 9.2

Sketch the following functions for the indicated intervals:

1. $y = -2\cos\frac{\theta}{2}$ for $-360^\circ \leq \theta \leq 360^\circ$
2. $y = \cos\left(-\frac{3x}{2}\right)$ for $-180^\circ \leq x \leq 180^\circ$
3. $y = \frac{1}{2}\cos 3\theta$ for $-180^\circ \leq \theta \leq 180^\circ$

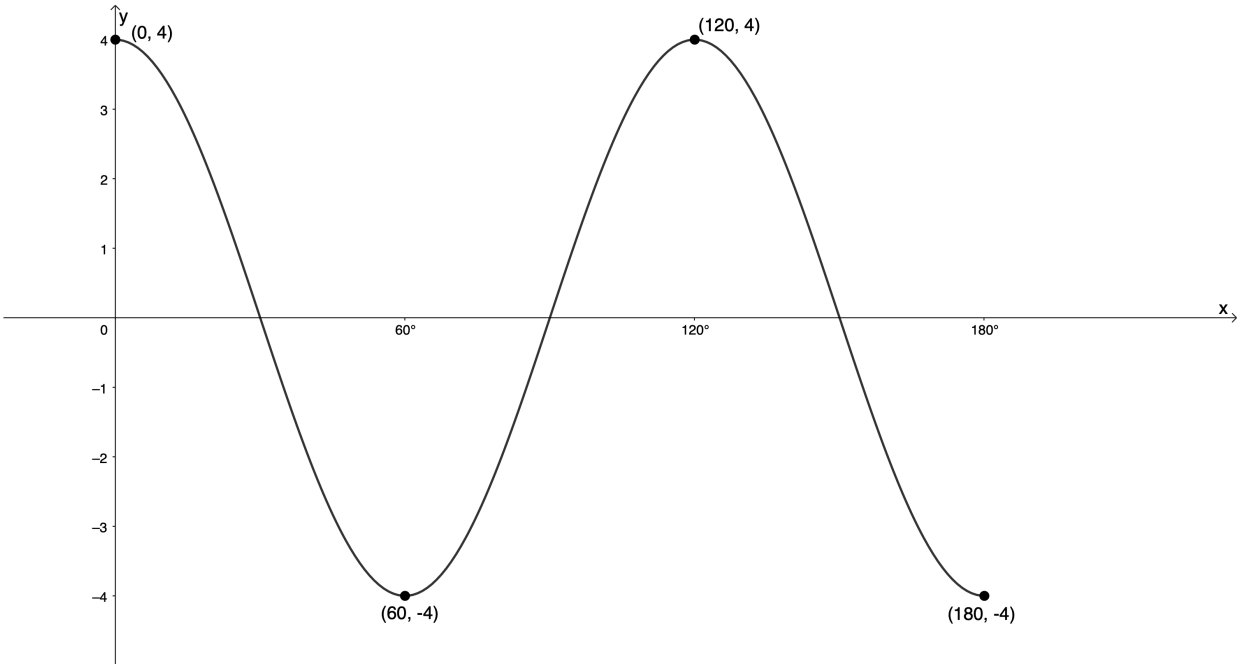
The full solutions are at the end of the unit.

Find the equation of a cosine function of the form $y = a\cos kx$

By examining the amplitude and period of a graph of the form $y = a\cos kx$, we can determine the values of a and k.

Example 9.4

Given the following graph of the form $y = a\cos kx$, determine the values of a and k:



Solution

From the graph we can see that the amplitude is 4. We can also see that the graph is not reflected about the x-axis. At zero, it is at its maximum turning point in the same way as the graph of $y = \cos x$. Therefore, $a = 4$ and $y = 4\cos kx$.

The period is $120°$.

$$\text{Period} = \frac{360°}{|k|}$$

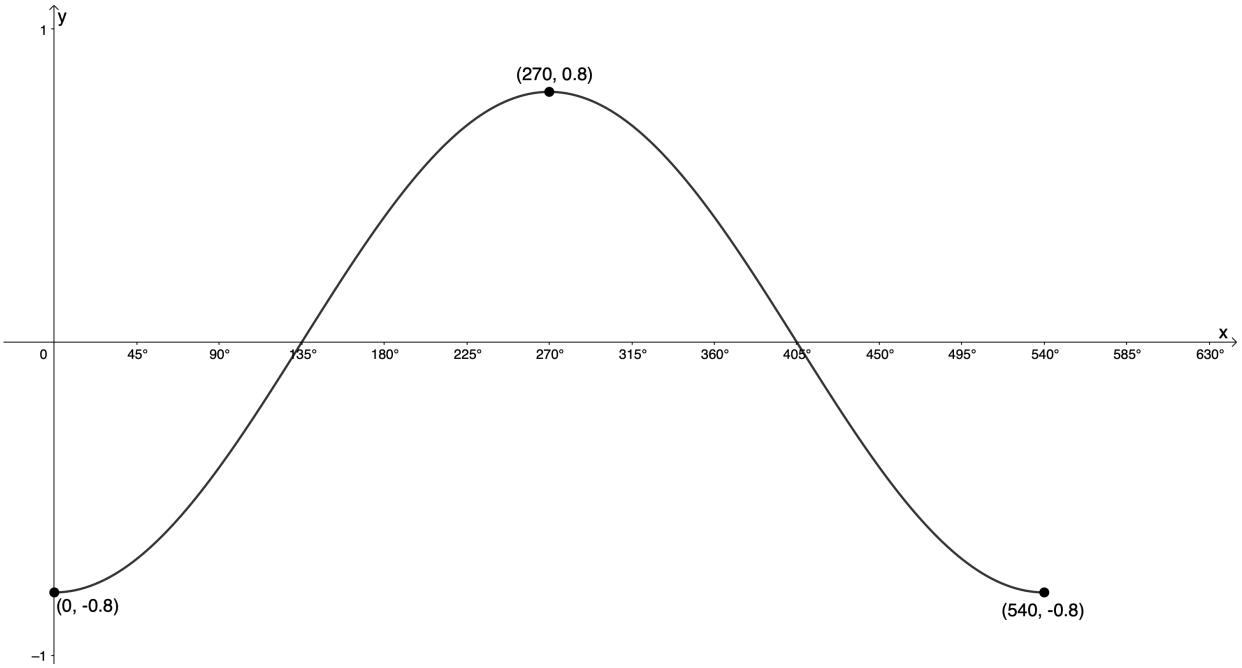
$$120° = \frac{360°}{|k|}$$

$$\therefore |k| = 3$$

$y = 4\cos 3x$

Example 9.5

Given the following graph of the form $y = a\cos kx$, determine the values of a and k:



Solution

From the graph we can see that the amplitude is $0.8 = \frac{4}{5}$. We can also see that the graph is reflected about the x-axis. At zero, it is at its minimum turning point unlike the graph of $y = \cos x$ which is at its maximum turning point. Therefore, $a = -\frac{4}{5}$ and $y = -\frac{4}{5}\cos kx$.

$$\text{Period} = \frac{360^\circ}{|k|}$$

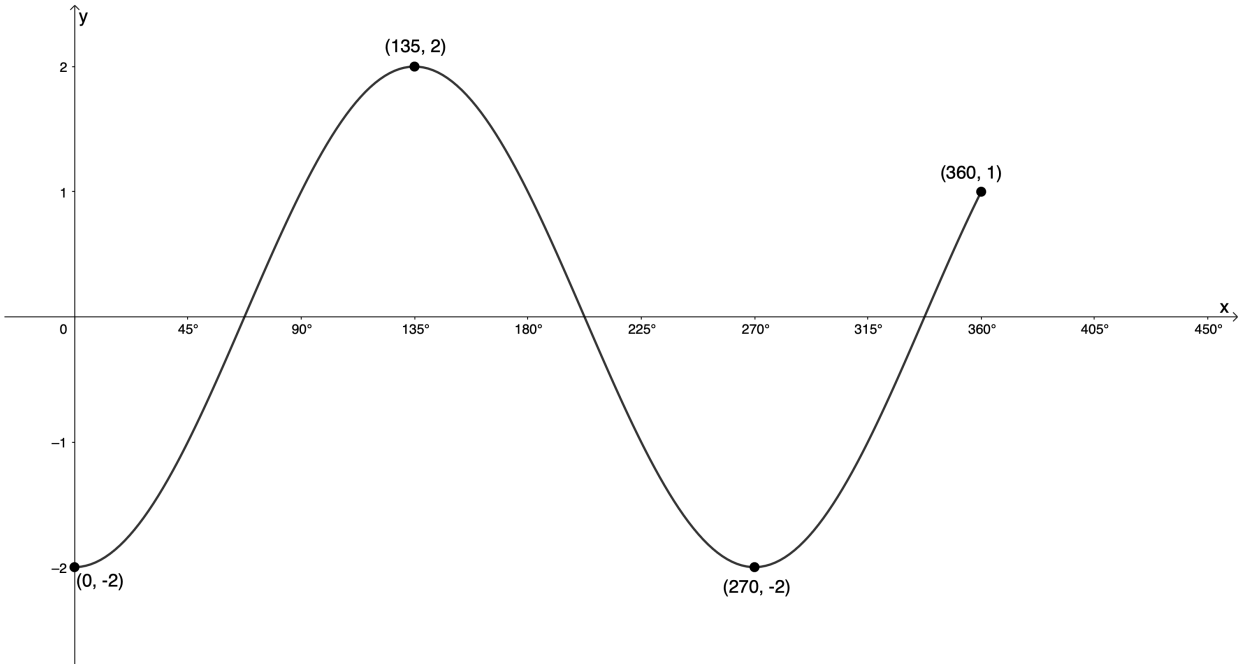
$$540^\circ = \frac{360^\circ}{|k|}$$

$$\therefore |k| = \frac{360^\circ}{540^\circ} = \frac{2}{3}$$

$$y = -\frac{4}{5}\cos\left(\frac{2}{3}x\right)$$

Exercise 9.3

Given the graph of the form $y = a\cos kx$:



1. Determine the values of a and k.
2. State the domain and range of the function.

The full solutions are at the end of the unit.

Summary

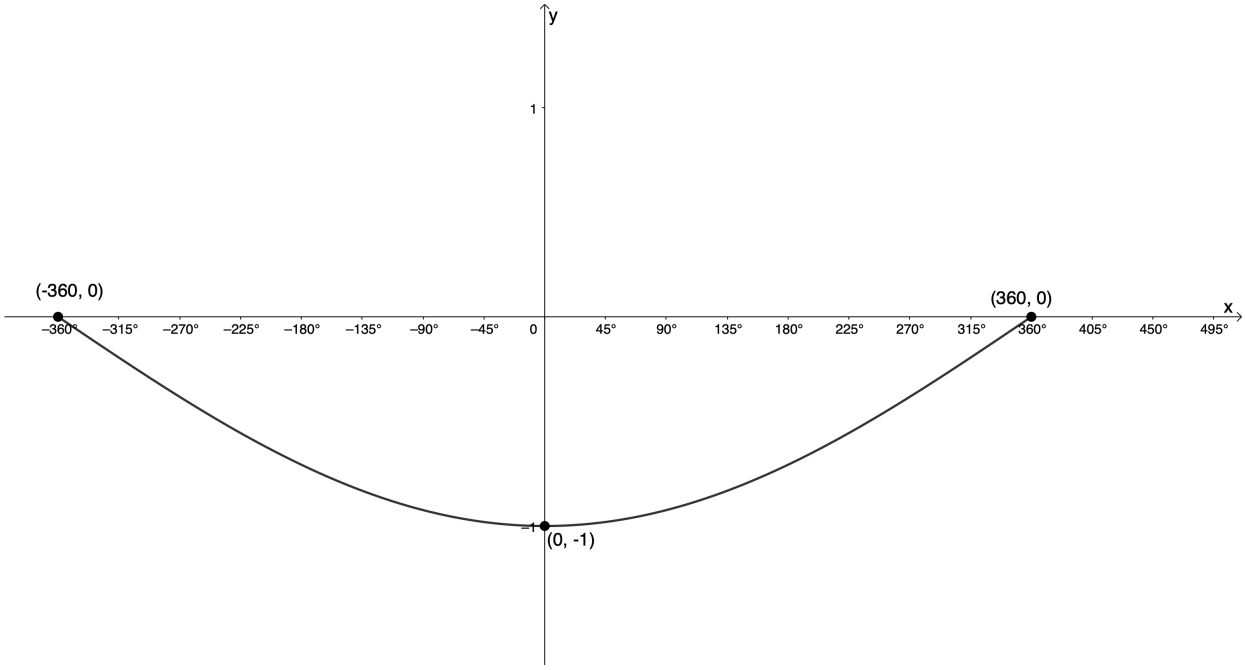
In this unit you have learnt the following:

- The effects of a and k on the cosine graph of the form $y = a\cos k\theta$.
- How to sketch functions of the form $y = a\cos k\theta$.
- How to find the values of a and k from a given cosine graph of the form $y = a\cos k\theta$.

Unit 9: Assessment

Suggested time to complete: 45 minutes

1. Sketch the following functions for the given intervals:
   a. $3y = \cos\left(\frac{1}{2}x\right)$ for $-720^\circ \leq x \leq 360^\circ$
   b. $g(x) = -3\cos 2x$ for $0^\circ \leq x \leq 270^\circ$
2. From the graph below of the form $y = a\cos kx$, determine the values of a and k:



The full solutions are at the end of the unit.

Unit 9: Solutions

Exercise 9.1

1.

a. $y = \cos(2x)$

$k = 2$

$\text{Period} = \dfrac{360^\circ}{|2|} = \dfrac{360^\circ}{2} = 180^\circ$

b. $y = \cos\left(-\dfrac{3}{5}x\right)$

$k = -\dfrac{3}{5}$

$\text{Period} = \dfrac{360^\circ}{\left|-\dfrac{3}{5}\right|} = \dfrac{360^\circ}{\dfrac{3}{5}} = \dfrac{5 \times 360^\circ}{3} = 600^\circ$

c. $y = \cos\left(-\dfrac{3}{2}x\right)$

$k = -\dfrac{3}{2}$

$\text{Period} = \dfrac{360^\circ}{\left|-\dfrac{3}{2}\right|} = \dfrac{2 \times 360^\circ}{3} = 240^\circ$

2.

a. The period of the graph is 720°.

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$720^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = \frac{360^\circ}{720^\circ}$$
$$\therefore |k| = \frac{1}{2}$$

b. The period of the graph is 288°.

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$288^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = \frac{360^\circ}{288^\circ} = \frac{5}{4}$$

c. The period of the graph is .

$$\text{Period} = \frac{360^\circ}{|k|}$$
$$120^\circ = \frac{360^\circ}{|k|}$$
$$\therefore |k| = \frac{360^\circ}{120^\circ} = 3$$

Back to Exercise 9.1

Exercise 9.2

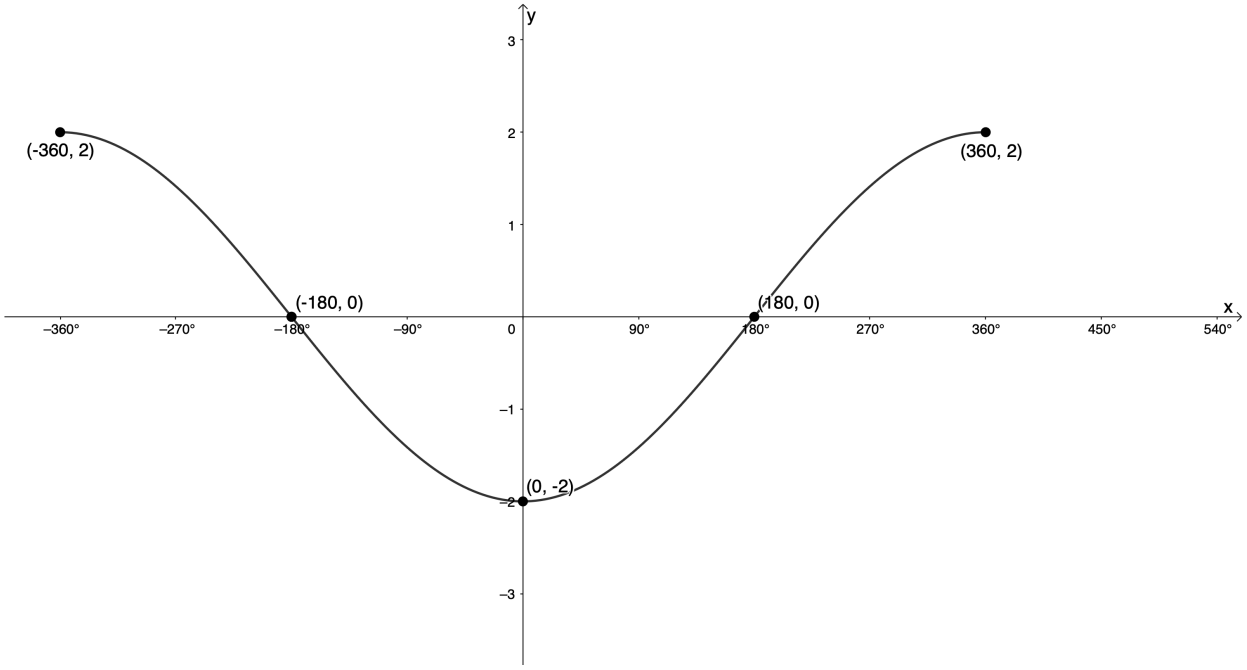
1. $y = -2\cos\frac{\theta}{2}$ for $-360^\circ \leq \theta \leq 360^\circ$

   $k = \frac{1}{2}$

   $\text{Period} = \frac{360^\circ}{|k|} = \frac{360^\circ}{\left|\frac{1}{2}\right|} = 720^\circ$

   $a = -2$. Amplitude is 2 and graph is reflected about the x-axis.

$\cos\theta$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$-2\cos\left(\frac{\theta}{2}\right)$	$(0^\circ, -2)$	$(180^\circ, 0)$	$(360^\circ, 2)$	$(540^\circ, 0)$	$(720^\circ, -2)$



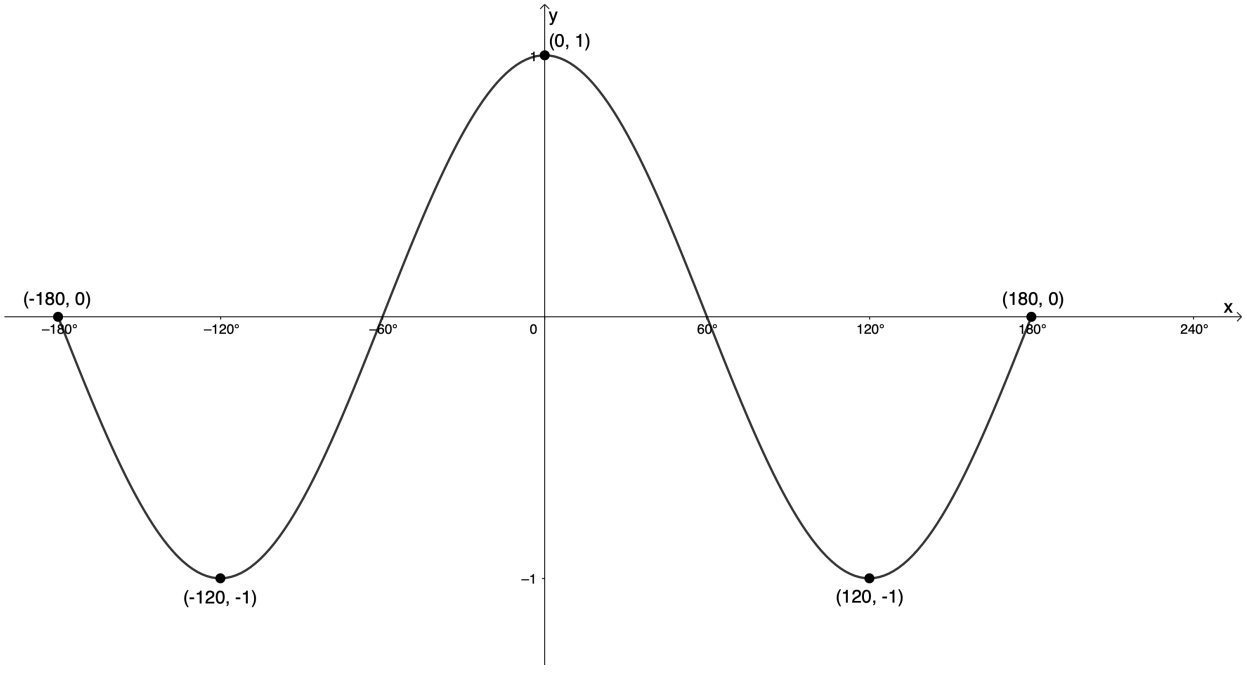
2. $y = \cos\left(-\frac{3x}{2}\right)$ for $-180^\circ \leq x \leq 180^\circ$

   $k = -\frac{3}{2}$

   $\text{Period} = \frac{360^\circ}{|k|} = \frac{360^\circ}{\left|-\frac{3}{2}\right|} = \frac{360^\circ}{\frac{3}{2}} = 240^\circ$

   $a = 1$. Amplitude is 1 and the graph is not reflected about the x-axis.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos\left(-\frac{3x}{2}\right)$	$(0^\circ, 1)$	$(60^\circ, 0)$	$(120^\circ, -1)$	$(180^\circ, 0)$	$(240^\circ, 1)$



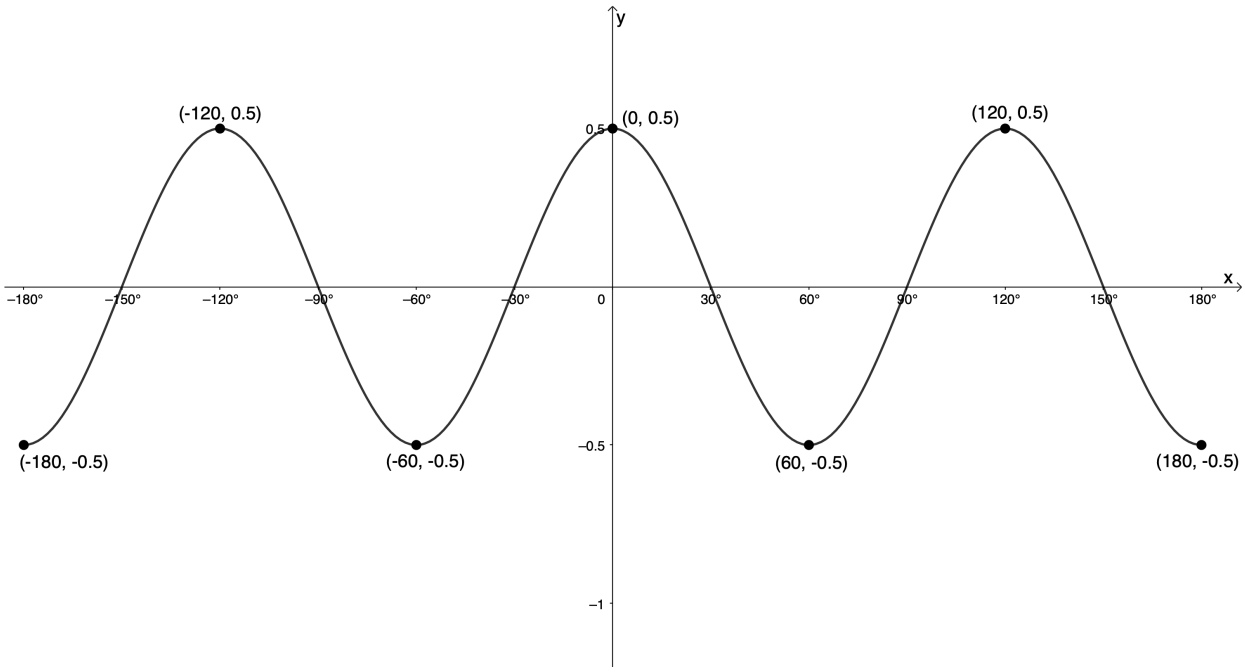
3. $y = \frac{1}{2}\cos 3\theta$ for $-180^\circ \leq x \leq 180^\circ$

$k = 3$

$\text{Period} = \dfrac{360^\circ}{|k|} = \dfrac{360^\circ}{|3|} = 120^\circ$

$a = \dfrac{1}{2}$. Amplitude is $\dfrac{1}{2}$ and graph is not reflected about the x-axis.

$\cos\theta$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\dfrac{1}{2}\cos(3\theta)$	$(0^\circ, \dfrac{1}{2})$	$(30^\circ, 0)$	$(60^\circ, -\dfrac{1}{2})$	$(90^\circ, 0)$	$(120^\circ, \dfrac{1}{2})$



Back to Exercise 9.2

Exercise 9.3

1. The amplitude of the function is $2$and graph is reflected about the x-axis. Therefore, $a = -2$ and $y = -2\cos kx$.
   The period is 270°. Therefore $k = \dfrac{360^\circ}{270^\circ} = \dfrac{4}{3}$.
   $y = -2\cos\left(\dfrac{4}{3}x\right)$
2. Domain: $\{x | x \in \mathbb{R}, 0^\circ \leq x \leq 360^\circ\}$
   Range: $\{y | y \in \mathbb{R}, \ -2 \leq x \leq 2\}$

Back to Exercise 9.3

Unit 9: Assessment

1.

   a. $3y = \cos\left(\dfrac{1}{2}x\right)$ for $-720^\circ \leq x \leq 360^\circ$

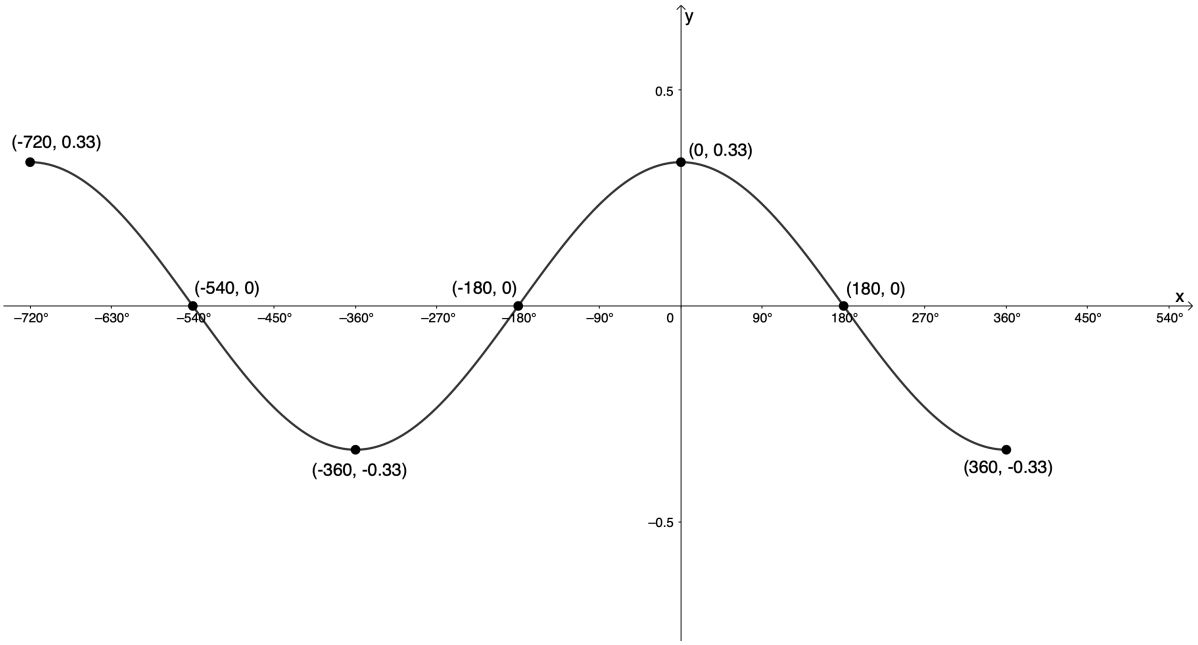
$3y = \cos\left(\frac{1}{2}x\right)$

$\therefore y = \frac{1}{3}\cos\left(\frac{1}{2}x\right)$

$k = \frac{1}{2}$. Therefore, $\text{period} = \frac{360^\circ}{\left|\frac{1}{2}\right|} = 720^\circ$

$a = \frac{1}{3}$. Therefore, amplitude is $\frac{1}{3}$.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\frac{1}{3}\cos\left(\frac{1}{2}x\right)$	$(0^\circ, \frac{1}{3})$	$(180^\circ, 0)$	$(360^\circ, -\frac{1}{3})$	$(540^\circ, 0)$	$(720^\circ, \frac{1}{3})$

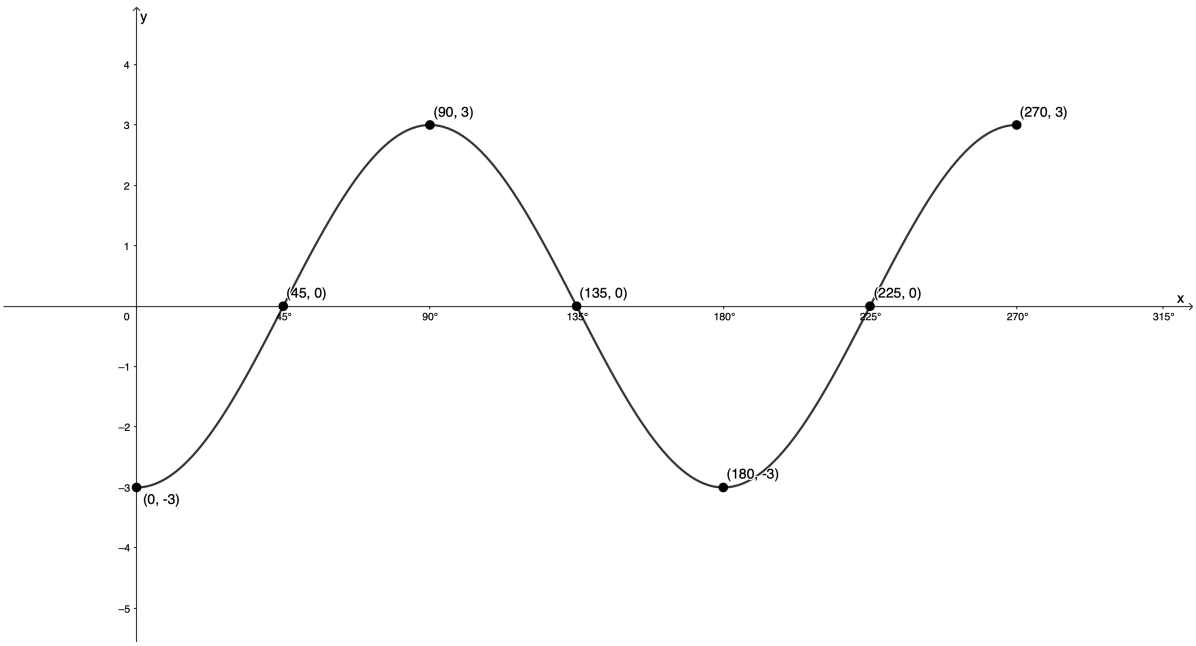


b. $g(x) = -3\cos 2x$ for $0^\circ \leq x \leq 270^\circ$

$k = 2$. Therefore, $\text{period} = \frac{360^\circ}{|2|} = 180^\circ$

$a = -3$. Therefore, amplitude is 3 and the graph is reflected about the x-axis.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$-3\cos 2x$	$(0^\circ, -3)$	$(45^\circ, 0)$	$(90^\circ, 3)$	$(135^\circ, 0)$	$(180^\circ, -3)$



2. The amplitude is 1 and the graph is reflected about the x-axis. Therefore, $a = -1$ and $y = -\cos kx$.
The graph completes half a period between -360° and 360°. Therefore, the period is $2 \times 720^\circ = 1\ 440^\circ$.

$$\text{Period} = \frac{360^\circ}{|k|}$$

$$1\ 440^\circ = \frac{360^\circ}{|k|}$$

$$\therefore |k| = \frac{360^\circ}{1\ 440^\circ} = \frac{1}{4}$$

$$y = -\cos\left(\frac{1}{4}x\right)$$

Back to Unit 9: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © DHET is licensed under a CC BY (Attribution) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- takenote © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.1Q2a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.1Q2b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.1Q2c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure6 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example9.2 © Geogebra is licensed under a CC BY (Attribution) license
- example9.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example9.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example9.5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.2A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise9.2A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 10: Horizontal transformation of the cosine graph

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = \cos(\theta + p)$.
- Determine the effects of positive and negative values of p on the cosine graph $y = \cos(\theta + p)$.
- Find the value of p from a given cosine graph of the form $y = \cos(\theta + p)$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \cos x$ and $y = \cos\theta$ are the same function.

What you should know

Before you start this unit, make sure you can:

- Sketch cosine functions of the form $y = a\cos\theta + q$. Refer to level 2 subject outcome 2.1 Unit 5 if you need help with this.

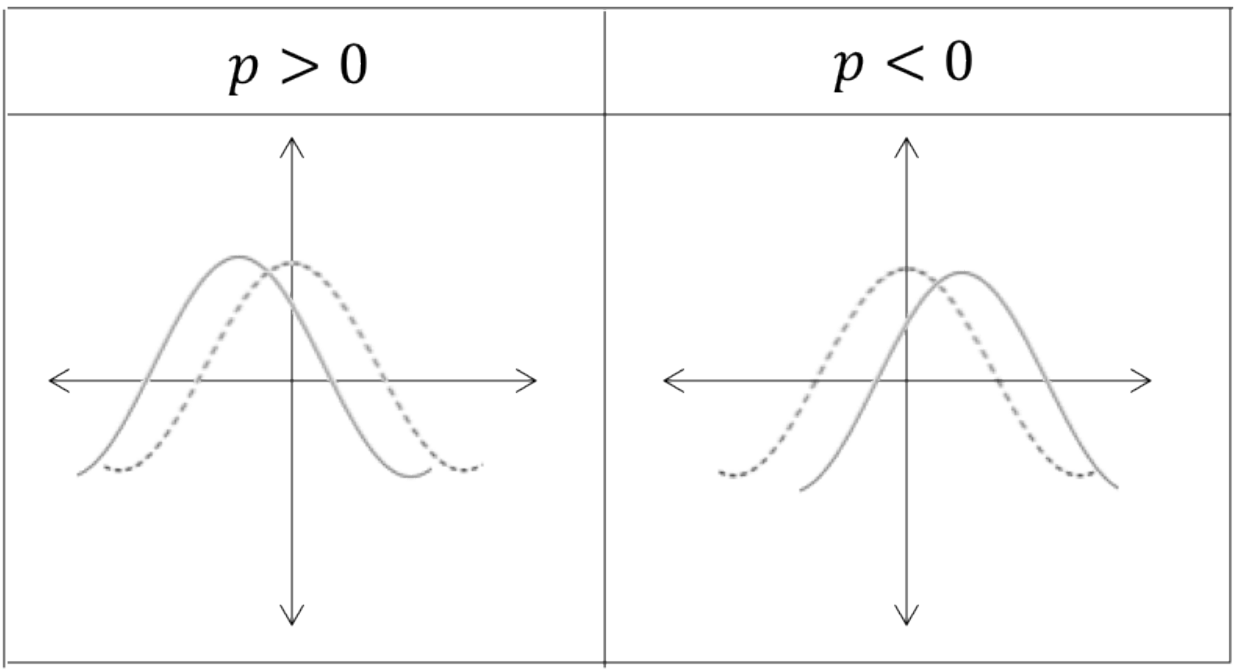
Introduction

It should not surprise you that the effect of p in $y = \cos(x + p)$ is to shift the graph horizontally by p units. If $p > 0$, the graph is shifted p units **to the left**. If $p < 0$, the graph is shifted p units **to the right**.

Take note!

In $y = \cos(x + p)$:

- if $p > 0$, the graph is shifted p units **to the left**
- if $p < 0$, the graph is shifted p units **to the right**.



Note

If you have an internet connection, spend some time playing with this interactive simulation.

Here you will find a graph of the function $y = \cos(x + p)$ with a slider to change the value of p. Pay particular attention to how changing the value of p affects the location of the turning points and the intercepts with the x-axis.

Sketch functions of the form $y = \cos(x + p)$

The best way to sketch functions of the form $y = \cos(x + p)$ is to transform the basic function of $y = \cos x$ depending on the value of p. To do this, you need to know the set of 'anchor points' of $y = \cos x$, as transformation of these points will help you to sketch functions of the form$y = \cos(x + p)$.

Example 10.1

Given the function $y = \cos(x - 30^\circ)$:

1. Sketch the function for the interval $0^\circ \leq x \leq 360^\circ$.
2. State the intercepts with the axes.
3. State the domain and range.
4. State the period.
5. State the amplitude.

Solutions

1. The function is of the form $y = \cos(x + p)$ with $p = -30^\circ$.

   We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 30° to the right.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos(x - 30^\circ)$	$(30^\circ, 1)$	$(120^\circ, 0)$	$(210^\circ, -1)$	$(300^\circ, 0)$	$(390^\circ, 1)$

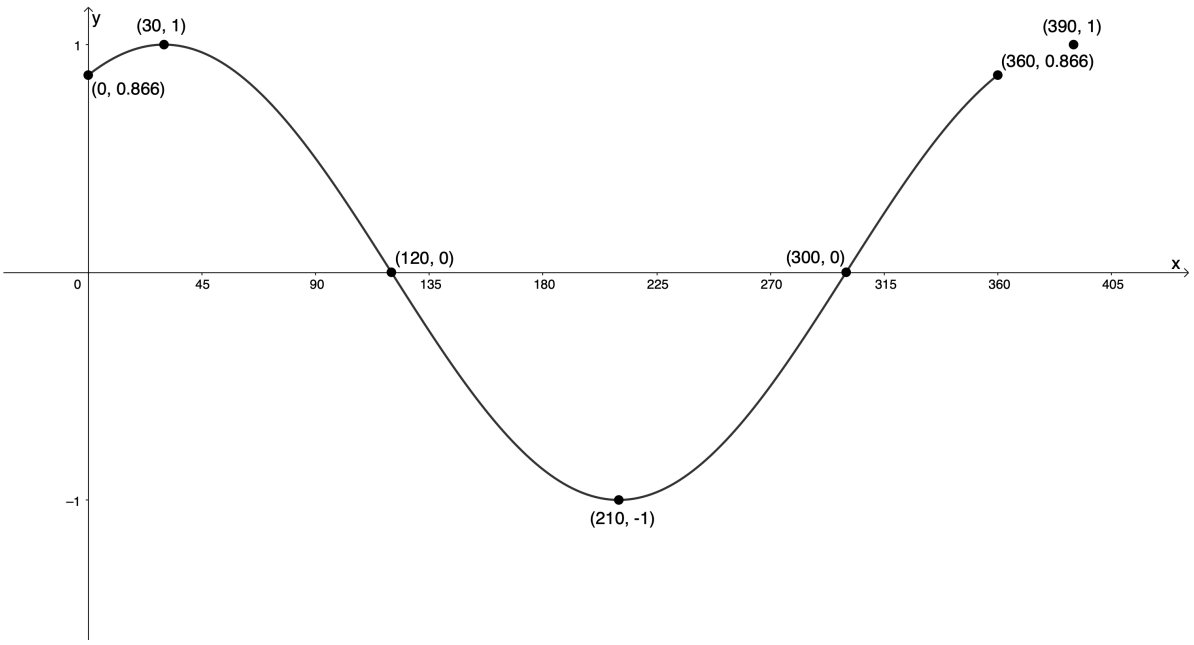
   To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.
   y-intercept (let $x = 0$):

   $$y = \cos(0^\circ - 30^\circ)$$

   $$\therefore y = \frac{\sqrt{3}}{2} = 0.866$$

   Because the period of the function is 360° we know that the function value at 360° will also be 0.866.

   We can now plot our transformed 'anchor points' and draw the graph for the interval $0^\circ \leq x \leq 360^\circ$.



2. y-intercept: $(0^\circ, 0.866)$
   x-intercepts: $(120^\circ, 0)$ and $(300^\circ, 0)$
3. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$
   Range: $\{y | y \in \mathbb{R},\ -1 \leq x \leq 1\}$
4. The period is 360°.
5. The amplitude is 1.

Take note!

In general, the domain of the function $y = \cos(x + p)$ is $x \in \mathbb{R}$. We had to restrict the domain in Example 10.1, because the interval in which we were working was restricted.

Exercise 10.1

Sketch the following functions for the indicated intervals on separate sets of axes:

1. $f(x) = \cos(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
2. $g(x) = \cos(x - 45^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
3. $h(x) = \cos(x - 40^\circ)$ for $-360^\circ \leq x \leq 360^\circ$

The full solutions are at the end of the unit.

Sketch functions of the form $y = a\cos(x + p)$

Now that we know that the value of p shifts the function of the form $y = \cos(x + p)$ horizontally by p degrees, we can combine this with our knowledge of the effects of a, and examine functions of the form $y = a\cos(x + p)$.

Example 10.2

Sketch the graph of $f(x) = 2.5\cos(x + 60^\circ)$ for the interval $-360^\circ \leq x \leq 360^\circ$.

Solution

The function is of the form $y = a\cos(x + p)$ where $a = 2.5$ and $p = 60^\circ$.

We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 60° to the left and each of the y-values is going to be multiplied by 2.5.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$2.5\cos(x + 60^\circ)$	$(-60^\circ, 2.5)$	$(30^\circ, 0)$	$(120^\circ, -2.5)$	$(210^\circ, 0)$	$(300^\circ, 2.5)$

To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.
y-intercept (let $x = 0$):

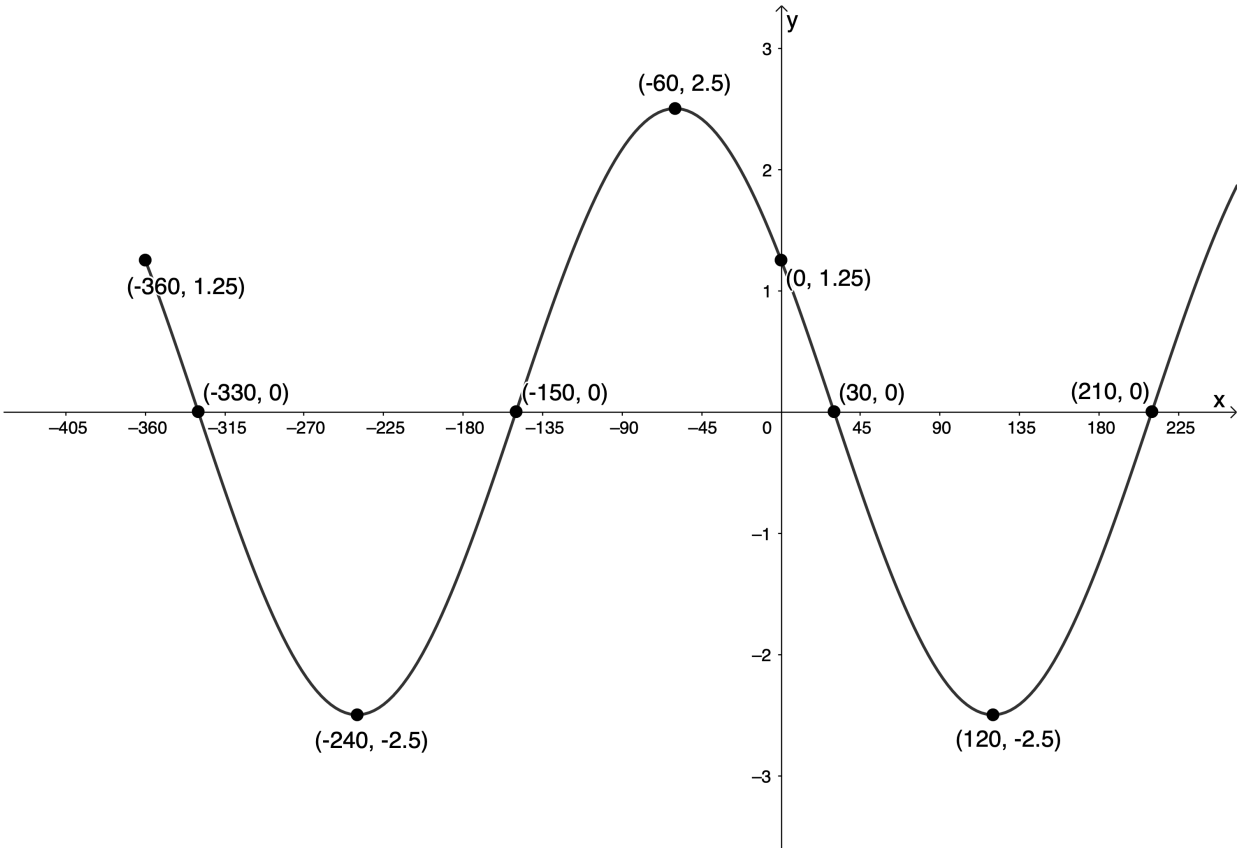
$$y = 2.5\cos(0 + 60^\circ)$$

$$\therefore y = 2.5 \times \frac{1}{2}$$

$$= 1.25$$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 1.25.

We can now plot our transformed 'anchor points' and draw the graph for the interval $-360^\circ \leq x \leq 360^\circ$.



Example 10.3

Given the function $f(x) = 2\cos(30^\circ - x)$:

1. Sketch the graph for the interval $0^\circ \leq x \leq 360^\circ$.

2. State the intercepts with the axes.
3. State the domain and range.
4. State the amplitude and period.

Solutions

1. The function is not in the form $y = a\cos(x + p)$. We first need to get it into this form.
$$\begin{aligned} f(x) &= 2\cos(30^\circ - x) \\ &= 2\cos(-x + 30^\circ) \\ &= 2\cos(-(x - 30^\circ)) \\ &= 2\cos(x - 30^\circ) \end{aligned}$$

Remember we learnt in unit 9 that $\cos(-x) = \cos x$ because the function is symmetrical about the y-axis and this meant that we get the same function value for $-x$ as we do for x. This is why $2\cos(-(x - 30^\circ)) = 2\cos(x - 30^\circ)$.

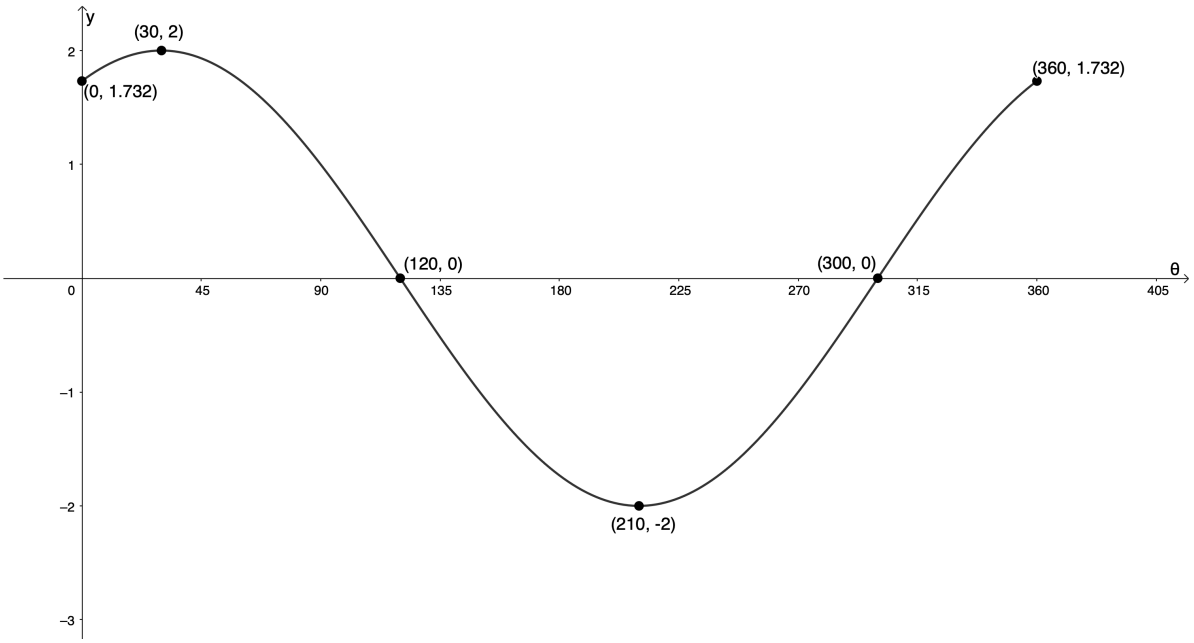
We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 30° to the right and each of the y-values of these points need to be multiplied by 2.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$2\cos(x - 30^\circ)$	$(30^\circ, 2)$	$(120^\circ, 0)$	$(210^\circ, -2)$	$(300^\circ, 0)$	$(390^\circ, 2)$

To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.
y-intercept (let $x = 0$):
$$f(0) = 2\cos(0^\circ - 30^\circ)$$
$$\therefore f(0) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} = 1.732$$

Because the period of the function is 360° we know that the function value at 360° will also be 1.732.

We can now plot our transformed 'anchor points' and draw the graph for the interval $0^\circ \leq x \leq 360^\circ$.



2. y-intercept: $(0^\circ, \sqrt{3})$ or $(0^\circ, 1.732)$
   x-intercepts: $(120^\circ, 0)$ and $(300^\circ, 0)$
3. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$
   Range: $\{f(x) | f(x) \in \mathbb{R},\ -2 \leq f(x) \leq 2\}$
4. The period is 360°.
   The amplitude is 2.

Exercise 10.2

For each of the following functions, sketch the graph for the indicated interval and state the domain and range and intercepts with the axes:

1. $y = -2\cos(x + 30^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
2. $2y = \cos(x + 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$
3. $\frac{1}{3}y = \cos(60^\circ - x)$ for $-360^\circ \leq x \leq 360^\circ$

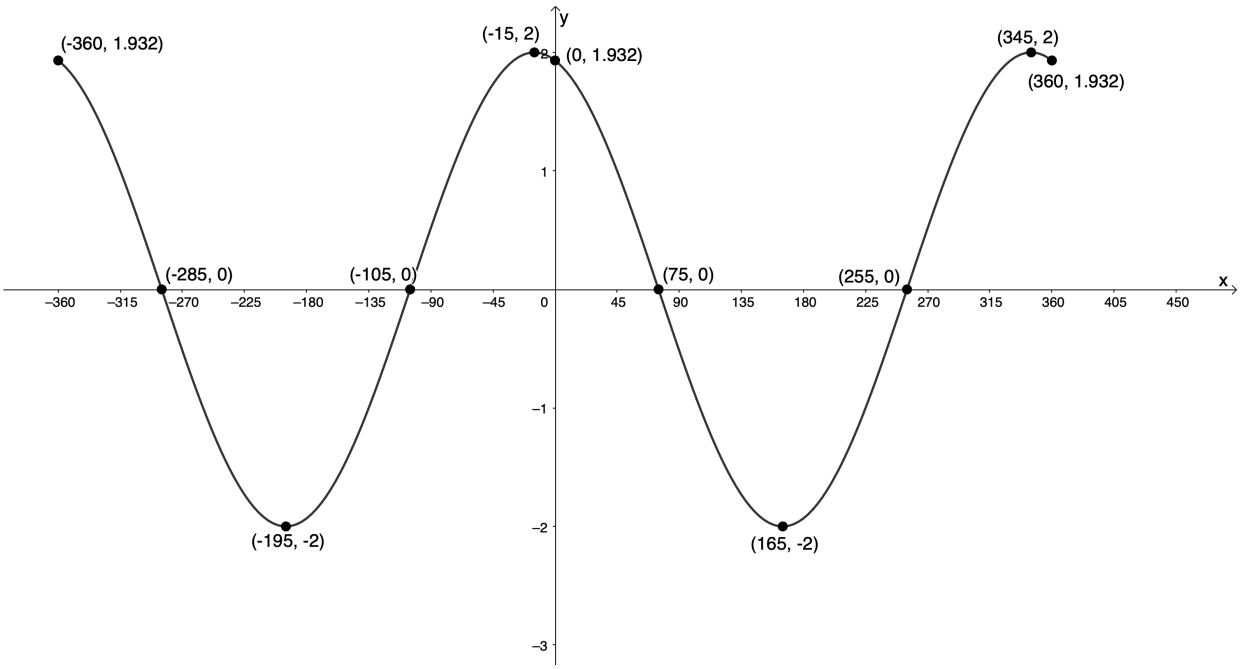
The full solutions are at the end of the unit.

Find the equation of a cosine function of the form $y = a\cos(x + p)$

By examining the amplitude and the horizontal shift of $y = a\cos(x + p)$, we can determine the values of a and p.

Example 10.4

The graph below is a function of the form $y = a\cos(x + p)$. Determine the values of a and p.



Solution

We are told that the function is of the form $y = a\cos(x + p)$.

Firstly, we can see that the amplitude of the graph is 2 units. We do not yet know whether $a = 2$ or $a = -2$.

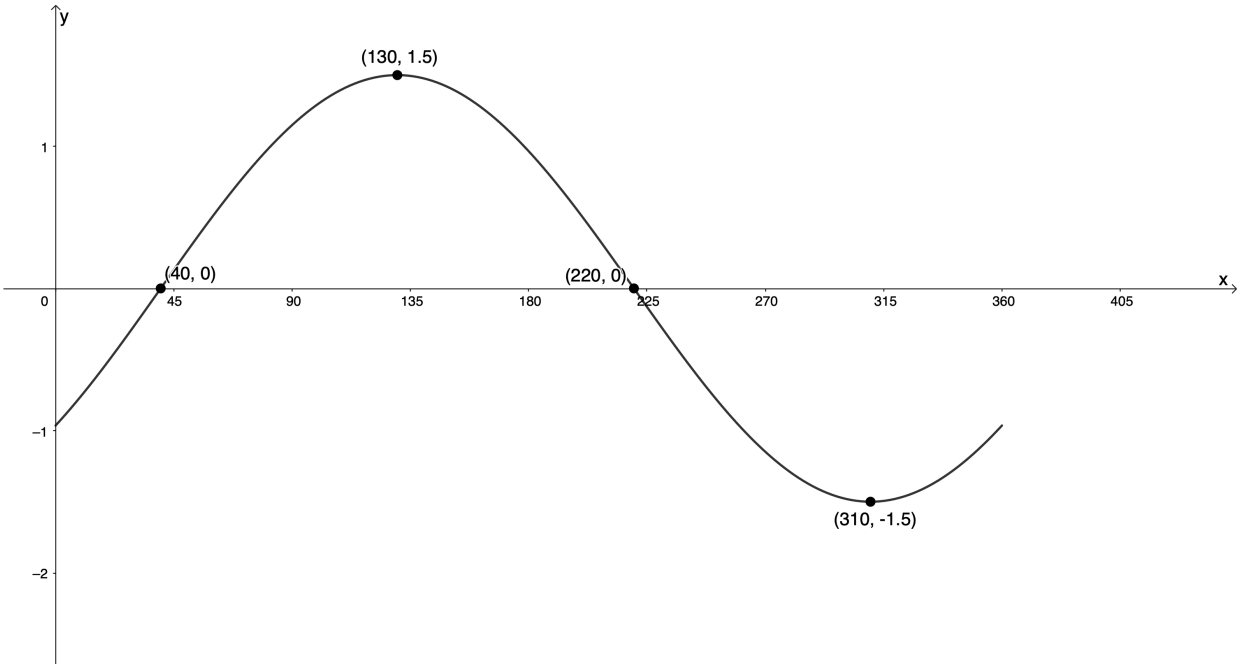
We know that the function $y = \cos x$ has a maximum turning point at $(0^\circ, 1)$. The function above has a maximum turning point at $(-15^\circ, 2)$. Therefore, the function has been shifted 15° to the left and $p = 15^\circ$.

The amplitude of the graph is 2 and has not been reflected about the x-axis. Therefore, $a = 2$.

The function is $y = 2\cos(x + 15^\circ)$.

Example 10.5

The graph below is a function of the form $y = a\cos(x + p)$. Determine the values of a and p.



Solution

We are told that the function is of the form $y = a\cos(x + p)$.

Firstly, we can see that the amplitude of the graph is 1.5 units. We do not yet know whether $a = 1.5$ or $a = -1.5$.

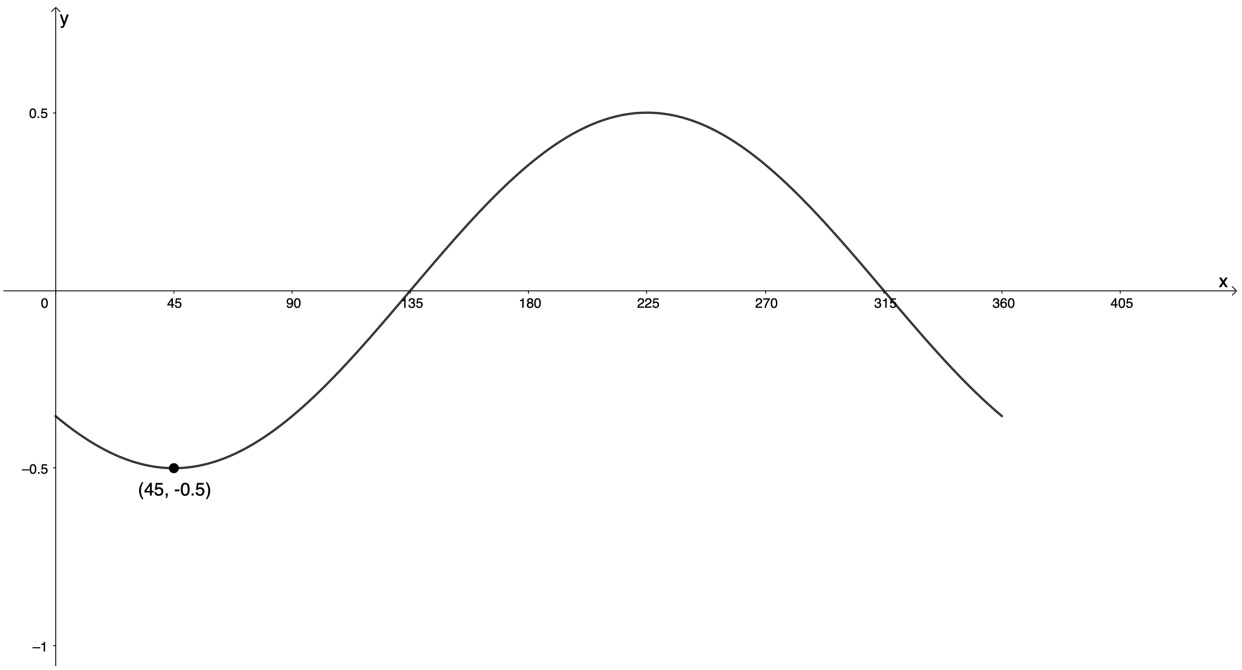
We know that the function $y = \cos x$ has a maximum turning point at $(0^\circ, 1)$ and then cuts the x-axis at $(90^\circ, 0)$. This function instead has a maximum turning point at $(130^\circ, 1.5)$. Therefore, we can say that the point $(0^\circ, 1)$ has been transformed into the point $(130^\circ, 1.5)$. This would indicate that that graph has been shifted 130° to the right and therefore that$p = -130^\circ$.

However, we generally prefer $-90^\circ \leq p \leq 90^\circ$. We also know that the function $y = \cos x$ has a minimum turning point at $(180^\circ, -1)$. Therefore, we can also say that the point $(130^\circ, 1.5)$ is the result of a 50° shift to the left and that the function has been reflected about the x-axis. Because this gives us a value for p that is $-90^\circ \leq p \leq 90^\circ$, this is the transformation we will use.

This means that $p = 50^\circ$, $a = -1.5$ and that the function is $y = -1.5\cos(x + 50^\circ)$.

Exercise 10.3

Given the graph of the form $y = a\cos(x + p)$:



1. Determine the values of a and p.
2. State the domain and range of the function.

The full solutions are at the end of the unit.

The relationship between $y = \sin x$ and $y = \cos x$

You may have noticed that the names of the functions sine and cosine are similar. Well, there is a very good reason for this as we will discover in Activity 10.1.

Activity 10.1: Co-functions

Time required: 20 minutes

What you need:

- a pen or pencil
- a calculator
- several pieces of paper

What to do:

1. Consider the function $f(x) = \cos(x - 90^\circ)$. Make a neat sketch of the graph of this function for the interval $0^\circ \leq x \leq 360^\circ$ on a piece of paper. Use the following table of 'anchor points' to help you.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos(x - 90^\circ)$					

2. What do you notice about the graph of the function $f(x) = \cos(x - 90^\circ)$ and the function $y = \sin x$?
3. Write a mathematical expression that captures this relationship.
4. Now consider the function $g(x) = \sin(x - 90^\circ)$. Make a neat sketch of the graph of this function $0^\circ \leq x \leq 360^\circ$ on a new piece of paper. Use the following table of 'anchor points' to help you.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x - 90^\circ)$					

5. What do you notice about the graph of the function $g(x) = \sin(x - 90^\circ)$ and the function $y = \cos x$?
6. Write a mathematical expression that captures this relationship.
7. Do these relationships between the sine and cosine functions hold for $y = \cos(x + 90^\circ)$ and $y = \sin(x + 90^\circ)$? Draw the graphs of these functions to help you.

What did you find?

1. Here is the completed table of values. The graph of $f(x) = \cos(x - 90^\circ)$ for the interval $0^\circ \leq x \leq 360^\circ$ is shown in Figure 7.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos(x - 90^\circ)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$	$(450^\circ, 1)$

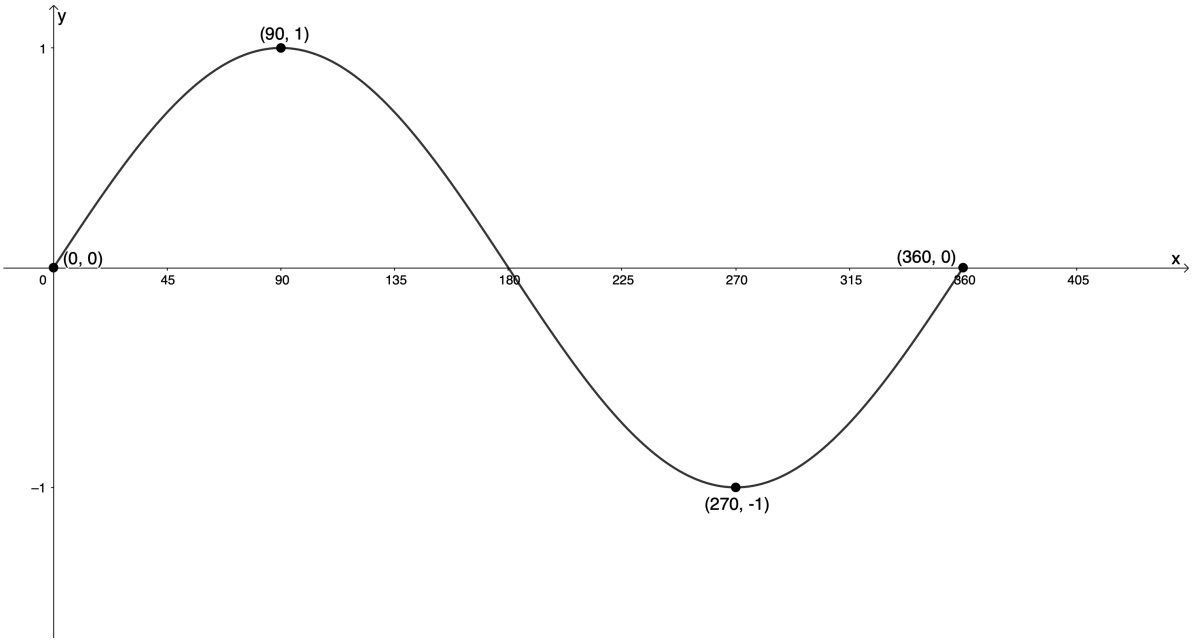


Figure 7: Graph of $f(x) = \cos(x - 90^\circ)$

2. The graph of $f(x) = \cos(x - 90^\circ)$ is the same as the graph of $y = \sin x$.
3. $\cos(x - 90^\circ) = \sin x$
4. Here is the completed table of values. The graph of $g(x) = \sin(x - 90^\circ)$ for the interval $0^\circ \leq x \leq 360^\circ$ is shown in Figure 8.

$\sin x$	$(0^\circ, 0)$	$(90^\circ, 1)$	$(180^\circ, 0)$	$(270^\circ, -1)$	$(360^\circ, 0)$
$\sin(x - 90^\circ)$	$(90^\circ, 0)$	$(180^\circ, 1)$	$(270^\circ, 0)$	$(360^\circ, -1)$	$(450^\circ, 0)$

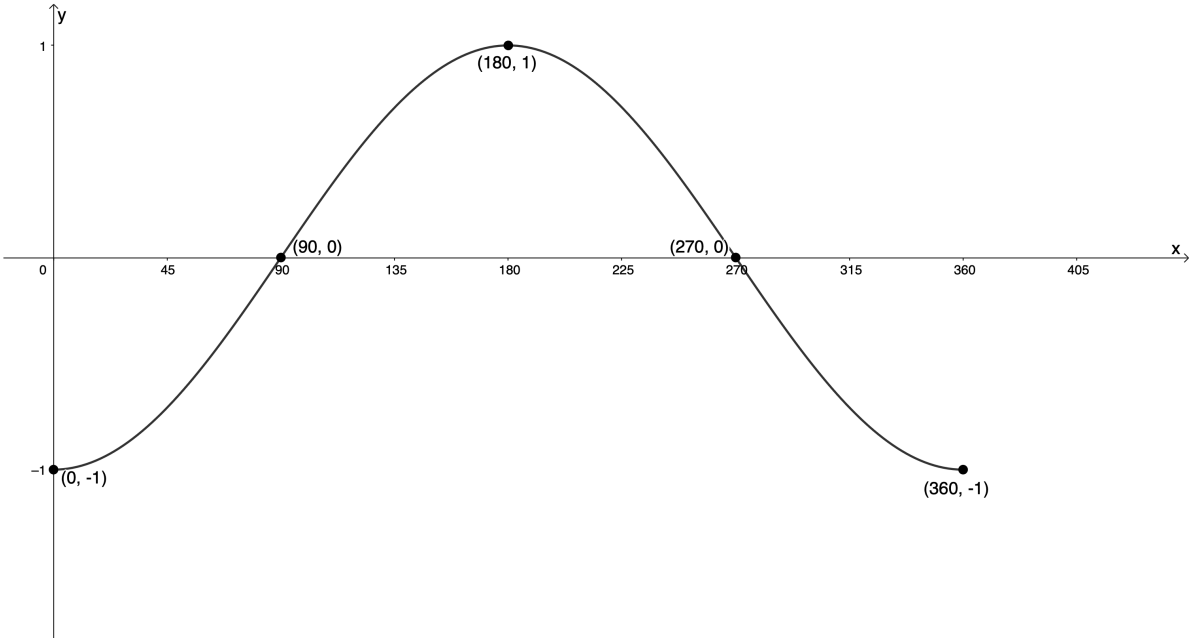


Figure 8: Graph of $f(x) = \sin(x - 90^\circ)$

5. The graph of $g(x) = \sin(x - 90^\circ)$ is the same as the graph of $y = -\cos x$.
6. $\sin(x - 90^\circ) = -\cos x$
7. If we sketch the function $y = \cos(x + 90^\circ)$ we get the graph in Figure 9. We can see that this is the same as the graph of $y = -\sin x$. Therefore, $\cos(x + 90^\circ) = -\sin x$.

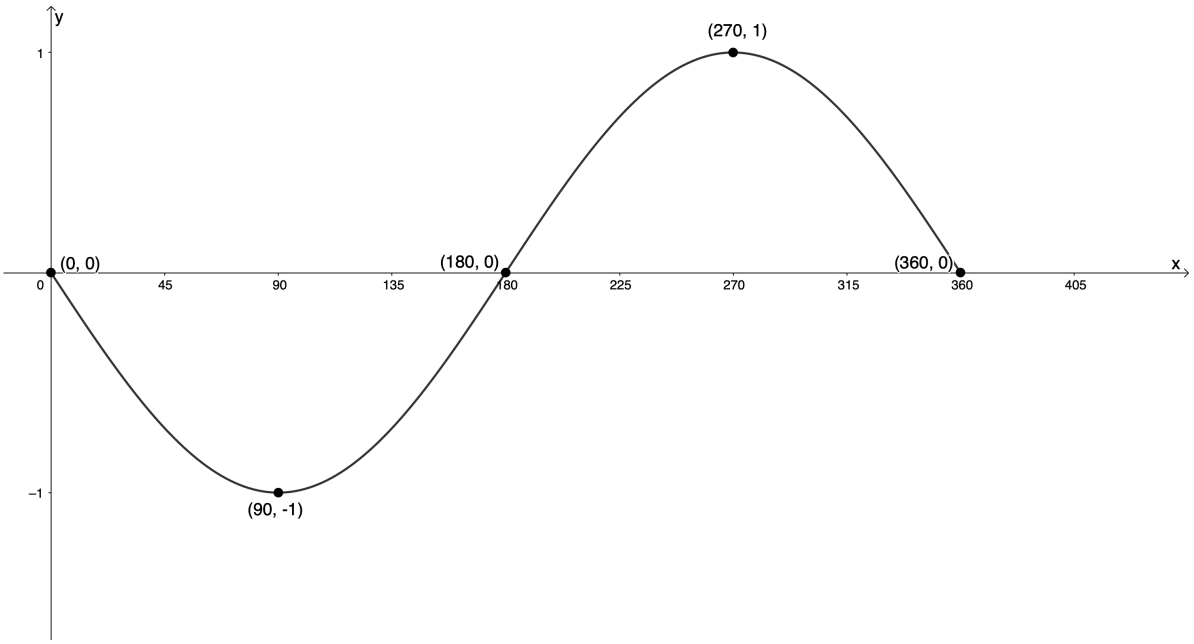


Figure 9: Graph of $y = \cos(x + 90^\circ)$

If we sketch the function $y = \sin(x + 90^\circ)$ we get the graph in Figure 10. We can see that this is the graph of $y = \cos x$. Therefore, $\sin(x + 90^\circ) = \cos x$.

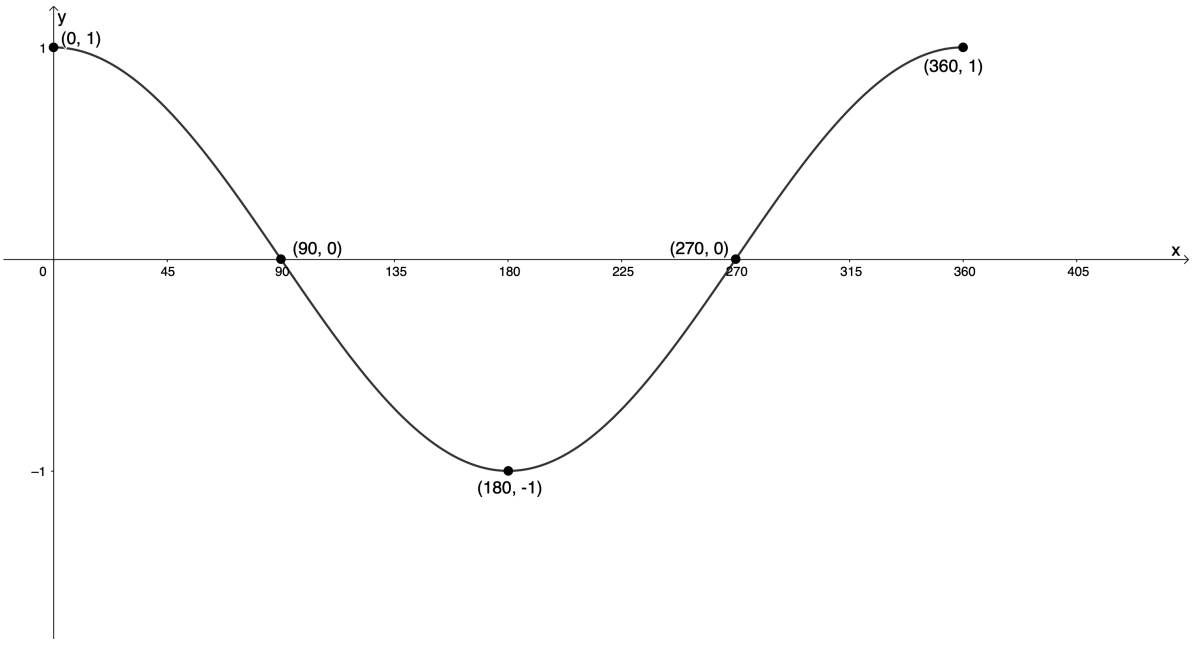


Figure 10: Graph of $y = \sin(x + 90^\circ)$

In Activity 10.1 we learnt of the following relationships between the sine and cosine functions:

- $\cos(x - 90^\circ) = \sin x$

- $\sin(x - 90^\circ) = -\cos x$
- $\cos(x + 90^\circ) = -\sin x$
- $\sin(x + 90^\circ) = \cos x$.

These relationships between the sine and cosine functions are precisely the reason for their names. We say that cosine is the **co-function** of sine because we can change one function into the other by adding or subtracting 90° from the function input values. The reason for the change in signs has to do with whether the original function value is positive or negative.

Take $\cos(x - 90^\circ)$ for example. If we assume that x is an acute angle ($0^\circ < x < 90^\circ$), then $(x - 90^\circ)$ will lie in the fourth quadrant where cosine is positive (see Figure 11). Therefore, because $\cos(x - 90^\circ)$is positive, it is equal to $\sin x$ [$\cos(x - 90^\circ) = \sin x$].

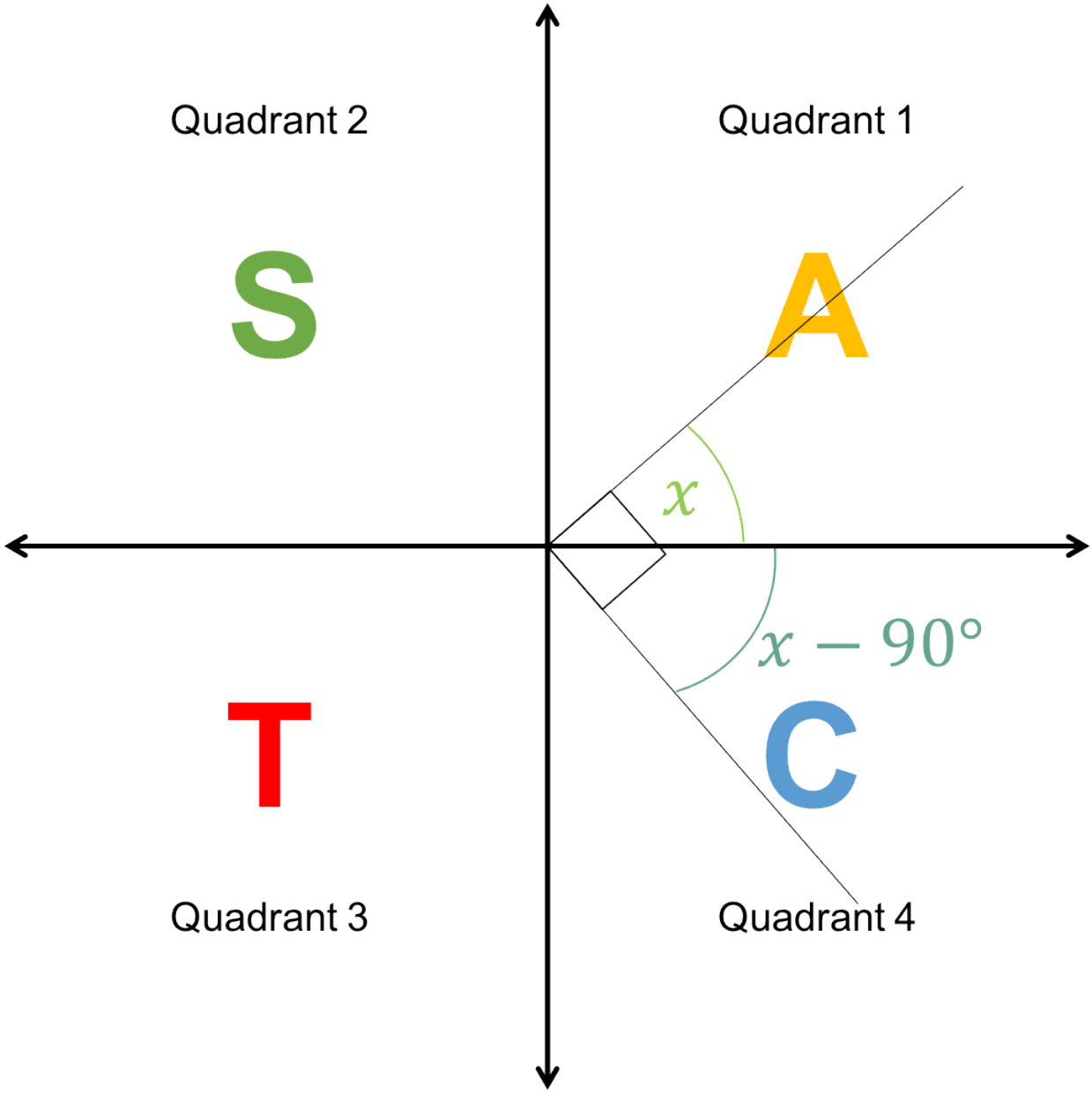


Figure 11: Location of the angle $(x - 90^\circ)$

However, if we take $\sin(x - 90^\circ)$ and still assume that $0^\circ < x < 90^\circ$, then $(x - 90^\circ)$ will still lie in the fourth

quadrant where sine is negative. Therefore, we need the negative value of $\cos x$ to ensure that the ratios are still the same [$\sin(x - 90^\circ) = -\cos x$].

Did you know?

The co-function of tangent is called cotangent or COT for short. Where $\tan\theta = \frac{y}{x}$ on the Cartesian plane, $\cot\theta = \frac{x}{y}$.

Summary

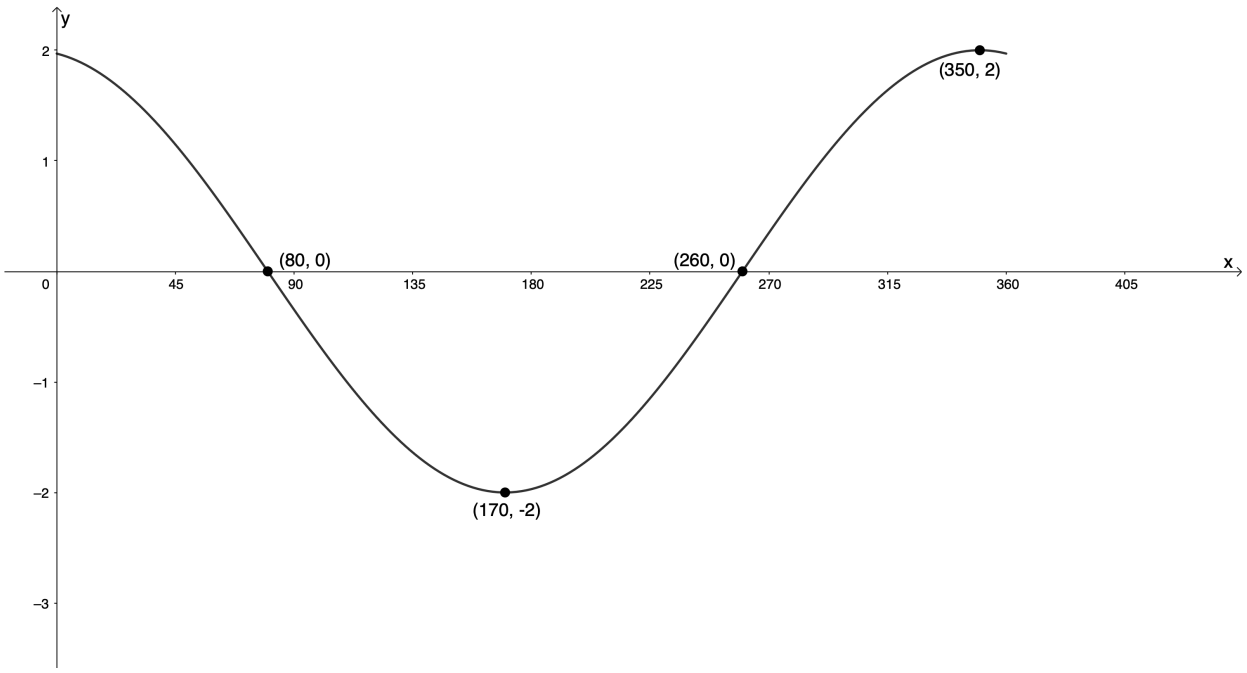
In this unit you have learnt the following:

- The effects of a and p on the cosine graph of the form $y = a\cos(x + p)$.
- How to sketch functions of the form $y = a\cos(x + p)$.
- How to find the values of a and p from a given cosine graph of the form $y = a\cos(x + p)$.
- That sine and cosine are co-functions, and that:
  - $\cos(x - 90^\circ) = \sin x$
  - $\sin(x - 90^\circ) = -\cos x$
  - $\cos(x + 90^\circ) = -\sin x$
  - $\sin(x + 90^\circ) = \cos x$.

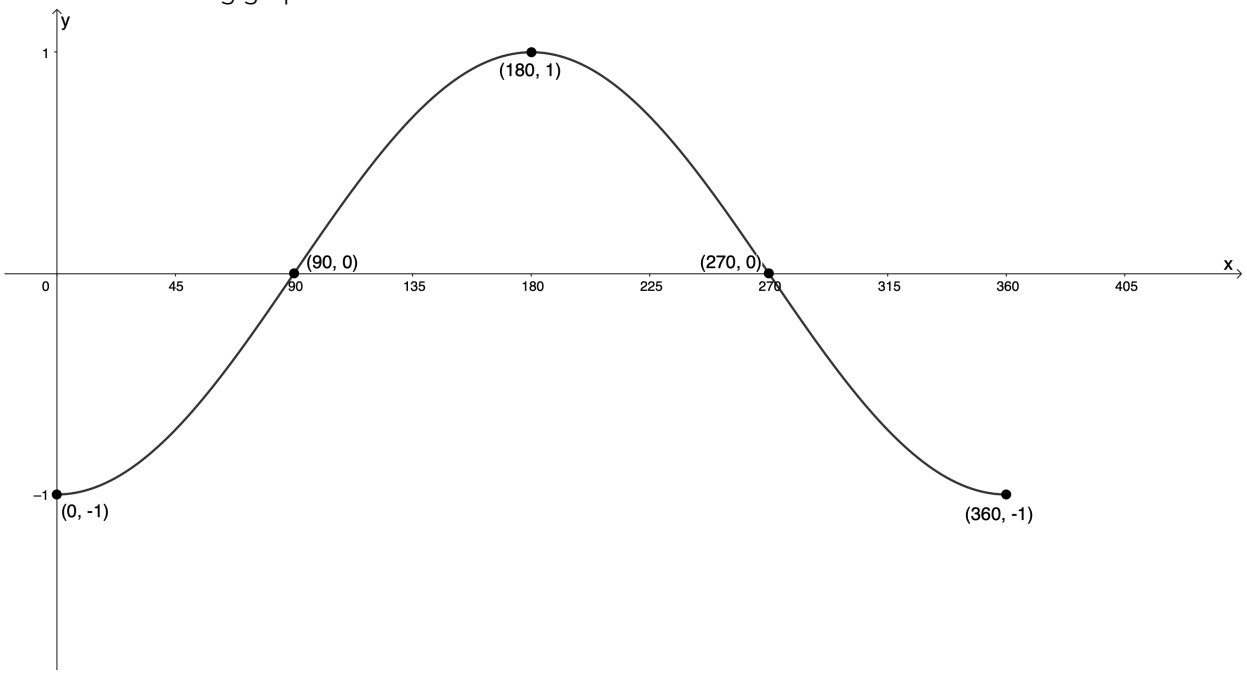
Unit 10: Assessment

Suggested time to complete: 55 minutes

1. Sketch the following functions for the given intervals:
   a. $4y = \cos(x - 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$
   b. $g(x) = -4\cos(45^\circ - x)$ for $-360^\circ \leq x \leq 360^\circ$
2. From the graph below of the form $y = a\cos(x + p)$, determine the values of a and p.



3. Given the following graph:



   a. Write the equation of this function in the form $y = a\cos x$.
   b. Write the equation of this function in the form $y = \cos(x + p)$.
   c. Write the equation of this function in the form $y = \sin(x + p)$.

The full solutions are at the end of the unit.

Unit 10: Solutions

Exercise 10.1

1. $f(x) = \cos(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$

$p = 30^\circ$. Therefore, the graph will be shifted 30° to the left.

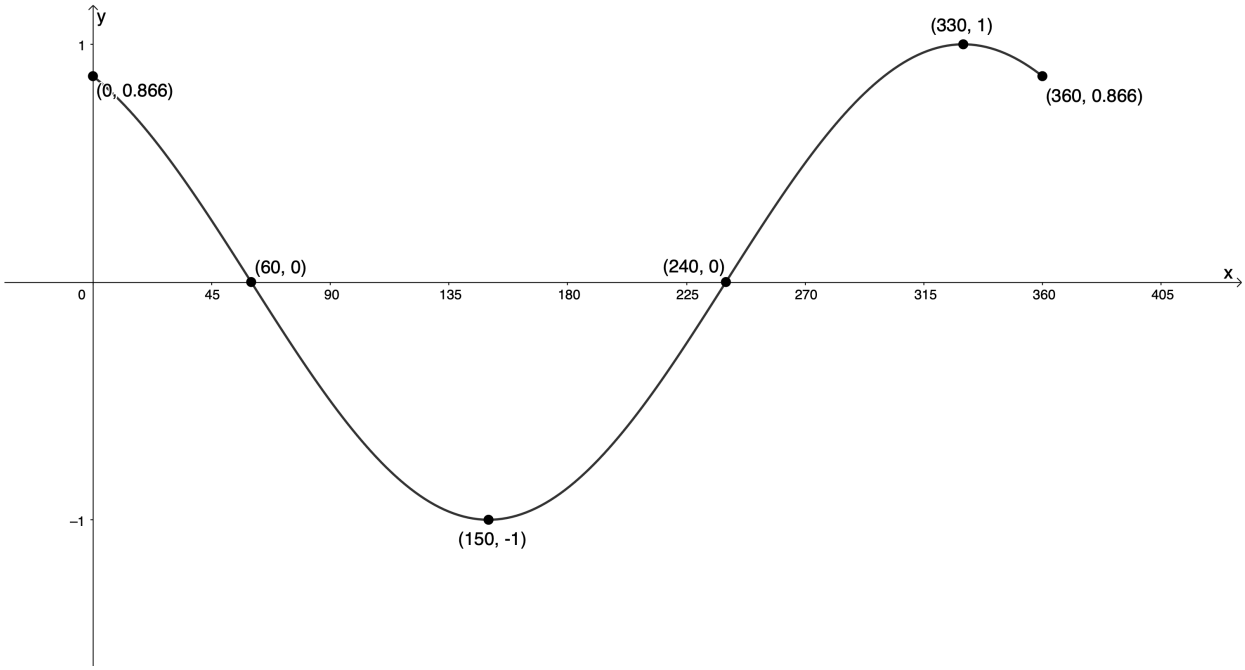
$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos(x + 30^\circ)$	$(-30^\circ, 1)$	$(60^\circ, 0)$	$(150^\circ, -1)$	$(240^\circ, 0)$	$(330^\circ, 1)$

y-intercept (let $x = 0$):

$f(0) = \cos(0^\circ + 30^\circ)$

$\therefore f(0) = \dfrac{\sqrt{3}}{2} = 0.866$

Because the period of the function is 360° we know that the function value at 360° will also be 0.866.



2. $g(x) = \cos(x - 45^\circ)$ for $-360^\circ \leq x \leq 360^\circ$

$p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

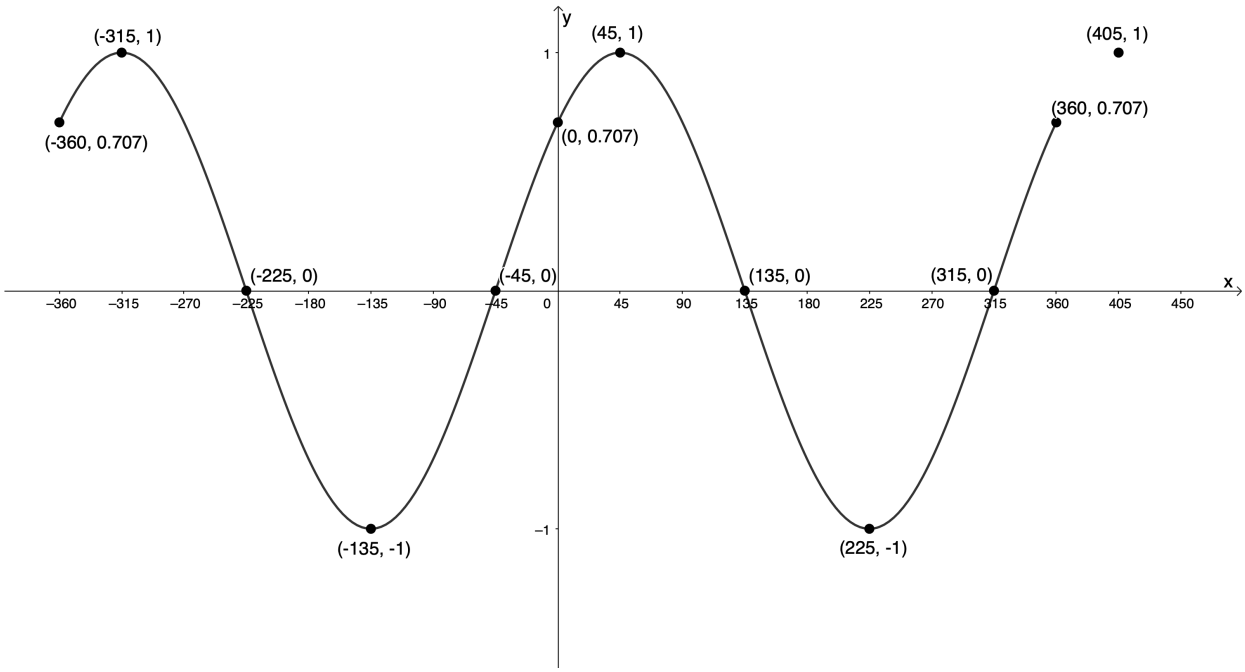
$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos(x - 45^\circ)$	$(45^\circ, 1)$	$(135^\circ, 0)$	$(225^\circ, -1)$	$(315^\circ, 0)$	$(405^\circ, 1)$

y-intercept (let $x = 0$):

$g(0) = \cos(0^\circ - 45^\circ)$

$\therefore g(0) = \dfrac{1}{\sqrt{2}} = 0.707$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 0.707.

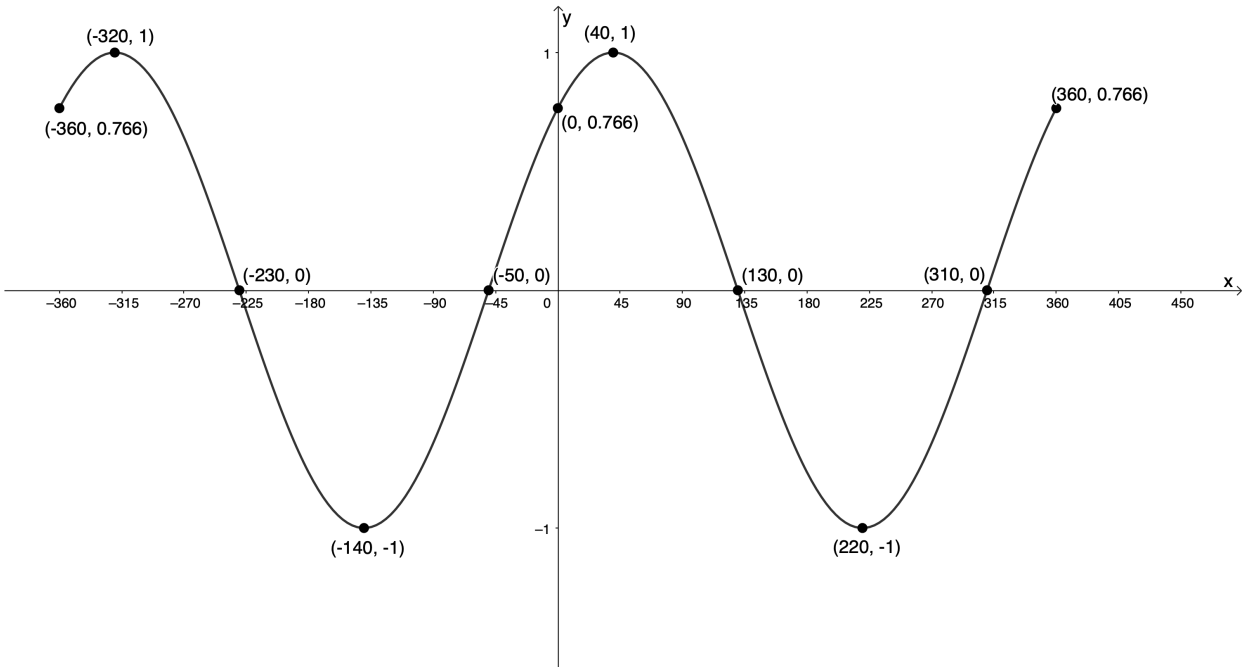


3. $h(x) = \cos(x - 40^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
   $p = -40^\circ$. Therefore, the graph will be shifted 40° to the right.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\cos(x - 40^\circ)$	$(40^\circ, 1)$	$(130^\circ, 0)$	$(220^\circ, -1)$	$(310^\circ, 0)$	$(400^\circ, 1)$

y-intercept (let $x = 0$):
$h(0) = \cos(0^\circ - 40^\circ)$
$\therefore h(0) = 0.766$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 0.766.



Back to Exercise 10.1

Exercise 10.2

1. $y = -2\cos(x + 30^\circ)$ for $-360^\circ \leq x \leq 360^\circ$
   $a = -2$. Therefore, the amplitude will be 2 and the graph will be reflected about the x-axis.
   $p = 30^\circ$. Therefore, the graph will be shifted 30° to the left.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$-2\cos(x + 30^\circ)$	$(-30^\circ, -2)$	$(60^\circ, 0)$	$(150^\circ, 2)$	$(240^\circ, 0)$	$(330^\circ, -2)$

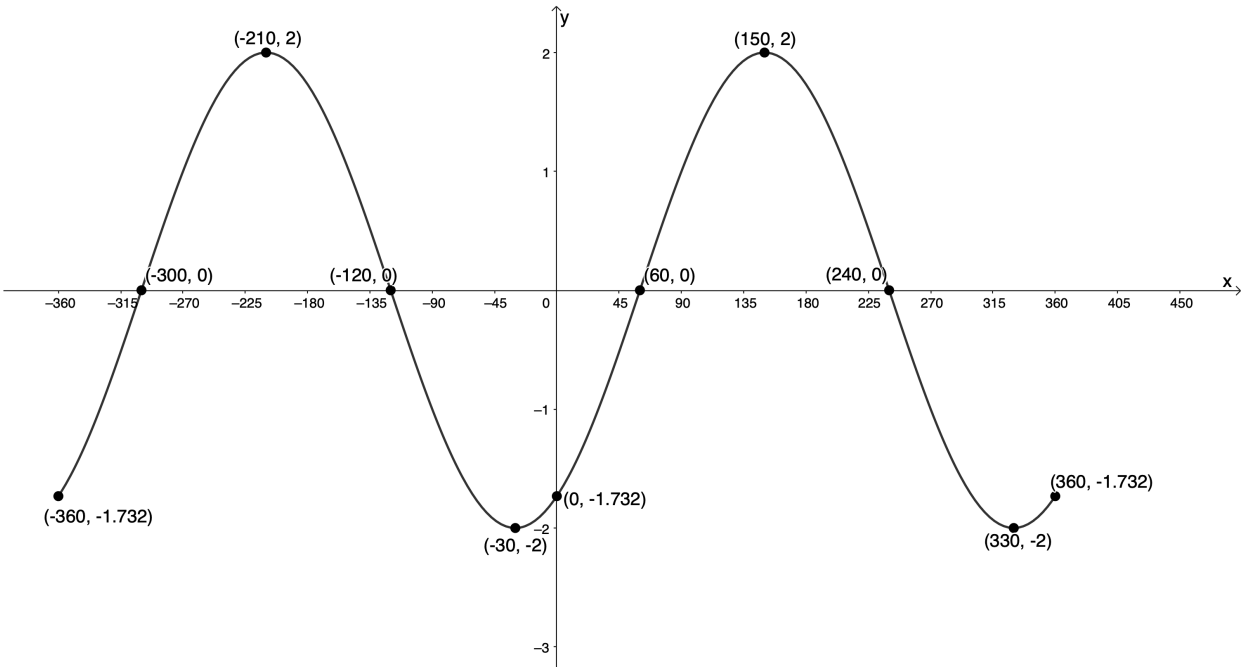
   y-intercept (let $x = 0$):

$$y = -2\cos(0^\circ + 30^\circ)$$

$$\therefore y = -2 \times \frac{\sqrt{3}}{2}$$

$$= -\sqrt{3} = -1.732$$

   Because the period of the function is 360° we know that the function values at -360° and 360° will also be -1.732.



   y-intercept: $(0^\circ, -1.732)$
   x-intercepts: $(-300^\circ, 0)$, $(-120^\circ, 0)$, $(60^\circ, 0)$, $(240^\circ, 0)$
   Domain: $\{x | x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$
   Range: $\{y | y \in \mathbb{R}, -2 \leq y \leq 2\}$

2. $2y = \cos(x + 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$

$$2y = \cos(x + 60^\circ)$$

$$\therefore y = \frac{1}{2}\cos(x + 60^\circ)$$

   $a = \frac{1}{2}$. Therefore, the amplitude will be $\frac{1}{2}$.
   $p = 60^\circ$. Therefore, the graph will be shifted 60° to the left.

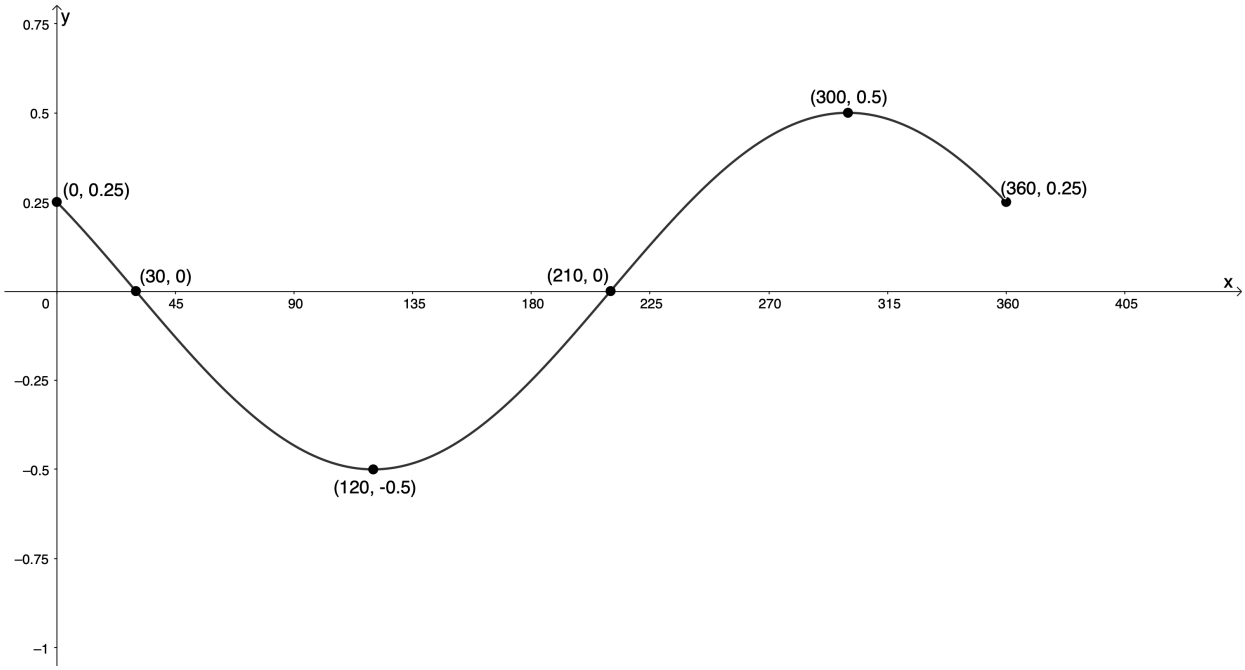
$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\frac{1}{2}\cos(x + 60^\circ)$	$(-60^\circ, \frac{1}{2})$	$(30^\circ, 0)$	$(120^\circ, -\frac{1}{2})$	$(210^\circ, 0)$	$(300^\circ, \frac{1}{2})$

y-intercept (let $x = 0$):

$$y = \frac{1}{2}\cos(0^\circ + 60^\circ)$$

$$\therefore y = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Because the period of the function is 360° we know that the function value at 360° will also be $\frac{1}{4}$.



y-intercept: $(0^\circ, 0.25)$ or $(0^\circ, \frac{1}{4})$

x-intercepts: $(30^\circ, 0)$, $(210^\circ, 0)$

Domain: $\{x | x \in \mathbb{R}, 0^\circ \leq x \leq 360^\circ\}$

Range: $\{y | y \in \mathbb{R}, -\frac{1}{2} \leq y \leq \frac{1}{2}\}$

3. $\frac{1}{3}y = \cos(60^\circ - x)$ for $-360^\circ \leq x \leq 360^\circ$

$$\frac{1}{3}y = \cos(60^\circ - x)$$

$$\begin{aligned}\therefore y &= 3\cos(60^\circ - x) \\ &= 3\cos(-x + 60^\circ) \\ &= 3\cos(-(x - 60^\circ)) \\ &= 3\cos(x - 60^\circ)\end{aligned}$$

$a = 3$. Therefore, the amplitude will be 3.

$p = -60^\circ$. Therefore, the graph will be shifted 60° to the right.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$3\cos(x - 60^\circ)$	$(60^\circ, 3)$	$(150^\circ, 0)$	$(240^\circ, -3)$	$(330^\circ, 0)$	$(420^\circ, 3)$

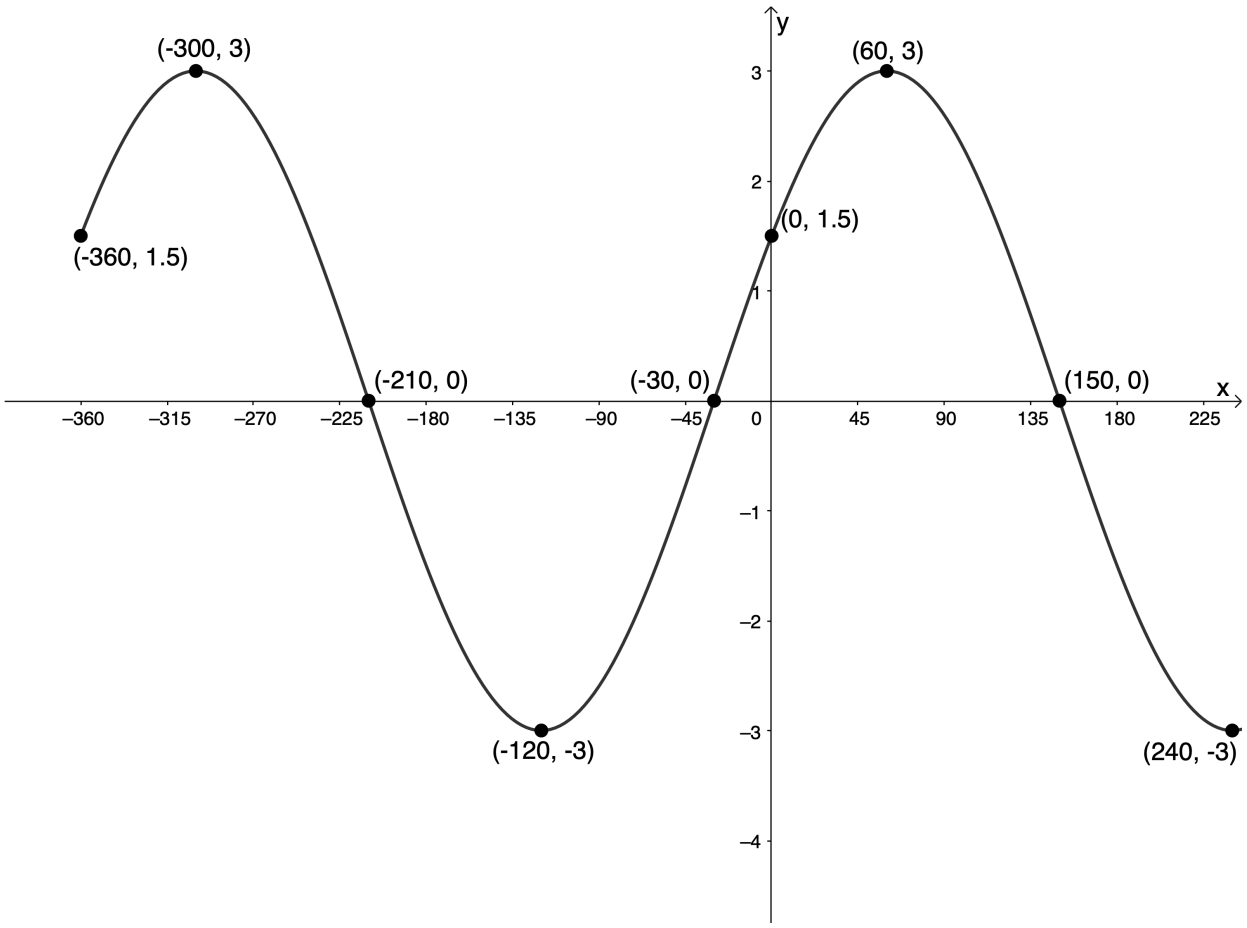
y-intercept (let $x = 0$):

$$y = 3\cos(0^\circ - 60^\circ)$$

$$\therefore y = -3 \times \frac{1}{2}$$

$$= \frac{3}{2} = 1.5$$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be 1.5.



y-intercept: $(0^\circ, 1.5)$ or $(0^\circ, \frac{3}{2})$

x-intercepts: $(-210^\circ, 0)$, $(-30^\circ, 0)$, $(150^\circ, 0)$, $(330^\circ, 0)$

Domain: $\{x | x \in \mathbb{R}, -360^\circ \leq x \leq 360^\circ\}$

Range: $\{y | y \in \mathbb{R}, -3 \leq y \leq 3\}$

Back to Exercise 10.2

Exercise 10.3

1. Amplitude is $0.5 = \frac{1}{2}$

   The maximum turning point has been transformed to $(45^\circ, -\frac{1}{2})$. The graph has been shifted 45° to the right and reflected about the x-axis. Therefore, $a = -\frac{1}{2}$ and $p = -45^\circ$.

$y = -\frac{1}{2}\cos(x - 45^\circ)$

2. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ\}$

   Range: $\{y | y \in \mathbb{R},\ -\frac{1}{2} \leq y \leq \frac{1}{2}\}$

Back to Exercise 10.3

Unit 10: Assessment

1.

a. $4y = \cos(x - 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$

$4y = \cos(x - 60^\circ)$

$\therefore y = \frac{1}{4}\cos(x - 60^\circ)$

$a = \frac{1}{4}$. Therefore, the amplitude will be $\frac{1}{4}$.

$p = -60^\circ$. Therefore, the graph will be shifted 60° to the right.

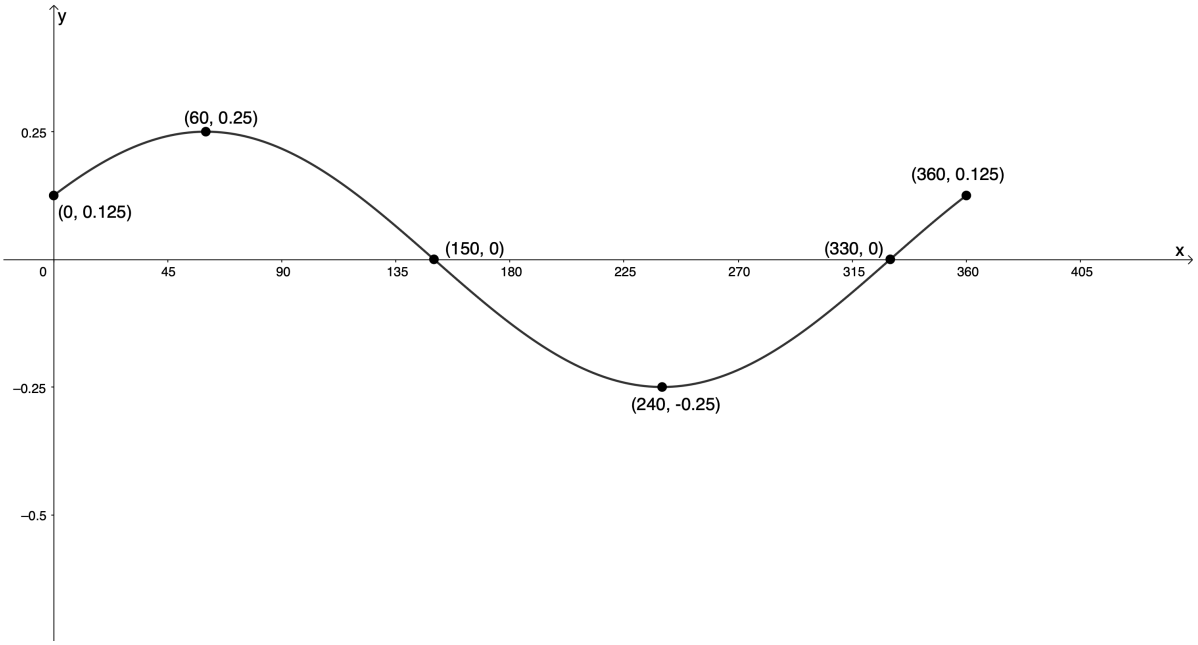
$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$\frac{1}{4}\cos(x - 60^\circ)$	$(60^\circ, \frac{1}{4})$	$(150^\circ, 0)$	$(240^\circ, -\frac{1}{4})$	$(330^\circ, 0)$	$(420^\circ, \frac{1}{4})$

y-intercept (let $x = 0$):

$y = \frac{1}{4}\cos(0 - 60^\circ)$

$\therefore y = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = 0.125$

Because the period of the function is 360° we know that the function value at 360° will also be 0.125.



b. $g(x) = -4\cos(45^\circ - x)$ for $-360^\circ \leq x \leq 360^\circ$

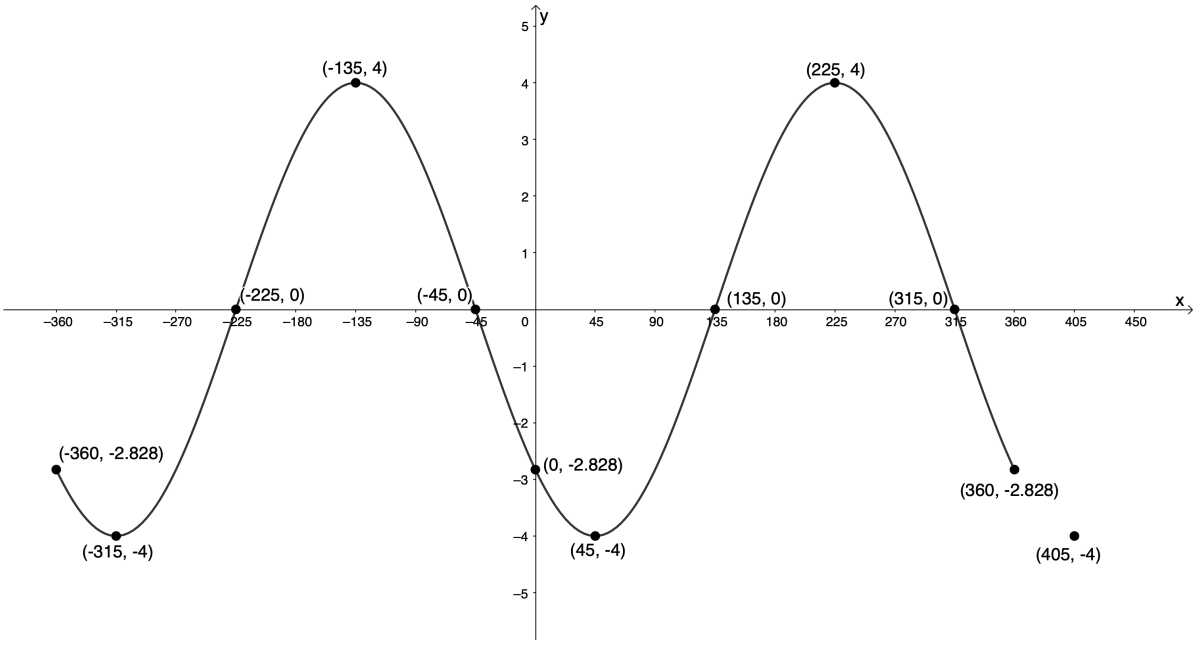
$$\begin{aligned} g(x) &= -4\cos(45^\circ - x) \\ &= -4\cos(-x + 45^\circ) \\ &= -4\cos(-(x - 45^\circ)) \\ &= -4\cos(x - 45^\circ) \end{aligned}$$

$a = -4$. Therefore, the amplitude will be 4 and the graph will be reflected about the x-axis.
$p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

$\cos x$	$(0^\circ, 1)$	$(90^\circ, 0)$	$(180^\circ, -1)$	$(270^\circ, 0)$	$(360^\circ, 1)$
$-4\cos(x - 45^\circ)$	$(45^\circ, -4)$	$(135^\circ, 0)$	$(225^\circ, 4)$	$(315^\circ, 0)$	$(405^\circ, -4)$

y-intercept (let $x = 0$):
$g(0) = -4\cos(0 - 45^\circ)$
$\therefore y = -4 \times \dfrac{1}{\sqrt{2}} = -2.828$

Because the period of the function is 360° we know that the function values at -360° and 360° will also be -2.828.



2. The function is of the form $y = a\cos(x + p)$. The amplitude is 2. The minimum turning point $(180^\circ, -1)$ has been transformed to $(170^\circ, -2)$. The graph has been shifted 10° to the left. Therefore, $a = 2$ and $p = 10^\circ$. The function is $y = 2\cos(x + 10^\circ)$.
3.
   a. $y = a\cos x$ shows no horizontal shift. The maximum turning point that was at $(0^\circ, 1)$ is now at $(0^\circ, -1)$. Therefore, the amplitude is 1 but the graph has been reflected about the x-axis. Therefore, $y = -\cos x$.
   b. $y = \cos(x + p)$ means that the function has undergone a horizontal shift. The maximum turning point that was at $(0^\circ, 1)$ is now at $(180^\circ, 1)$. In other words, the graph has been shifted 180° to the right. Therefore $y = \cos(x - 180^\circ)$.
   Alternatively, we can say that the maximum turning point that was at $(360^\circ, 1)$ is now at $(180^\circ, 1)$ and that the graph has been shifted 180° to the left and $y = \cos(x + 180^\circ)$
   c. $y = \sin(x + p)$ means that the function has undergone a horizontal shift. The maximum turning point that was at $(90^\circ, 1)$ is now at $(180^\circ, 1)$. Therefore, the graph has been shifted 90° to the right

and $y = \sin(x - 90^\circ)$

Back to Unit 10: Assessment

Media Attributions

- takenote © DHET is licensed under a CC BY (Attribution) license
- example10.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example10.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example10.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example10.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example10.5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure7 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure8 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure9 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure10 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure11 © DHET is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.1A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.1A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.1A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.2A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise10.2A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 11: The tangent function

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = a \tan k\theta$.
- Determine the effects of a and k on the tangent graph of the form $y = a \tan k\theta$.
- Find the values of a and k from a given tangent graph of the form $y = a \tan k\theta$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \tan x$ and $y = \tan \theta$ are the same function.

What you should know

Before you start this unit, make sure you can:

- Sketch tangent functions of the form $y = a \tan \theta + q$. Refer to level 2 subject outcome 2.1 Unit 5 if you need help with this.

Introduction

We know that the **period** of the tangent function $y = \tan \theta$ is $180°$ (see Figure 1). The period of the tangent function is different from the sine and cosine functions and, as we can see in Figure 1, the shape of the function is also different. Instead of having regular repeated maximum and minimum turning points, the tangent function increases to positive and negative infinity on either side of vertical asymptotes. Remember that the function $y = \tan x$ is undefined for the values of $x$at these asymptotes.

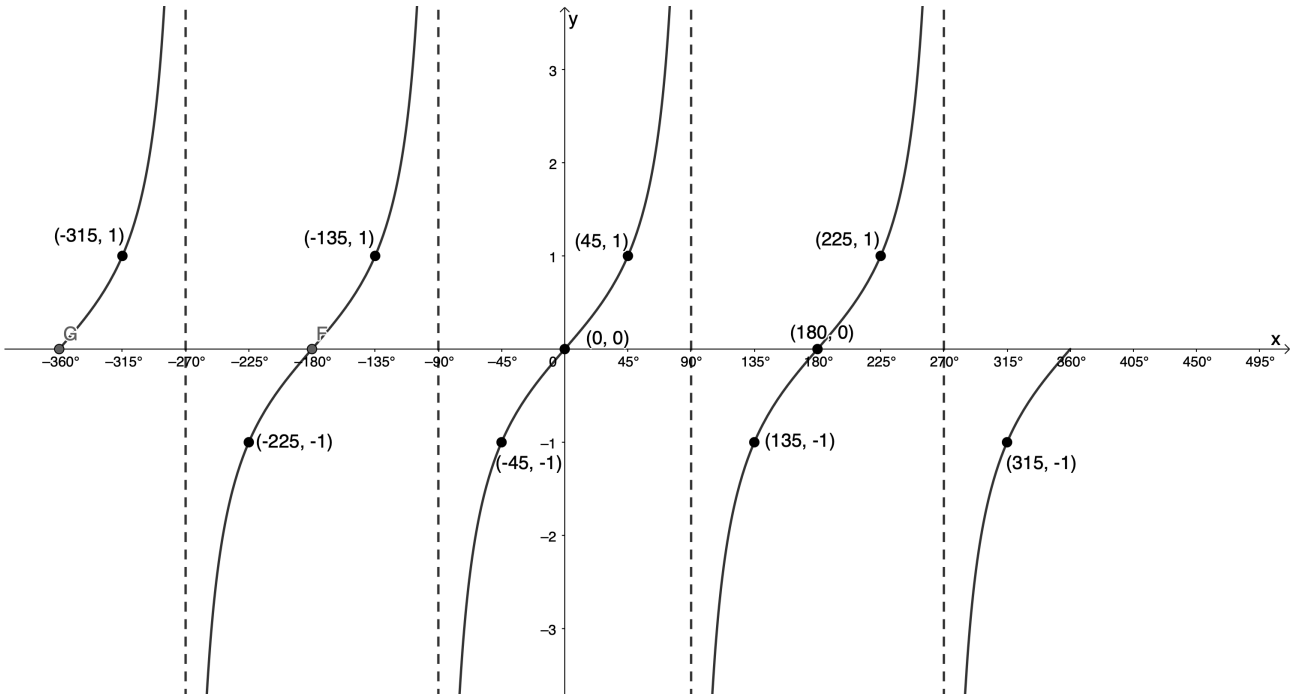


Figure 1: $y = \tan x$ *for the interval* $-360^\circ \leq x \leq 360^\circ$

Because the tangent function does not have maximum and minimum turning points and increases to positive and negative infinity, we cannot say that it has an **amplitude**. However, the function $y = \tan x$ does pass through the points $(45^\circ, 1)$ and $(225^\circ, 1)$ and $(135^\circ, -1)$ and $(225^\circ, -1)$ in the interval $0^\circ \leq x \leq 360^\circ$ (see Figure 1) which are important 'anchor points' when it comes to sketching the tangent function.

In level 2 we were introduced to the tangent function of the form $y = a \tan \theta + q$. We discovered that the value of q shifted the graph vertically up and down while the value of a changed the shape of the graph.

For example, for the function $y = \tan x + 1$, each of these 'anchor points' is shifted 1 unit up (see Figure 2). The point $(0^\circ, 0)$ becomes the point $(0^\circ, 1)$, the point $(45^\circ, 1)$ becomes the point $(45^\circ, 2)$, the point $(135^\circ, -1)$ becomes the point $(135^\circ, 0)$ and so on.

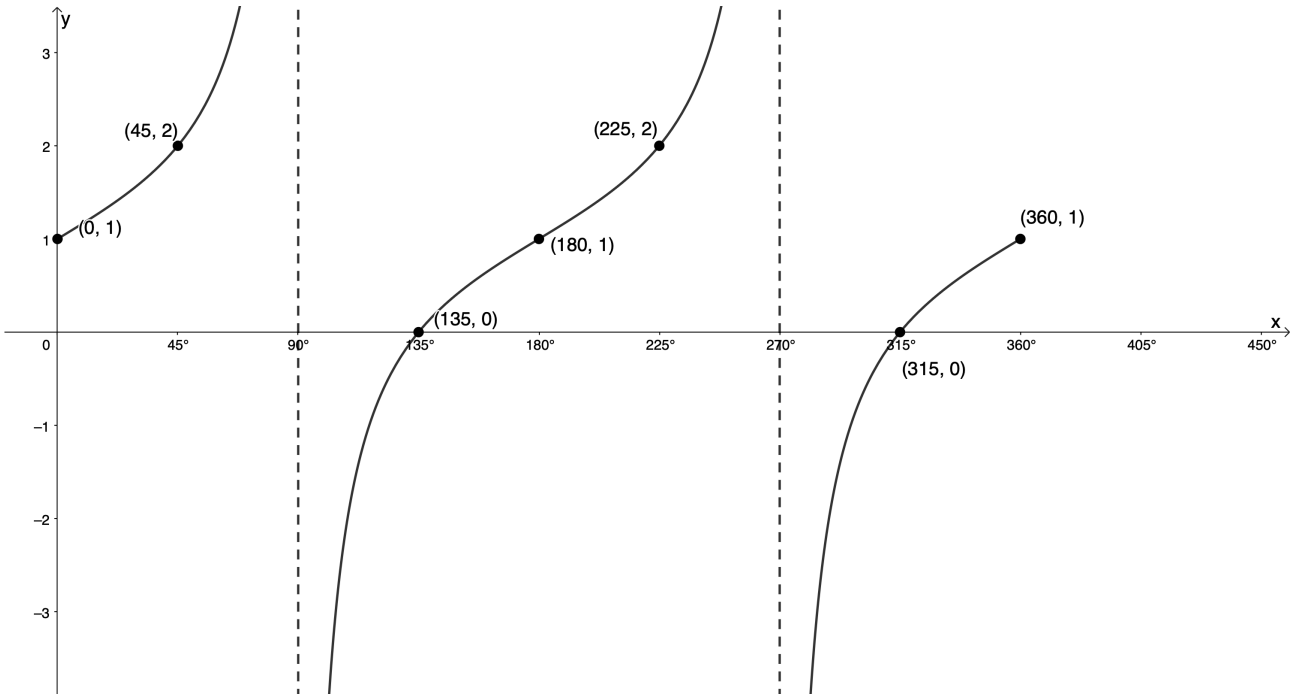


Figure 2: $y = \tan x + 1$ *illustrating the effect of* q *on the graph of* $y = a \tan x + q$

And for the function $y = 2 \tan x$, each of these 'anchor points' have y-values of 2 or -2 respectively (see Figure 3).

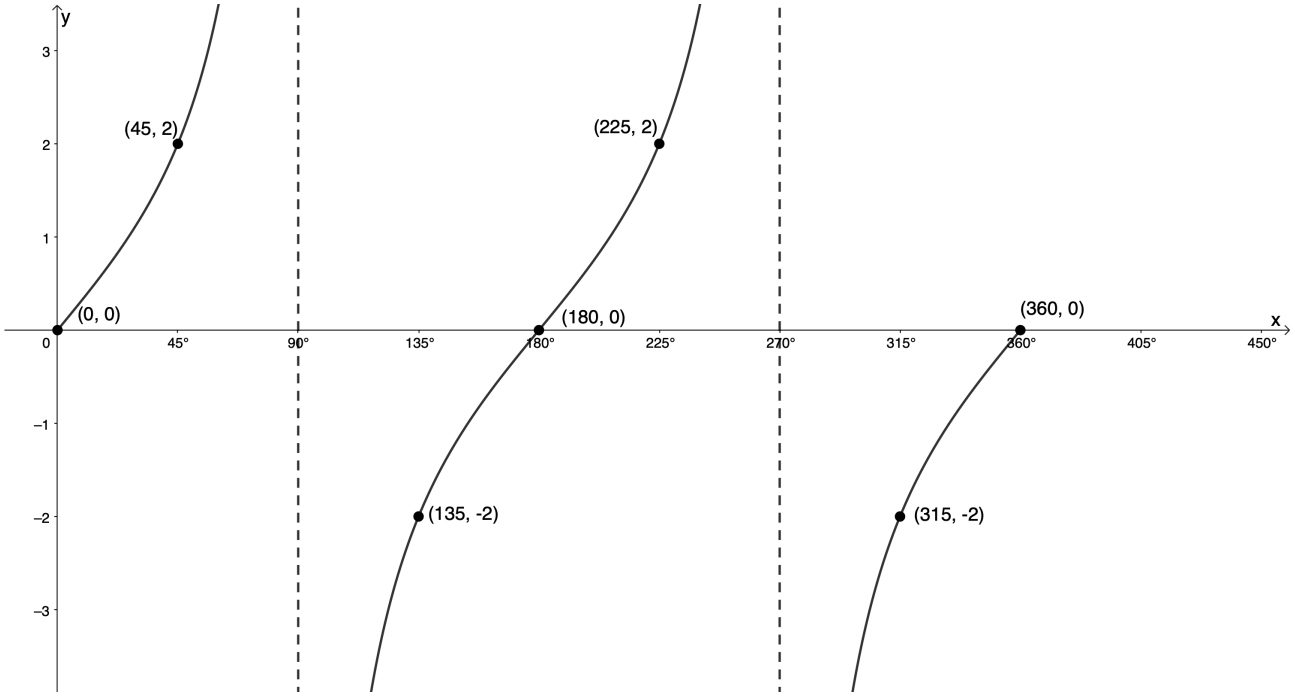


Figure 3: $y = 2 \tan x$ *illustrating the effect of* a *on the graph of* $y = a \tan x + q$

Take note!

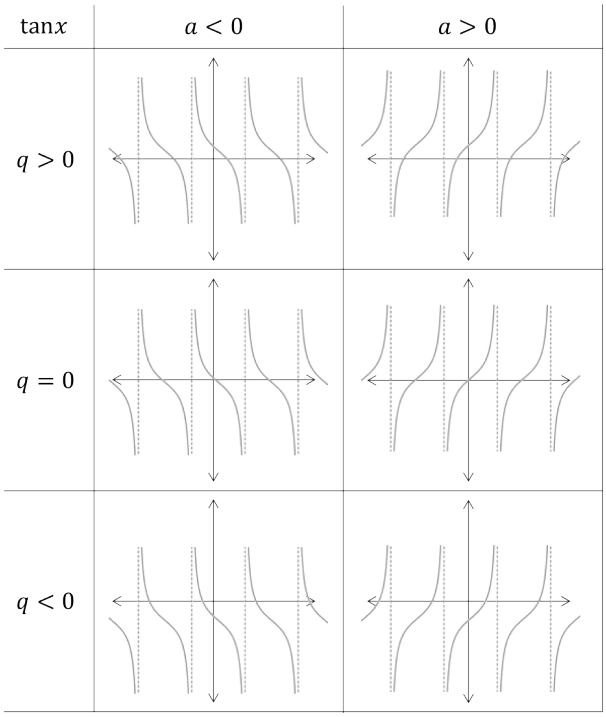
Period of $y = a\tan\theta + q$ is $180°$.

The effect of q:

- if $q > 0$, the function is shifted **up** by q units
- if $q < 0$, the function is shifted **down** by q units.

The effect of a:

- if $a > 1$, the branches of the function **are steeper**
- if $0 < a < 1$, the branches of the function are **less steep** (the function bends more)
- if $a < 0$, the function is reflected about the x-axis
- if $-1 < a < 0$, the function is reflected about the x-axis and the branches of the function are **less steep** (the function bends more)
- if $a < -1$, the function is reflected about the x-axis and the branches of the function **are steeper**.



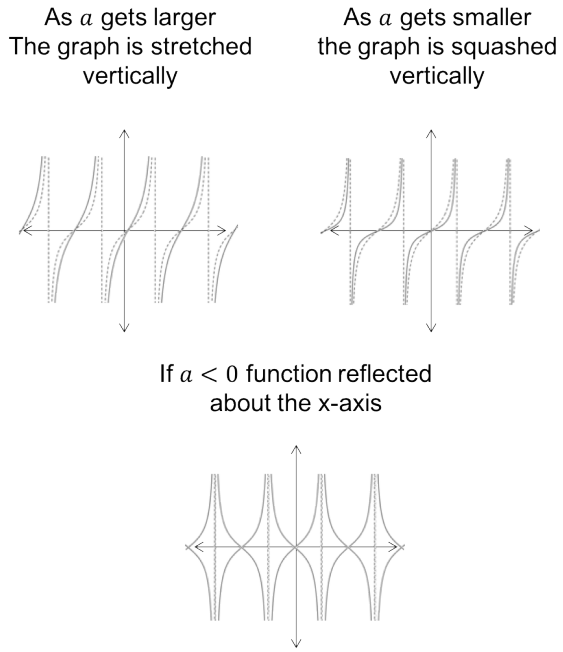


Figure 4: Summary of effects of a and q on $y = a\tan\theta + q$

Note

If you have access to the internet, visit this interactive simulation.



Here you will find a tangent function of the form $y = a\tan x + q$ with sliders to change the values of a and q. Spend some time playing with it to make sure that you understand how changing the values of a and q affects the shape and location of the tangent function of the form $y = a\tan x + q$.

The tangent function of the form $y = \tan kx$

In Activity 7.1 in unit 7, we saw that the value of k in $y = \sin kx$ affects the period of the function and that we can calculate the period as $\text{period} = \dfrac{360^\circ}{|k|}$ where $|k|$ is the absolute value of k, in other words the positive value of k.

This means that if $k > 1$, the period **decreases** and if $0 < k < 1$, the period **increases**.

The same is true for the tangent function $y = \tan kx$. Remember that the period of $y = \tan x$ is 180°. Therefore, the period can be calculated as follows:

$\text{Period} = \dfrac{180^\circ}{|k|}$ where $|k|$ is the absolute value of k, in other words the positive value of k.

This means that if $k > 1$, the period **decreases** and if $0 < k < 1$, the period **increases**.

If $k < 0$, it changes the period AND reflects the graph about the x-axis because $\tan(-x) = -\tan(x)$.

Note

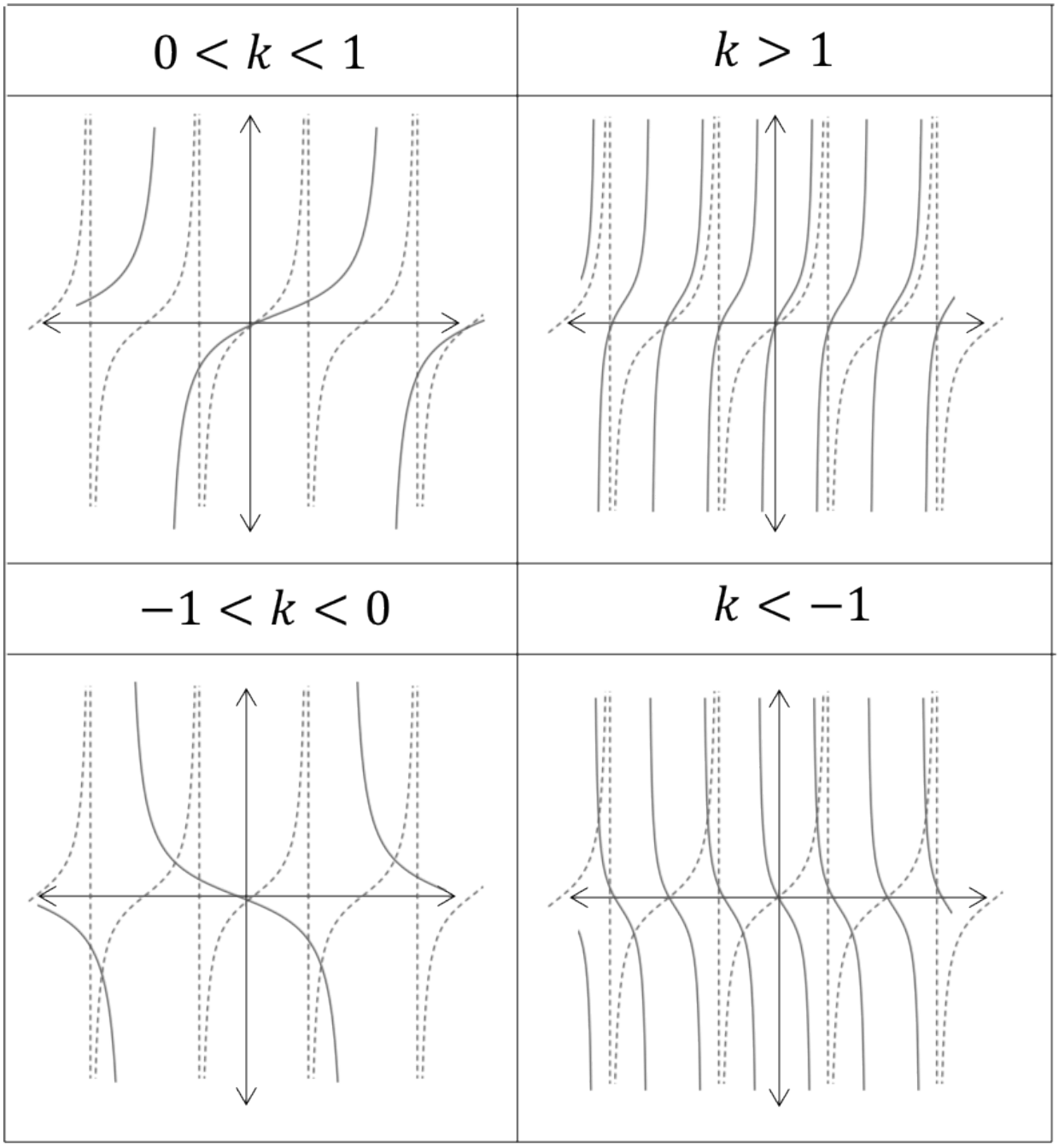
If you have an internet connection, visit this online simulation to compare the effects values of $k < 0$ have on the sine, cosine and tangent functions.



Take note!

In $y = \tan kx$ the $\text{Period} = \dfrac{180^\circ}{|k|}$.

$\tan(-x) = -\tan(x)$. Therefore, if $k < 0$, it changes the period AND reflects the graph about the x-axis.



Example 11.1

Given the function $y = \tan\left(\frac{2x}{3}\right)$ for the interval $0^\circ \leq x \leq 360^\circ$, determine the:

1. period
2. location of the vertical asymptotes
3. domain and range.

Solutions

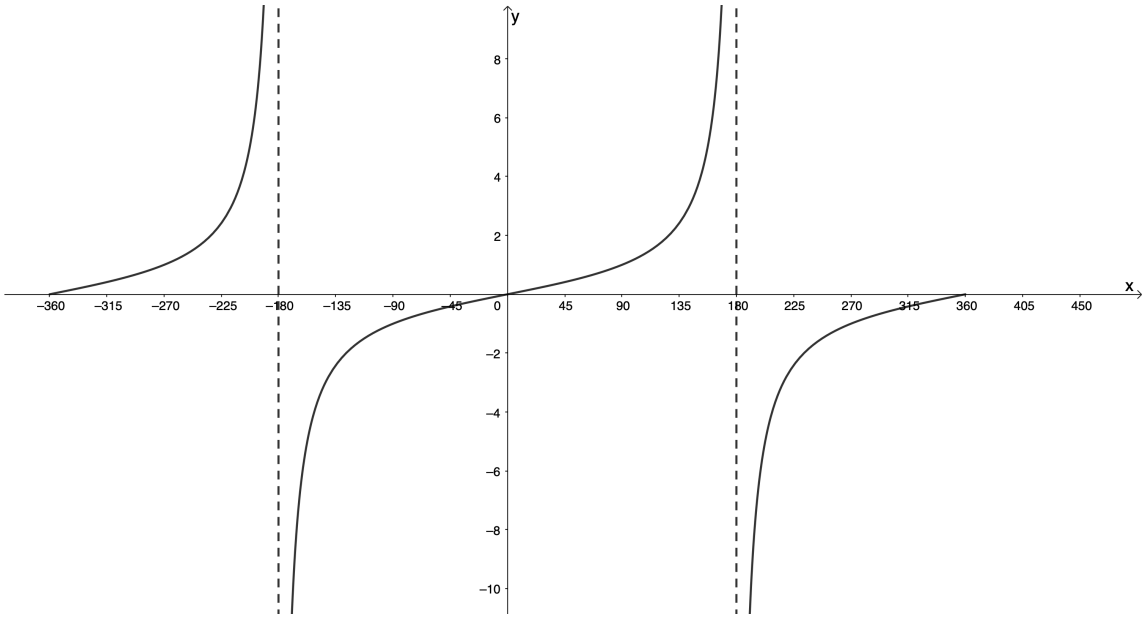
1. $k = \frac{2}{3}$. Therefore, the period of the graph is $\frac{180^\circ}{\left|\frac{2}{3}\right|} = \frac{3 \times 180^\circ}{2} = 270^\circ$.
2. The new period is 270°. This means that the asymptotes have also changed. There used to be an asymptote at 90°, in the middle of the original period of 180°. The new asymptote will still be in the middle of the period. There are two ways to calculate the value of this new asymptote.
   $\text{new asymptote} = \frac{\text{period}}{2} = \frac{270^\circ}{2} = 135^\circ$, or
   $\text{new asymptote} = \frac{\text{original asymptote}}{|k|} = \frac{90^\circ}{\frac{2}{3}} = \frac{3 \times 90^\circ}{2} = 135^\circ$

   In $y = \tan x$ we also know that the vertical asymptotes are 180° or one period apart. The new asymptotes will also be one period apart. Therefore, the next asymptote to the right will be at $135^\circ + 270^\circ = 405^\circ$ which is outside of the given interval. The next asymptote to the left will be $135^\circ - 270^\circ = -135^\circ$ which is also outside the given interval. Therefore, the only asymptote within our interval is $x = 135^\circ$.
3. Domain: $\{x | x \in \mathbb{R}, 0^\circ \leq x \leq 360^\circ, x \neq 135^\circ\}$
   Range: $\{y | y \in \mathbb{R}\}$
   Remember that the tangent function does not have maximum and minimum turning points. Therefore, the range is all real values.

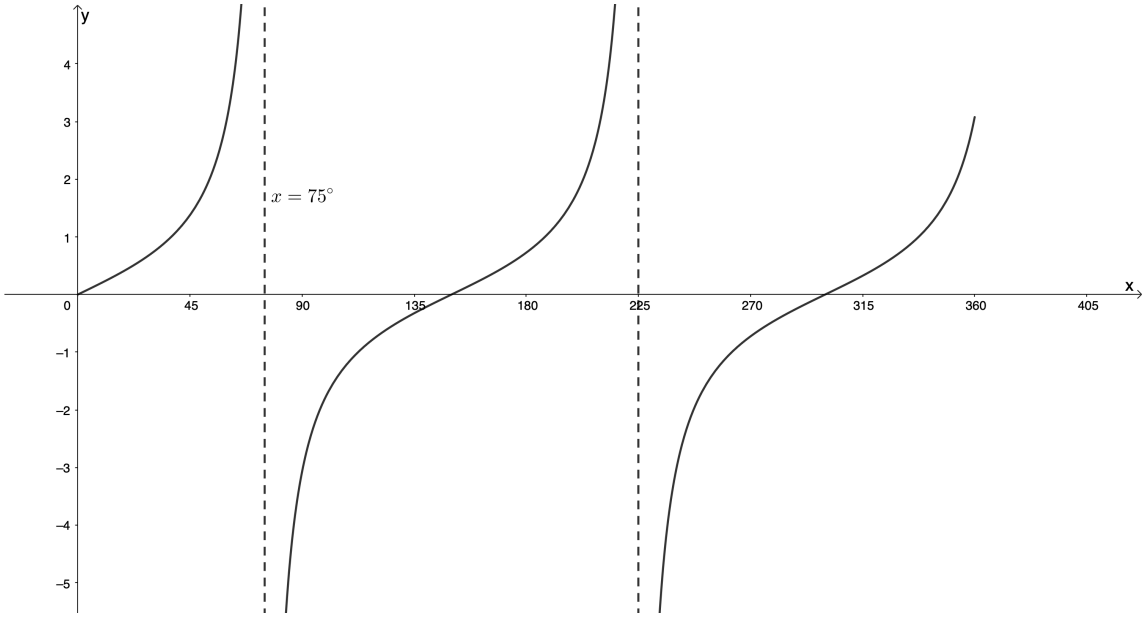


Exercise 11.1

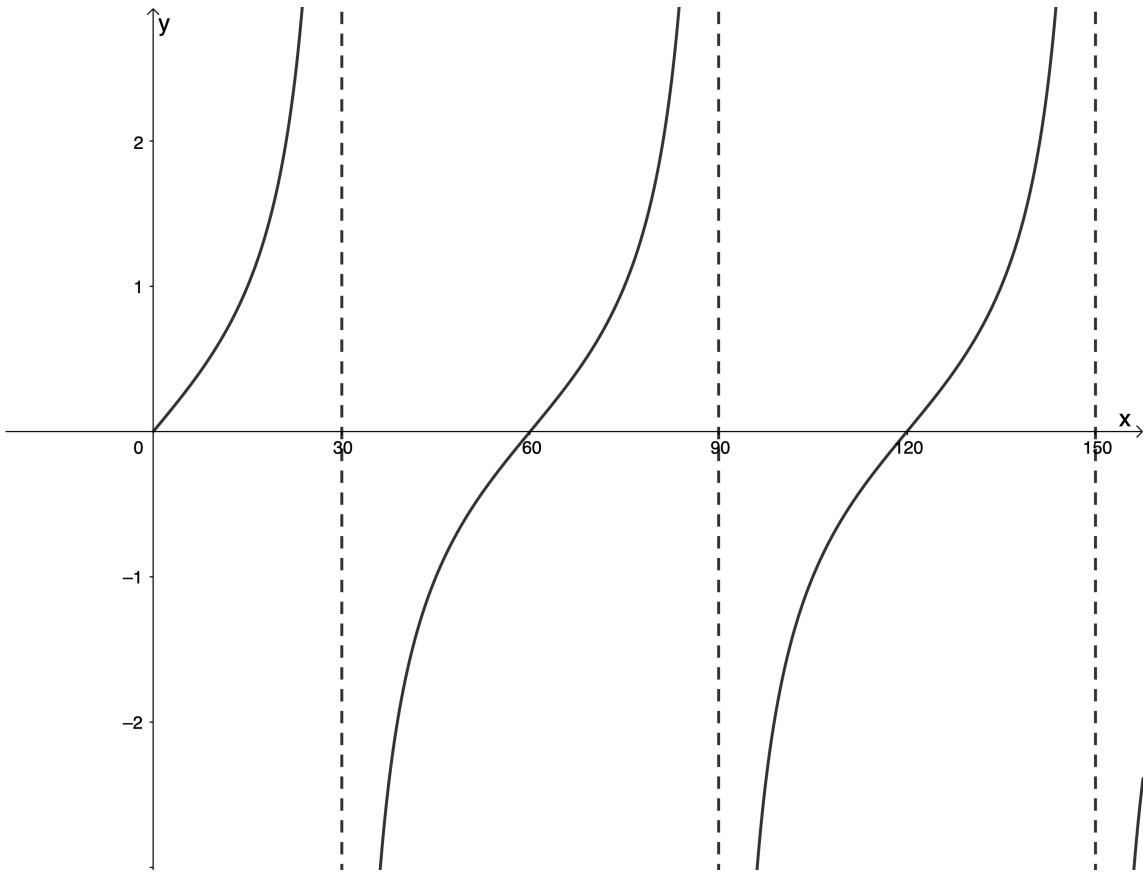
1. Determine the periods and the location of the asymptotes for the following functions:
   a. $y = \tan(2x)$
   b. $y = \tan\left(-\frac{3}{5}x\right)$
   c. $y = \tan\left(-\frac{3}{2}x\right)$
2. For each function of the form $y = \tan kx$, determine the value of k:
   a.


y
x
8
6
4
2
−2
−4
−6
−8
−10
−360
−315
−270
−225
−180
−135
−90
−45
0
45
90
135
180
225
270
315
360
405
450

b.


y
x
$x = 75°$
4
3
2
1
−1
−2
−3
−4
−5
0
45
90
135
180
225
270
315
360
405

c.



The full solutions are at the end of the unit.

Sketch functions of the form $y = a \tan kx$

Now that we know that the value of k affects the period of the function $y = \tan kx$, we can combine this with our knowledge of the effects of a, and examine functions of the form $y = a \tan kx$.

The effects of a on $y = a \tan kx$ are exactly the same as they were on $y = a \tan x + q$. Before we learn how to sketch functions of the form $y = a \tan kx$, spend some time playing with this online simulation.

Here you will find a graph of the function $y = a \tan kx + q$ for the interval $-360^\circ \leq x \leq 360^\circ$. Pay particular attention to how the position of the asymptotes, the value of the 'anchor points' (including the intercepts with the x-axis), the shape of the branches of the graph and the period change as you change the values of a, k and q.

The best way to sketch functions of the form $y = a \tan kx$ is to transform the basic function of $y = \tan x$ depending on the values of a and k. To do this, you need to know the set of 'anchor points' of $y = \tan x$, as transformation of these points will help you to sketch functions of the form $y = a \tan kx$.

Take note!

The 'anchor points' for $y = \tan x$ are illustrated in Figure 5 for the interval $-180^\circ \leq x \leq 180^\circ$.

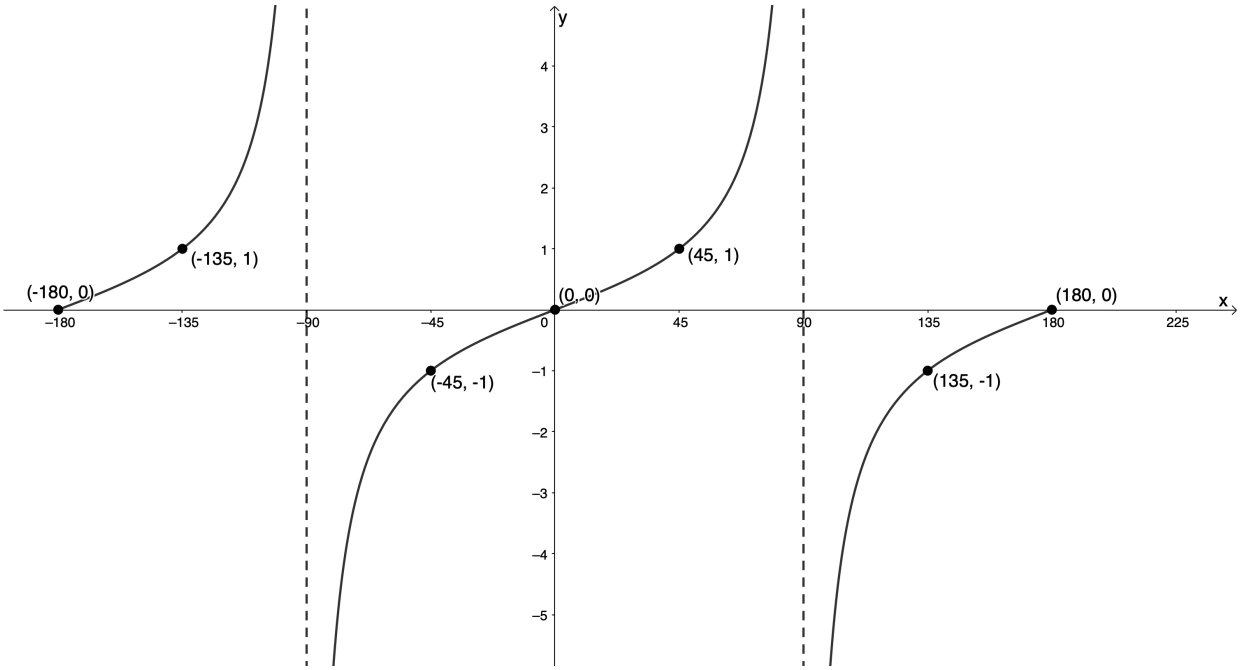


Figure 5: The 'anchor points' of $y = \tan x$

Example 11.2

Sketch the function $y = 2\tan 2x$ for the interval $-90^\circ \leq x \leq 90^\circ$.

Solution

The function is of the form $y = a\tan kx$. Therefore, $a = 2$ and $k = 2$. Because $k = 2$, we know that the period of the function will be $\frac{180^\circ}{|2|} = 90^\circ$. We need to transform one period of 'anchor points' as follows:

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan 2x$	$(0^\circ, 0)$	$(22.5^\circ, 1)$	$x = 45^\circ$ asymptote	$(67.5^\circ, -1)$	$(90^\circ, 0)$

Because the period has been halved, the value of each of the x-coordinates is also halved. We can see that the function will start at 0, pass through $(22.5^\circ, 1)$ as it increases to infinity approaching the asymptote from the left , decrease to negative infinity as it approaches the asymptote from the right, pass through the point $(67.5^\circ, -1)$, rise back to zero at 90°, the new period.

Because $a = 2$, we know that each function value (or y-value) is going to be multiplied by 2 thereby

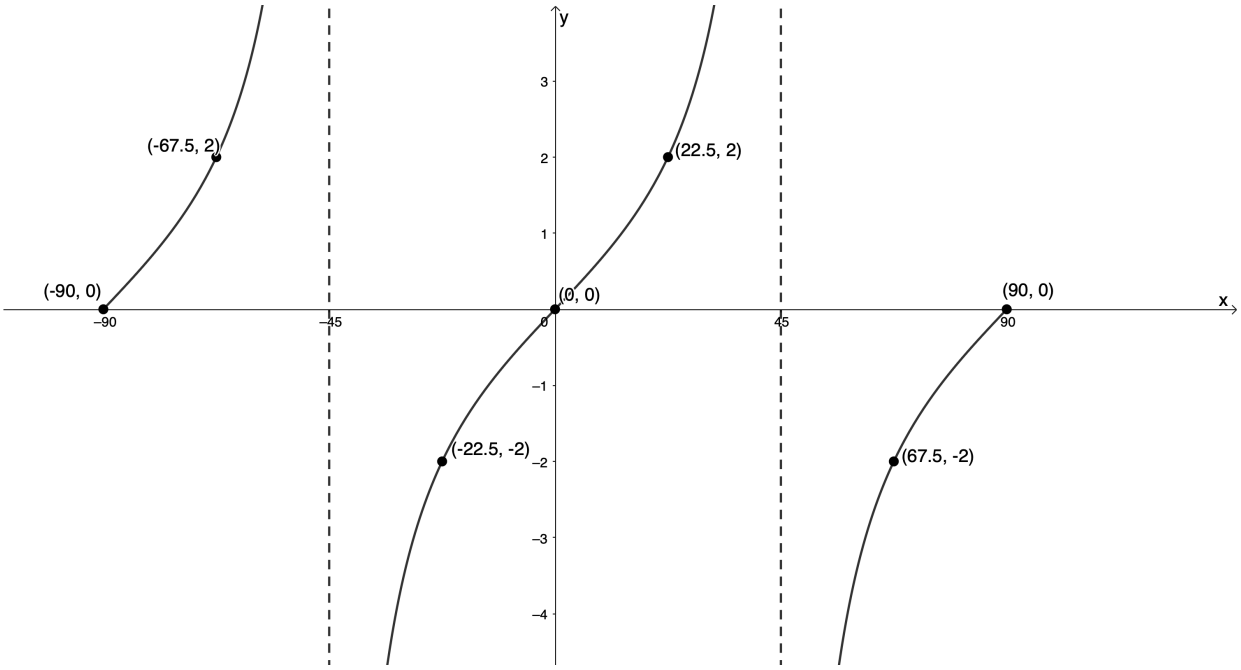
stretching the branches of the function out by a factor of 2. We need to further transform one period of 'anchor points' as follows:

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan 2x$	$(0^\circ, 0)$	$(22.5^\circ, 1)$	$x = 45^\circ$ asymptote	$(67.5^\circ, -1)$	$(90^\circ, 0)$

$2\tan 2x$ $(0^\circ, 0)$ $(22.5^\circ, 2)$ $x = 45^\circ$ asymptote $(67.5^\circ, -2)$ $(90^\circ, 0)$

We can see that the function will start at 0, pass through $(22.5^\circ, 2)$ as it increases to infinity approaching the asymptote from the left , decrease to negative infinity as it approaches the asymptote from the right, pass through the point $(67.5^\circ, -2)$, rise back to zero at 90°, the new period.

We can now plot our transformed 'anchor points' and draw the graph. We first draw it for the interval $0^\circ \leq x \leq 90^\circ$ and then continue to pattern for the full interval of $-90^\circ \leq x \leq 90^\circ$.



Example 11.3

Given the function $f(x) = 3\tan\left(-\frac{1}{2}x\right)$.

1. Sketch the graph of $f(x)$ for the interval $-180^\circ \leq x \leq 180^\circ$.
2. State the asymptotes within the interval.
3. State the domain and range of $f(x)$.
4. What is the period of $f(x)$?

Solutions

1. The function is of the form $y = a\tan kx$ Therefore, $a = 3$ and $k = -\frac{1}{2}$.

   Because $k = -\frac{1}{2}$, we know that the period of the function will be $\frac{180^\circ}{\left|-\frac{1}{2}\right|} = \frac{180^\circ}{\frac{1}{2}} = 360^\circ$ .But $k < 0$ and we know that $\tan(-x) = -\tan x$. Therefore, we need to multiply the x-values of the 'anchor points' by 2 for the new period and multiply the y-values of the 'anchor points' by -1 to reflect the fact that $k < 0$.

   We need to transform one period of 'anchor points' as follows:

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan\left(-\frac{1}{2}x\right)$	$(0^\circ, 0)$	$(90^\circ, -1)$	$x = 180^\circ$ asymptote	$(270^\circ, 1)$	$(360^\circ, 0)$

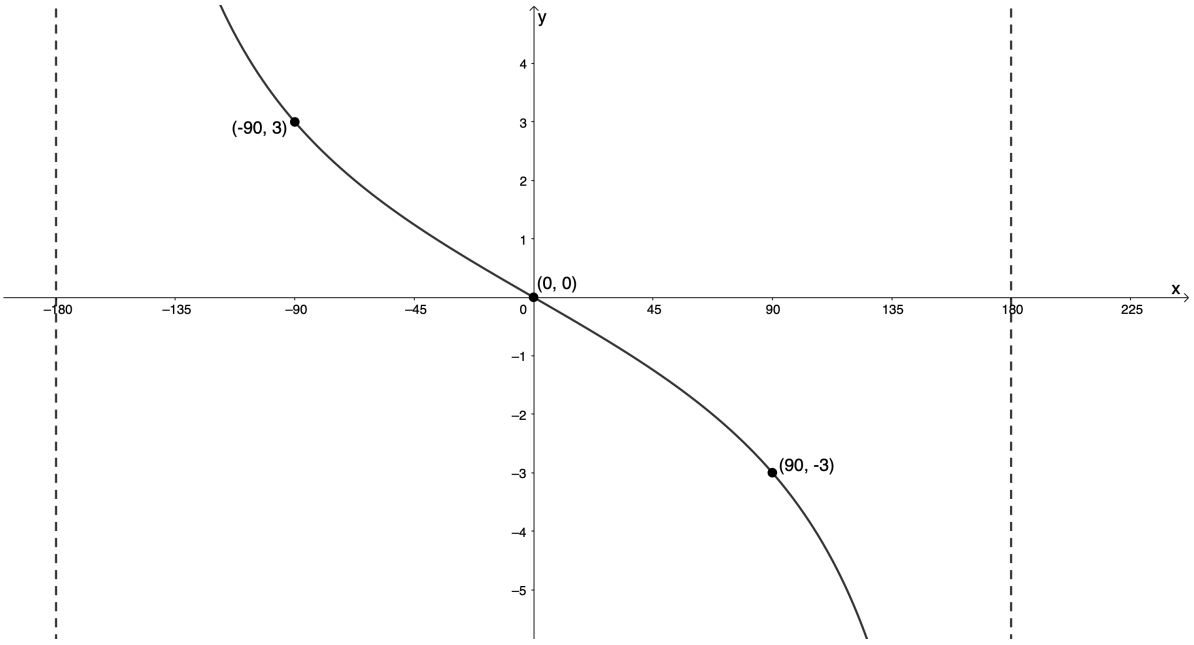
   Because the period has been doubled, the value of each of the x-coordinates is also doubled.

   Because $a = 3$, we know that each function value (or y-value) is going to be multiplied by 3. We need to further transform one period of 'anchor points' as follows:

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan\left(-\frac{1}{2}x\right)$	$(0^\circ, 0)$	$(90^\circ, -1)$	$x = 180^\circ$ asymptote	$(270^\circ, 1)$	$(360^\circ, 0)$
$3\tan\left(-\frac{1}{2}x\right)$	$(0^\circ, 0)$	$(90^\circ, -3)$	$x = 180^\circ$	$(270^\circ, 3)$	$(360^\circ, 0)$

   Because the period has been doubled, the value of each of the x-coordinates is also doubled. We can see that the function will start at 0, pass through $(90^\circ, -3)$, decrease to negative infinity as it approaches the asymptote from the left, increase to infinity as it approaches the asymptote from the right, pass through $(270^\circ, 3)$ and fall back to zero at 360°, the new period.

   We can now plot our transformed 'anchor points' and draw the graph for the interval $-180^\circ \leq x \leq 180^\circ$.



2. The asymptotes within the interval are at $x = -180^\circ$ and $x = 180^\circ$.
3. Domain: $\{x | x \in \mathbb{R}, \ -180^\circ < x < 180^\circ\}$
   Range$\{f(x) | f(x) \in \mathbb{R}\}$
4. The period is 360°.

Exercise 11.2

Sketch the following functions for the indicated intervals:

1. $y = -2\tan\dfrac{\theta}{2}$ for $-180^\circ \leq \theta \leq 180^\circ$
2. $y = \tan\left(-\dfrac{3x}{2}\right)$ for $-90^\circ \leq x \leq 90^\circ$
3. $y = \dfrac{1}{2}\tan 3\theta$ for $0^\circ \leq \theta \leq 180^\circ$

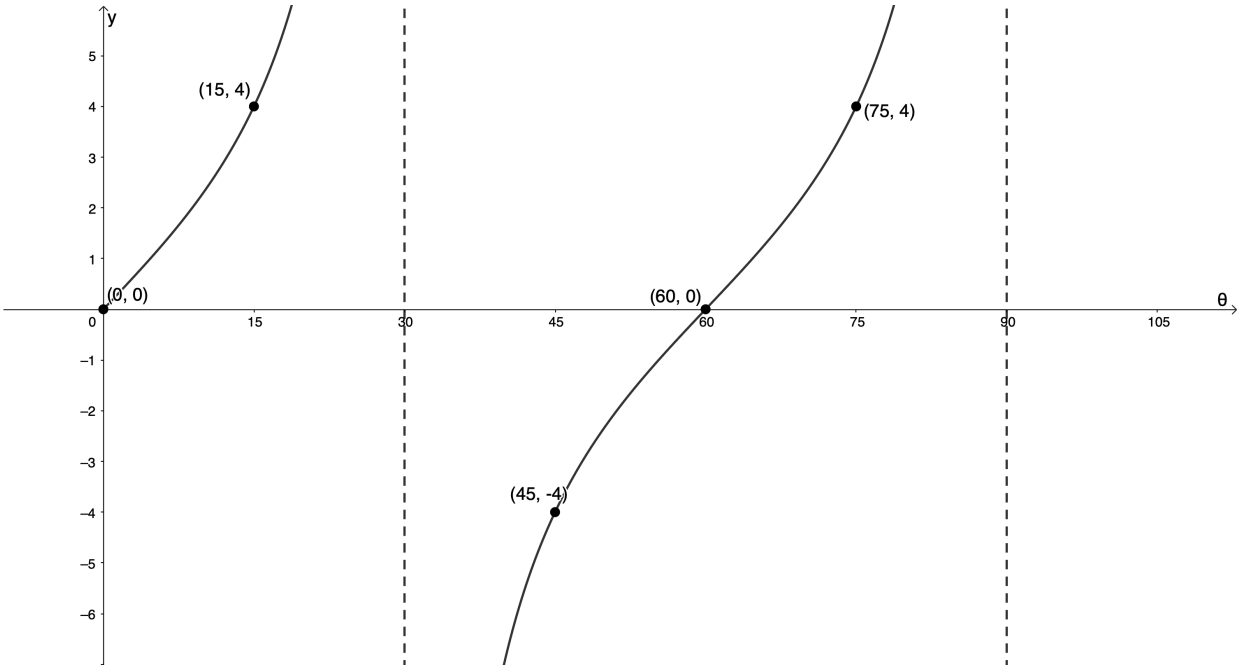
The full solutions are at the end of the unit.

Find the equation of a tangent function of the form $y = a\tan kx$

By examining the location of the 'anchor points' and period of a graph of the form $y = a\tan kx$, we can determine the values of a and k.

Example 11.4

Given the following graph of the form $y = a\cos kx$, determine the values of a and k.



Solution

From the graph we can see that the 'anchor points' have y-values of 4. We can also see that the graph is not reflected about the x-axis. It increases to infinity as it approaches the asymptote from the left just like $y = \tan x$. Therefore, $a = 4$ and $y = 4\tan kx$.

The period is $60°$.

$$\text{Period} = \frac{180°}{|k|}$$

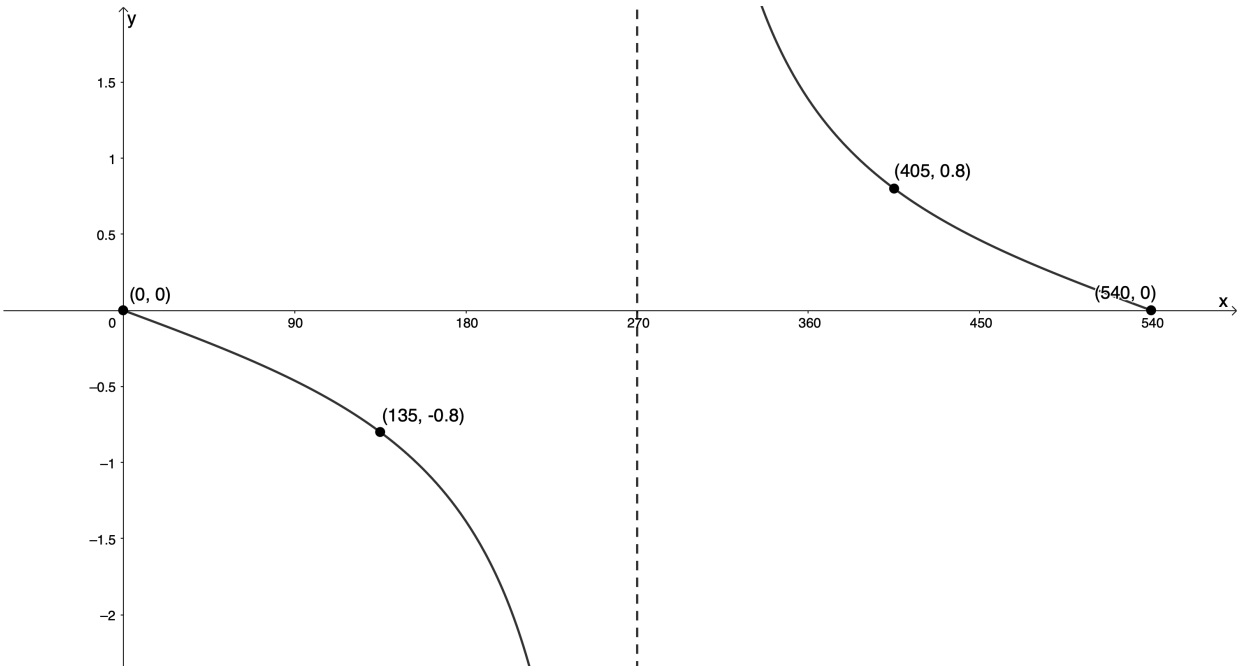
$$60° = \frac{180°}{|k|}$$

$$\therefore |k| = 3$$

$y = 4\tan 3x$

Example 11.5

Given the following graph of the form $y = a\tan kx$, determine the values of a and k.



Solution

From the graph we can see that the y-values of the 'anchor points' are $0.8 = \frac{4}{5}$. We can also see that the graph is reflected about the x-axis. The function decreases to negative infinity as it approaches the asymptote from the left unlike the graph of $y = \tan x$, which increases to positive infinity as it approaches the asymptote from the left. Therefore, $a = -\frac{4}{5}$ and $y = -\frac{4}{5}\tan kx$.

$$\text{Period} = \frac{180^\circ}{|k|}$$

$$540^\circ = \frac{180^\circ}{|k|}$$

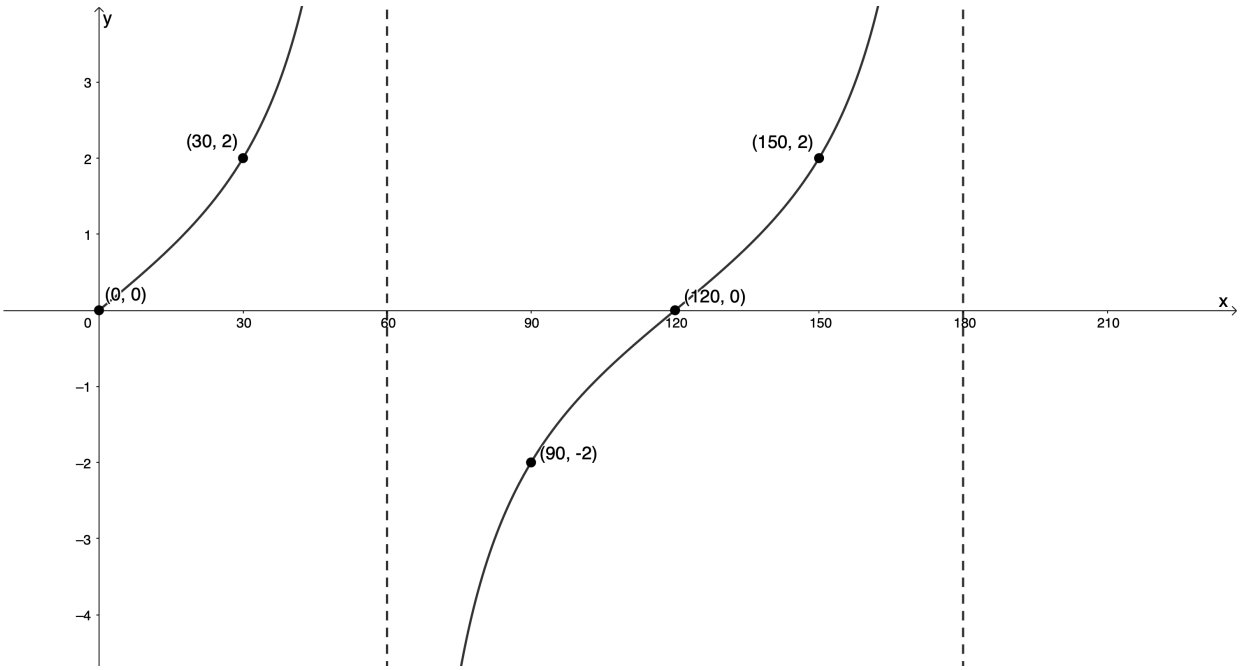
$$\therefore |k| = \frac{180^\circ}{540^\circ} = \frac{1}{3}$$

$$y = -\frac{4}{5}\tan\left(\frac{1}{3}x\right)$$

Note: We could have assigned the negative sign to the value of k but it is more common to keep $k > 0$ if the graph is reflected about the x-axis and make $a < 0$.

Exercise 11.3

Given the graph of the form $y = a\tan kx$:



1. Determine the values of a and k.
2. State the domain and range of the function.

The full solutions are at the end of the unit.

Summary

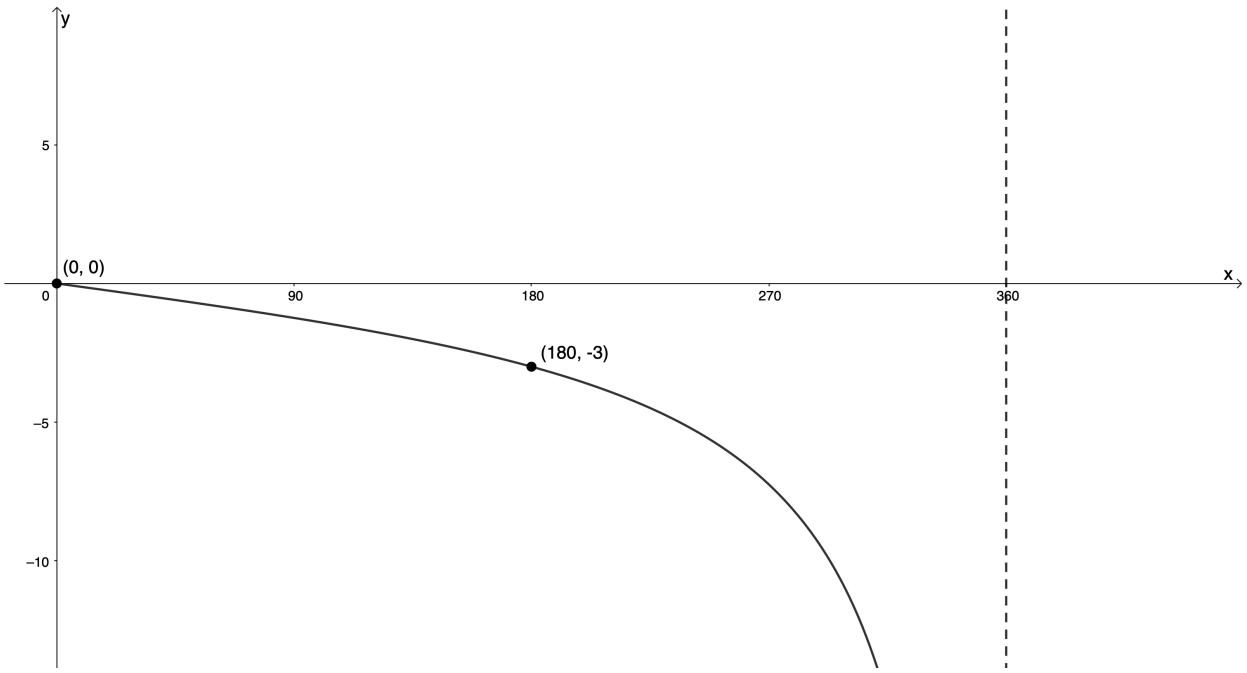
In this unit you have learnt the following:

- The effects of a and k on the tangent graph of the form $y = a\tan k\theta$.
- How to sketch functions of the form $y = a\tan k\theta$.
- How to find the values of a and k from a given tangent graph of the form $y = a\tan k\theta$.

Unit 11: Assessment

Suggested time to complete: 45 minutes

1. Sketch the following functions for the given intervals:
   a. $2y = \tan\left(-\frac{1}{3}x\right)$ for $-270^\circ \leq x \leq 270^\circ$
   b. $g(x) = -3\tan 2x$ for $0^\circ \leq x \leq 135^\circ$
2. From the graph below of the form $y = a\tan kx$, determine the values of a and k.



The full solutions are at the end of the unit.

Unit 11: Solutions

Exercise 11.1

1.

a. $y = \tan(2x)$

$k = 2$

$\text{Period} = \dfrac{180^\circ}{|2|} = \dfrac{180^\circ}{2} = 90^\circ$

$\text{Asymptote} = \dfrac{90^\circ}{|k|} = \dfrac{90^\circ}{2} = 45^\circ$

All the new asymptotes can be found by adding or subtracting multiples of the new period.

Asymptotes: $45^\circ \pm m.90^\circ, m \in \mathbb{Z}$

b. $y = \tan\left(-\dfrac{3}{5}x\right)$

$k = -\dfrac{3}{5}$

$\text{Period} = \dfrac{180^\circ}{\left|-\dfrac{3}{5}\right|} = \dfrac{180^\circ}{\dfrac{3}{5}} = \dfrac{5 \times 180^\circ}{3} = 300^\circ$

$\text{Asymptote} = \dfrac{90^\circ}{|k|} = \dfrac{90^\circ}{\dfrac{3}{5}} = 150^\circ$

Asymptotes: $150^\circ \pm m.90^\circ, m \in \mathbb{Z}$

c. $y = \tan\left(-\dfrac{3}{2}x\right)$

$k = -\dfrac{3}{2}$

$$\text{Period} = \frac{180^\circ}{\left|-\frac{3}{2}\right|} = \frac{2 \times 180^\circ}{3} = 120^\circ$$

$$\text{Asymptote} = \frac{90^\circ}{|k|} = \frac{90^\circ}{\frac{3}{2}} = 60^\circ$$

Asymptotes: $60^\circ \pm m.90^\circ, m \in \mathbb{Z}$

2.

a. The period of the graph is 360°.

$$\text{Period} = \frac{180^\circ}{|k|}$$

$$360^\circ = \frac{180^\circ}{|k|}$$

$$\therefore |k| = \frac{180^\circ}{360^\circ}$$

$$\therefore |k| = \frac{1}{2}$$

b. There is an asymptote at $x = 75^\circ$. This means that the period is $2 \times 75^\circ = 150^\circ$.

$$\text{Period} = \frac{180^\circ}{|k|}$$

$$150^\circ = \frac{180^\circ}{|k|}$$

$$\therefore |k| = \frac{180^\circ}{150^\circ} = \frac{6}{5}$$

c. The period of the graph is 60°.

$$\text{Period} = \frac{180^\circ}{|k|}$$

$$60^\circ = \frac{180^\circ}{|k|}$$

$$\therefore |k| = \frac{180^\circ}{60^\circ} = 3$$

Back to Exercise 11.1

Exercise 11.2

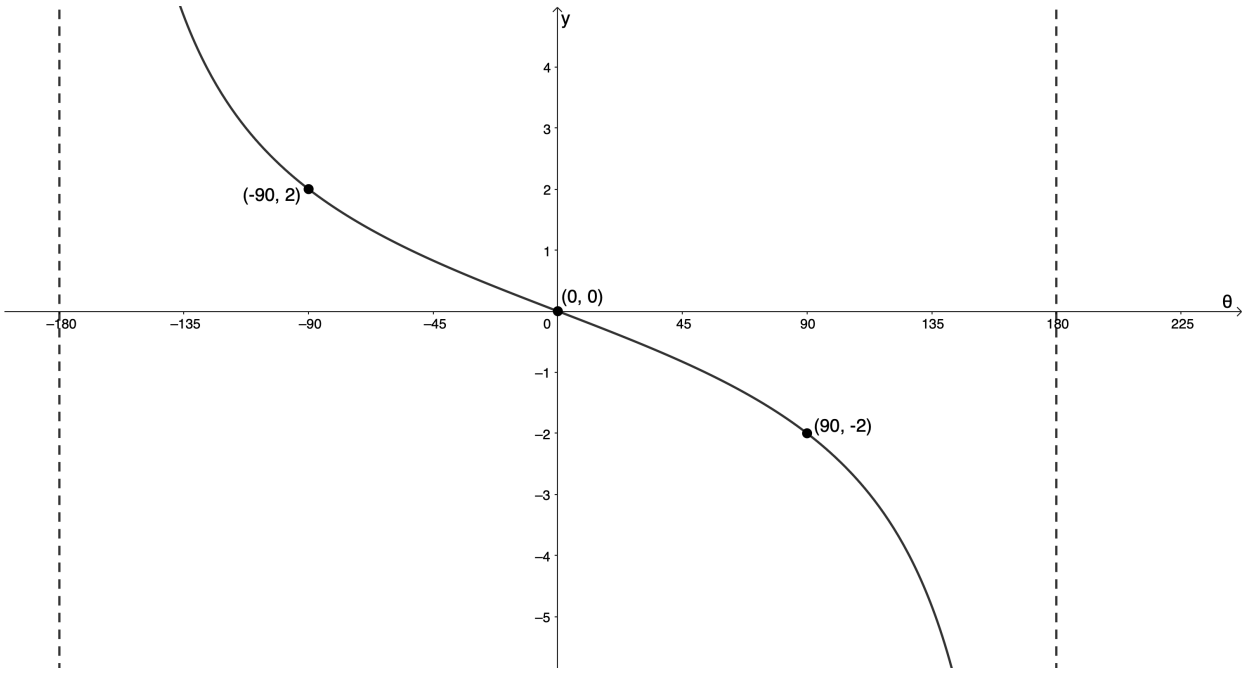
1. $y = -2\tan\frac{\theta}{2}$ for $-180^\circ \leq \theta \leq 180^\circ$

$k = \frac{1}{2}$

$$\text{Period} = \frac{180^\circ}{|k|} = \frac{180^\circ}{\left|\frac{1}{2}\right|} = 360^\circ$$

$a = -2$. Graph is reflected about the x-axis.

$\tan\theta$	$(0^\circ, 0)$	$(45^\circ, 1)$	$\theta = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$-2\tan\left(\frac{\theta}{2}\right)$	$(0^\circ, 0)$	$(90^\circ, -2)$	$\theta = 180^\circ$ asymptote	$(270^\circ, 2)$	$(360^\circ, 0)$



2. $y = \tan\left(-\frac{3x}{2}\right)$ for $-90^\circ \leq x \leq 90^\circ$

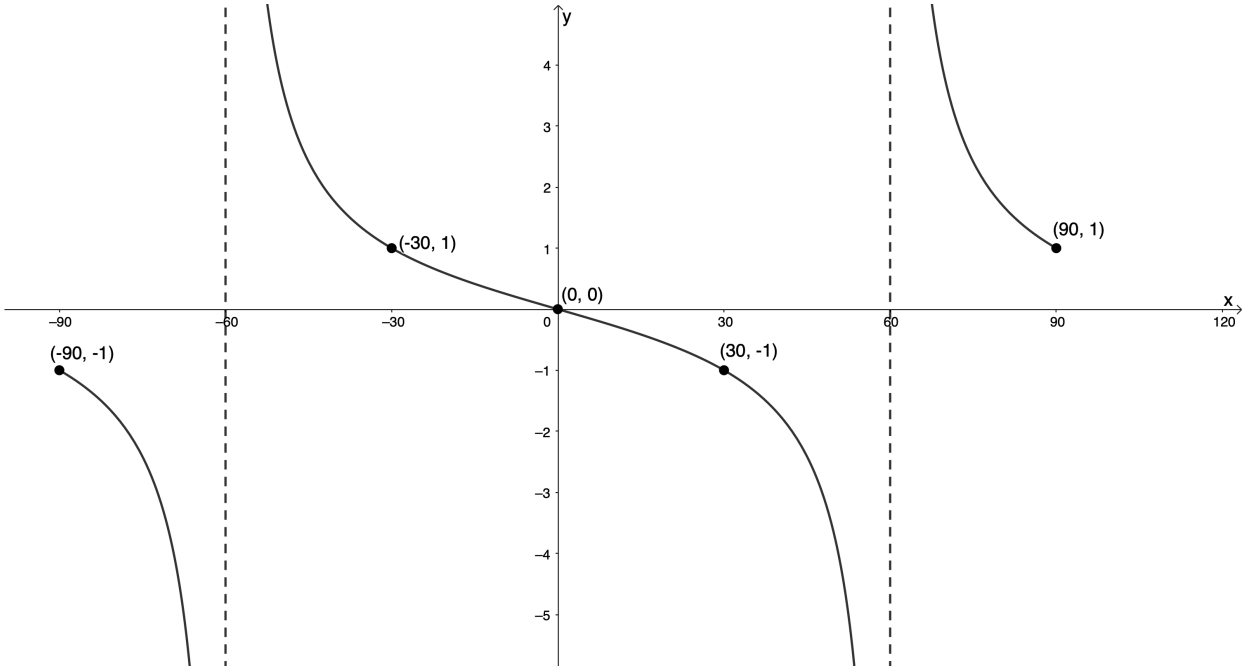
   $k = -\frac{3}{2}$

   $\text{Period} = \frac{180^\circ}{|k|} = \frac{180^\circ}{\left|-\frac{3}{2}\right|} = \frac{180^\circ}{\frac{3}{2}} = 120^\circ$

   $k < 0$, so graph is reflected about the x-axis.

   $a = 1$

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan\left(-\frac{3x}{2}\right)$	$(0^\circ, 0)$	$(30^\circ, -1)$	$x = 60^\circ$ asymptote	$(90^\circ, 1)$	$(120^\circ, 0)$



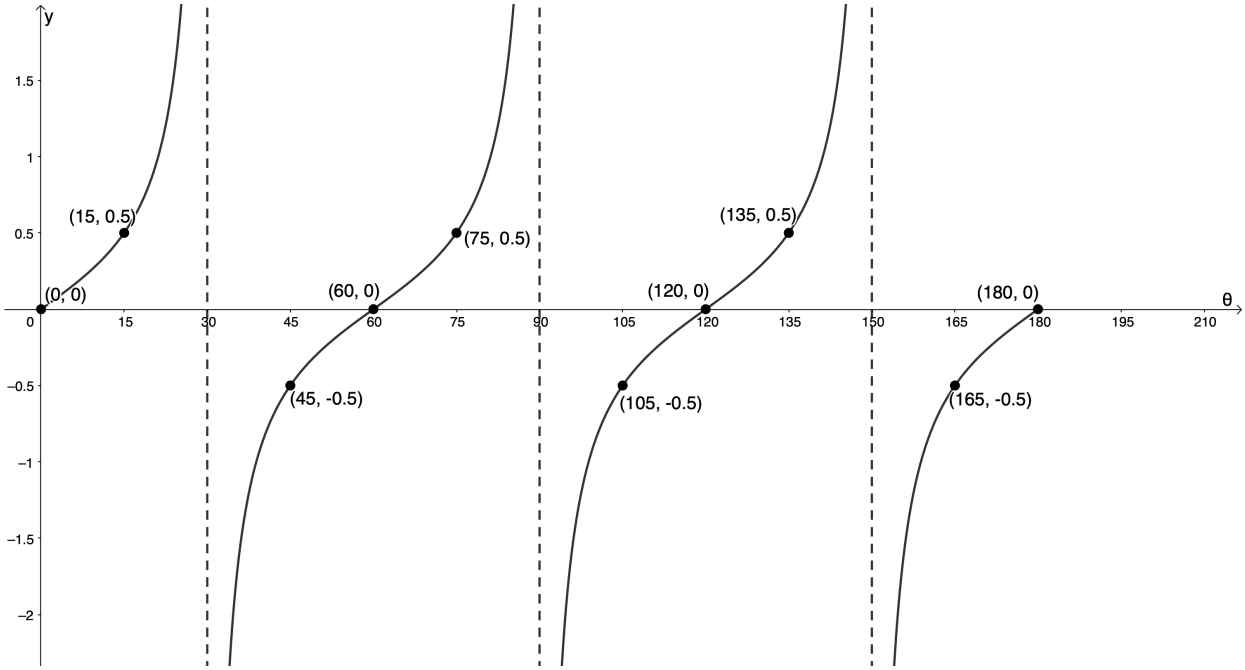
3. $y = \frac{1}{2}\tan 3\theta$ for $0^\circ \leq x \leq 180^\circ$

   $k = 3$

   $\text{Period} = \frac{180^\circ}{|k|} = \frac{180^\circ}{|3|} = 60^\circ$

   $a = \frac{1}{2}$. Graph is not reflected about the x-axis.

$\tan\theta$	$(0^\circ, 0)$	$(45^\circ, 1)$	$\theta = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\frac{1}{2}\tan 3\theta$	$(0^\circ, 0)$	$(15^\circ, \frac{1}{2})$	$\theta = 30^\circ$ asymptote	$(45^\circ, -\frac{1}{2})$	$(60^\circ, 0)$



Back to Exercise 11.2

Exercise 11.3

1. The y-values of the 'anchor points' are 2and the graph is not reflected about the x-axis. Therefore, $a = 2$ and $y = 2\tan kx$.

   The period is 120°. Therefore $k = \frac{180^\circ}{120^\circ} = \frac{3}{2}$.

   $y = 2\tan\left(\frac{3}{2}x\right)$

2. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x < 180^\circ; x \neq 60^\circ\}$
   Range: $\{y | y \in \mathbb{R}\}$

Back to Exercise 11.3

Unit 11: Assessment

1.

   a. $2y = \tan\left(-\frac{1}{3}x\right)$ for $-270^\circ \leq x \leq 270^\circ$

$2y = \tan\left(-\frac{1}{3}x\right)$

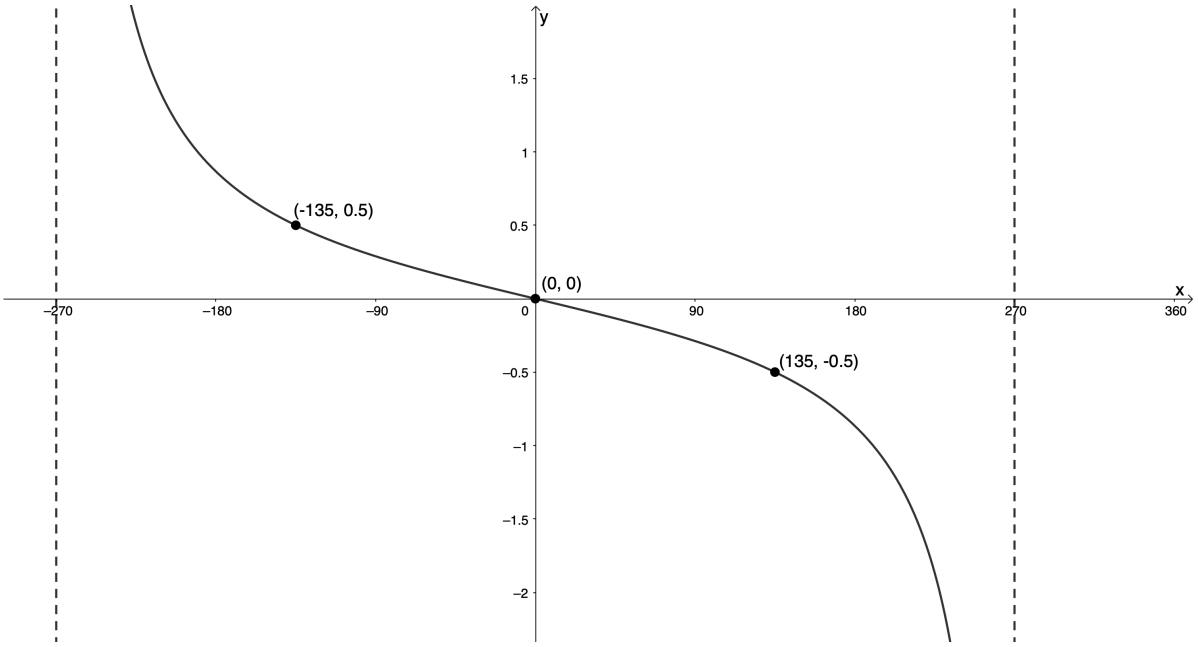
$\therefore y = \frac{1}{2}\tan\left(-\frac{1}{3}x\right)$

$k = -\frac{1}{3}$. Therefore, $\text{period} = \frac{180^\circ}{\left|-\frac{1}{3}\right|} = 540^\circ$

$k < 0$. Therefore, the graph is reflected about the x-axis.

$a = \frac{1}{2}$.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\frac{1}{2}\tan\left(-\frac{1}{3}x\right)$	$(0^\circ, 0)$	$(135^\circ, -\frac{1}{2})$	$x = 270^\circ$ asymptote	$(405^\circ, \frac{1}{2})$	$(540^\circ, 0)$

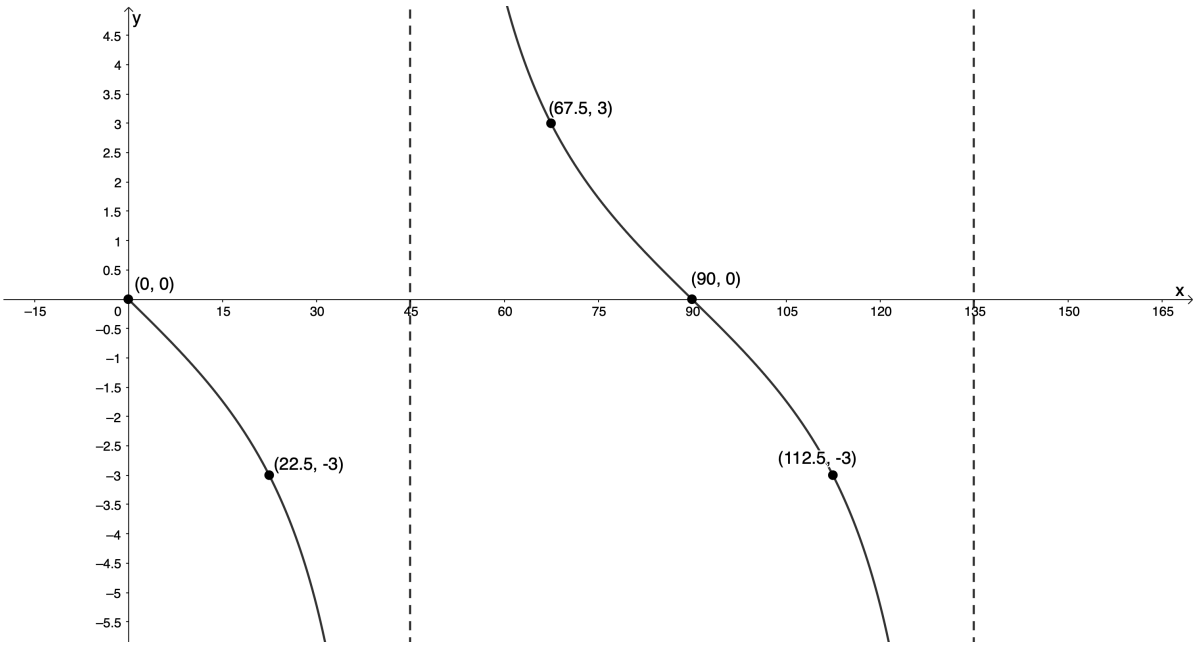


b. $g(x) = -3\tan 2x$ for $0^\circ \leq x \leq 135^\circ$

$k = 2$. Therefore, $\text{period} = \frac{180^\circ}{|2|} = 90^\circ$

$a = -3$. Therefore, the graph is reflected about the x-axis.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$-3\tan 2x$	$(0^\circ, 0)$	$(22.5^\circ, -3)$	$x = 45^\circ$ asymptote	$(67.5^\circ, 3)$	$(90^\circ, 0)$



2. The y-value of the indicated 'anchor point' is -3 and the graph is reflected about the x-axis (it decreases to negative infinity as it approaches the asymptote from the left). Therefore, $a = -3$ and $y = -3\tan kx$.
The graph completes half a period between 0° and 360°. Therefore, the period is $2 \times 360^\circ = 720^\circ$.

$$\text{Period} = \frac{180^\circ}{|k|}$$
$$720^\circ = \frac{180^\circ}{|k|}$$
$$\therefore |k| = \frac{180^\circ}{720^\circ} \qquad = \frac{1}{4}$$
$$y = -3\tan\left(\frac{1}{4}x\right)$$

Back to Unit 11: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © DHET is licensed under a CC BY (Attribution) license
- takenote © DHET is licensed under a CC BY (Attribution) license
- exercise11.1Q2a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise11.1Q2b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise11.1Q2c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example11.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example11.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example11.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example11.5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise11.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise11.2A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise11.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise11.2A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

- assessmentA1a © Geogebra is licensed under a CC BY (Attribution) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 12: Horizontal transformation of the tangent graph

DYLAN BUSA



Unit outcomes

By the end of this unit you will be able to:

- Sketch functions of the form $y = \tan(\theta + p)$.
- Determine the effects of positive and negative values of p on the tangent graph $y = \tan(\theta + p)$.
- Find the value of p from a given tangent graph of the form $y = \tan(\theta + p)$.

Remember that the domain of trigonometric functions can be represented as x or θ. Therefore, $y = \tan x$ and $y = \tan \theta$ are the same function.

What you should know

Before you start this unit, make sure you can:

- Sketch tangent functions of the form $y = a \tan \theta + q$. Refer to level 2 subject outcome 2.1 Unit 5 if you need help with this.

Introduction

It should not surprise you that the effect of p on $y = \tan(x + p)$ is also to shift the graph horizontally by p units. If $p > 0$, the graph is shifted p units **to the left**. If $p < 0$, the graph is shifted p units **to the right**.

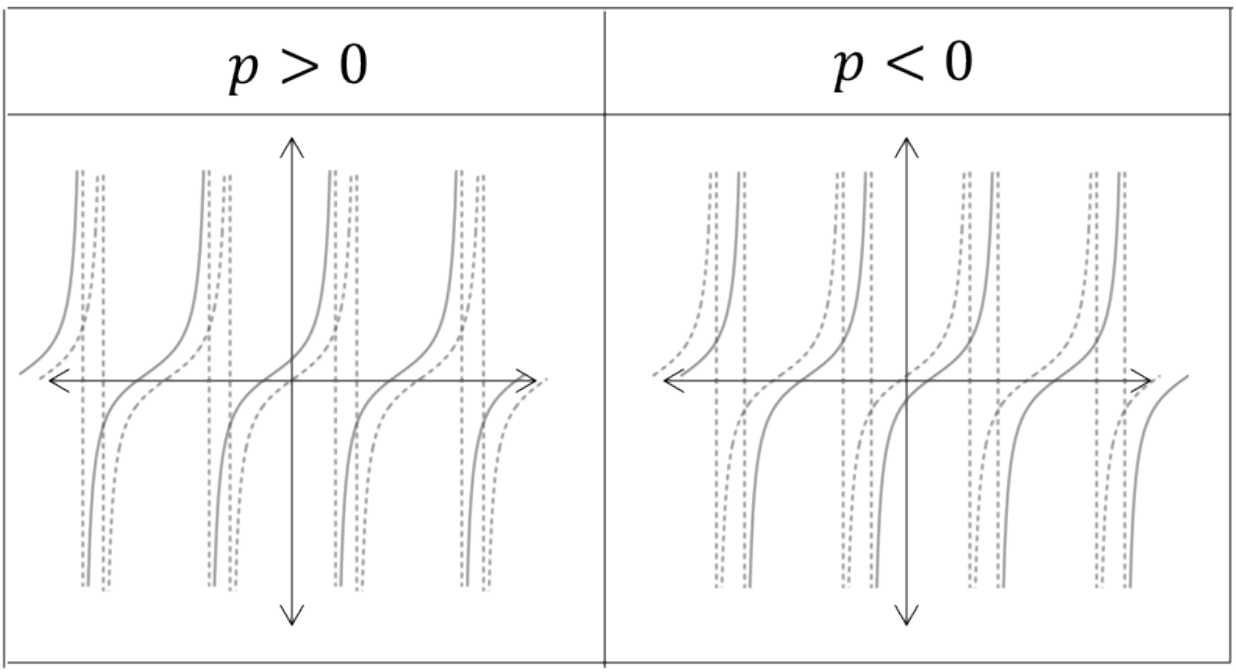
In the case of the sine and cosine graphs, we can track these horizontal shifts by looking at how the maximum and minimum turning points and x-intercepts are shifted, however with the tangent graph we track the vertical asymptotes, x-intercepts and other 'anchor points'.



Take note!

In $y = \tan(x + p)$:

- if $p > 0$, the graph is shifted p units **to the left**
- if $p < 0$, the graph is shifted p units **to the right**.



Note

If you have an internet connection, spend some time playing with this interactive simulation.

Here you will find a graph of the function $y = \tan(x + p)$ with a slider to change the value of p. Pay particular attention to how changing the value of p affects the location of the x-intercepts, the 'anchor points' and the vertical asymptotes.

Sketch functions of the form $y = \tan(x + p)$

The best way to sketch functions of the form $y = \tan(x + p)$ is to transform the basic function of $y = \tan x$ depending on the value of p. To do this, you need to know the set of 'anchor points' of $y = \tan x$, as transformation of these points will help you to sketch functions of the form$y = \tan(x + p)$.

Example 12.1

Given the function $y = \tan(x + 45^\circ)$:

1. Sketch the function for the interval $0^\circ \leq x \leq 360^\circ$.
2. State the intercepts with the axes.
3. State the domain and range.
4. State the period.
5. State the amplitude.

Solutions

1. The function is of the form $y = \tan(x + p)$ Therefore, $p = 45^\circ$.

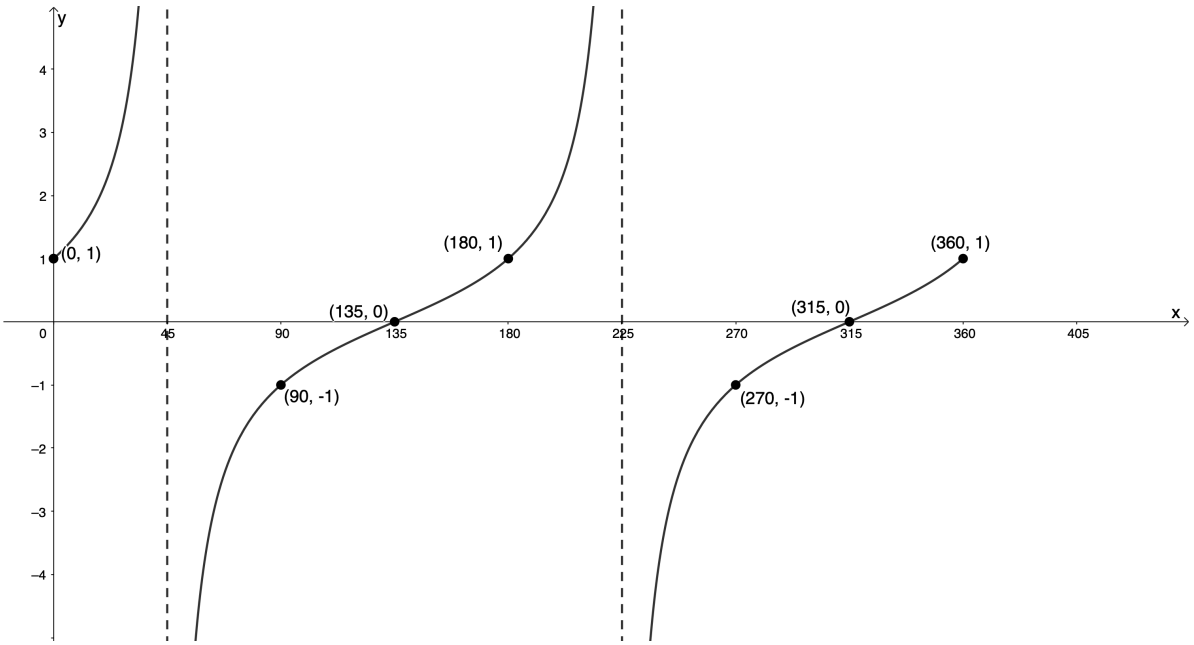
   We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 45° to the left.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan(x + 45^\circ)$	$(-45^\circ, 0)$	$(0^\circ, 1)$	$x = 45^\circ$ asymptote	$(90^\circ, -1)$	$(135^\circ, 0)$

   To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.
   y-intercept (let $x = 0$):
   $y = \tan(0^\circ + 45^\circ)$
   $\therefore y = 1$

   Because the period of the function is 180° we know that the function value at 180° will also be 1.

   We can now plot our transformed 'anchor points' and draw the graph for the interval $0^\circ \leq x \leq 360^\circ$.



2. y-intercept: $(0^\circ, 1)$
   x-intercepts: $(135^\circ, 0)$ and $(315^\circ, 0)$

3. Domain: $\{x|x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ; x \neq 45^\circ, 225^\circ\}$
   Range: $\{y|y \in \mathbb{R}\}$
4. The period is 180°.
5. The amplitude is undefined because the tangent graph increases to positive and negative infinity at the asymptotes.

Take note!

In general, the domain of the function $y = \tan(x + p)$ is $x \in \mathbb{R}$, excluding the x-values of the asymptotes. We had to restrict the domain in Example 12.1, because the interval in which we were working was restricted.

Exercise 12.1

Sketch the following functions for the indicated intervals on separate sets of axes:

1. $f(x) = \tan(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
2. $g(x) = \tan(x - 45^\circ)$ for $-180^\circ \leq x \leq 180^\circ$
3. $h(x) = \tan(x - 40^\circ)$ for $-180^\circ \leq x \leq 180^\circ$

The full solutions are at the end of the unit.

Sketch functions of the form $y = a\tan(x + p)$

Now that we know that the value of p shifts the function of the form $y = \tan(x + p)$ horizontally by p degrees, we can combine this with our knowledge of the effects of a, and examine functions of the form $y = a\tan(x + p)$.

Example 12.2

Sketch the graph of $f(x) = 1.5\tan(x + 60^\circ)$ for the interval $-180^\circ \leq x \leq 180^\circ$.

Solution

The function is of the form $y = a\tan(x + p)$ where $a = 1.5$ and $p = 60^\circ$.

We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 60° to the left and each of the y-values is going to be multiplied by 1.5.

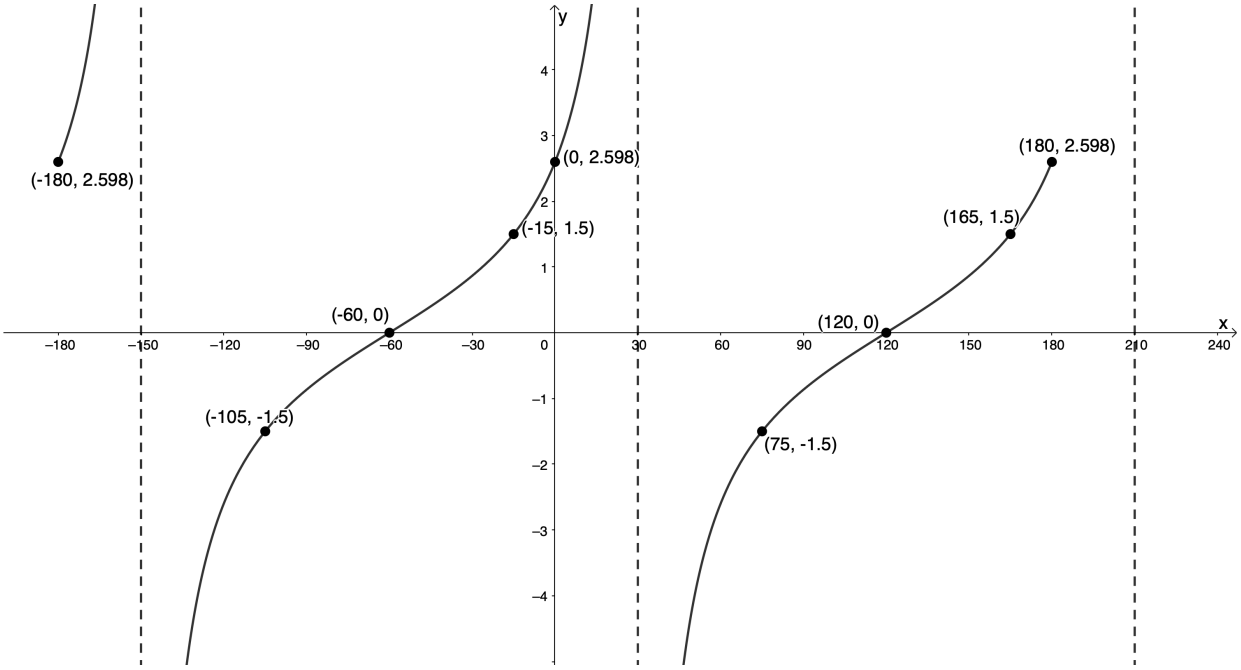
$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$1.5\tan(x + 60^\circ)$	$(-60^\circ, 0)$	$(-15^\circ, 1.5)$	$x = 30^\circ$ asymptote	$(75^\circ, -1.5)$	$(120^\circ, 0)$

To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.
y-intercept (let $x = 0$):

$$y = 1.5\tan(0^\circ + 60^\circ)$$
$$\therefore y = 1.5 \times \frac{\sqrt{3}}{1} = \frac{3\sqrt{3}}{2}$$
$$= 2.598$$

Because the period of the function is 180° we know that the function values at -180° and 180° will also be 2.598.

We can now plot our transformed 'anchor points' and draw the graph for the interval $-180^\circ \leq x \leq 180^\circ$.



Example 12.3

Given the function $f(x) = 2\tan(30^\circ - x)$:

1. Sketch the graph for the interval $0^\circ \leq x \leq 180^\circ$.
2. State the intercepts with the axes.
3. State the domain and range.
4. State the amplitude and period.

Solutions

1. The function is not in the form $y = a\tan(x + p)$. We first need to get it into this form.

$$\begin{aligned} f(x) &= 2\tan(30^\circ - x) \\ &= 2\tan(-x + 30^\circ) \\ &= 2\tan(-(x - 30^\circ)) \\ &= -2\tan(x - 30^\circ) \end{aligned}$$

We know from the previous unit that $\tan(-x) = -\tan(x)$. In other words, the graph of $f(x) = \tan(30^\circ - x)$ is going to be a reflection of the standard tangent graph about the x-axis because $a = -1$.

We need to transform one period of 'anchor points' noting that each of these points is going to be shifted 30° to the right and each of the y-values of these points needs to be multiplied by -2.

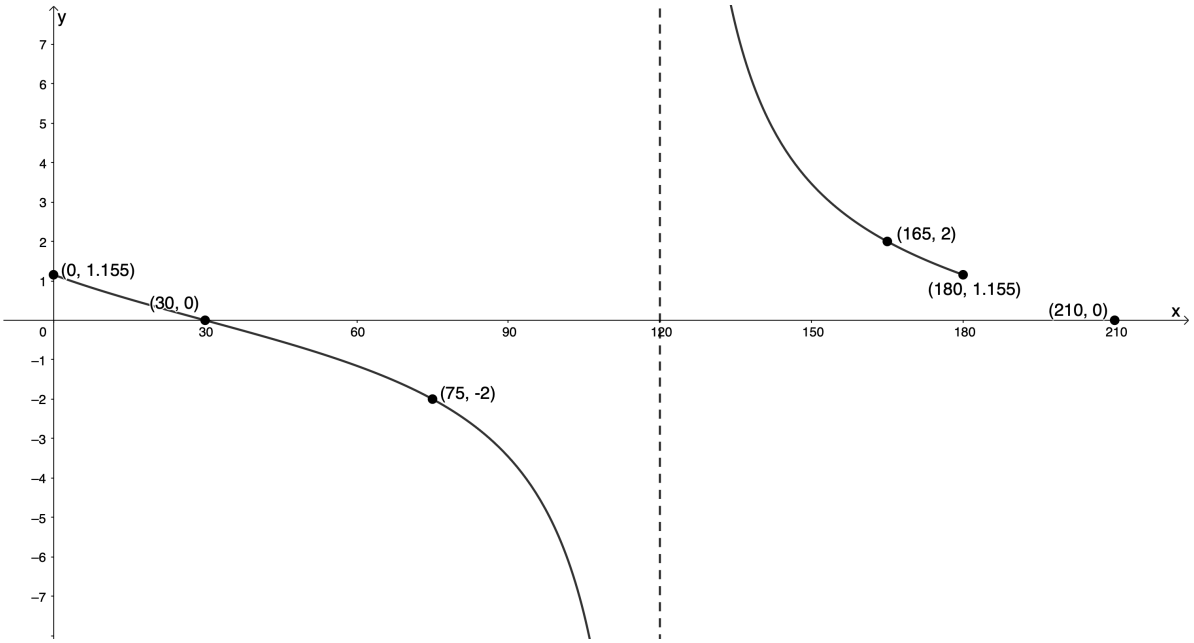
$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$-2\tan(x - 30^\circ)$	$(30^\circ, 0)$	$(75^\circ, -2)$	$x = 120^\circ$ asymptote	$(165^\circ, 2)$	$(210^\circ, 0)$

To make sure that the shape of the graph is as accurate as possible, we must also find the y-intercept.
y-intercept (let $x = 0$):

$$\begin{aligned} f(0) &= -2\tan(0^\circ - 30^\circ) \\ \therefore f(0) &= -2 \times -\frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \\ &= 1.155 \end{aligned}$$

Because the period of the function is 180° we know that the function value at 180° will also be 1.155.

We can now plot our transformed 'anchor points' and draw the graph for the interval $0^\circ \leq x \leq 180^\circ$. Remember that because $a = -1$, the graph is reflected about the x-axis. This means that it decreases to negative infinity as it approaches the asymptote from the left and increases to positive infinity as it approaches the asymptote from the right, the opposite to the function $y = \tan x$.



2. y-intercept: $(0^\circ, \frac{2}{\sqrt{3}})$ or $(0^\circ, 1.155)$
   x-intercepts: $(30^\circ, 0)$
3. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 180^\circ; x \neq 120^\circ\}$
   Range: $\{f(x) | f(x) \in \mathbb{R}\}$
4. The period is 180°.
   The amplitude is undefined because the function values increase to positive and negative infinity as the function approaches the asymptote from the right and left respectively.

Exercise 12.2

For each of the following functions, sketch the graph for the indicated interval and state the domain and range and intercepts with the axes:

1. $y = -2\tan(x + 45^\circ)$ for $0^\circ \leq x \leq 180^\circ$
2. $2y = \tan(x - 45^\circ)$ for $0^\circ \leq x \leq 360^\circ$
3. $\frac{1}{3}y = \tan(60^\circ - x)$ for $-180^\circ \leq x \leq 180^\circ$

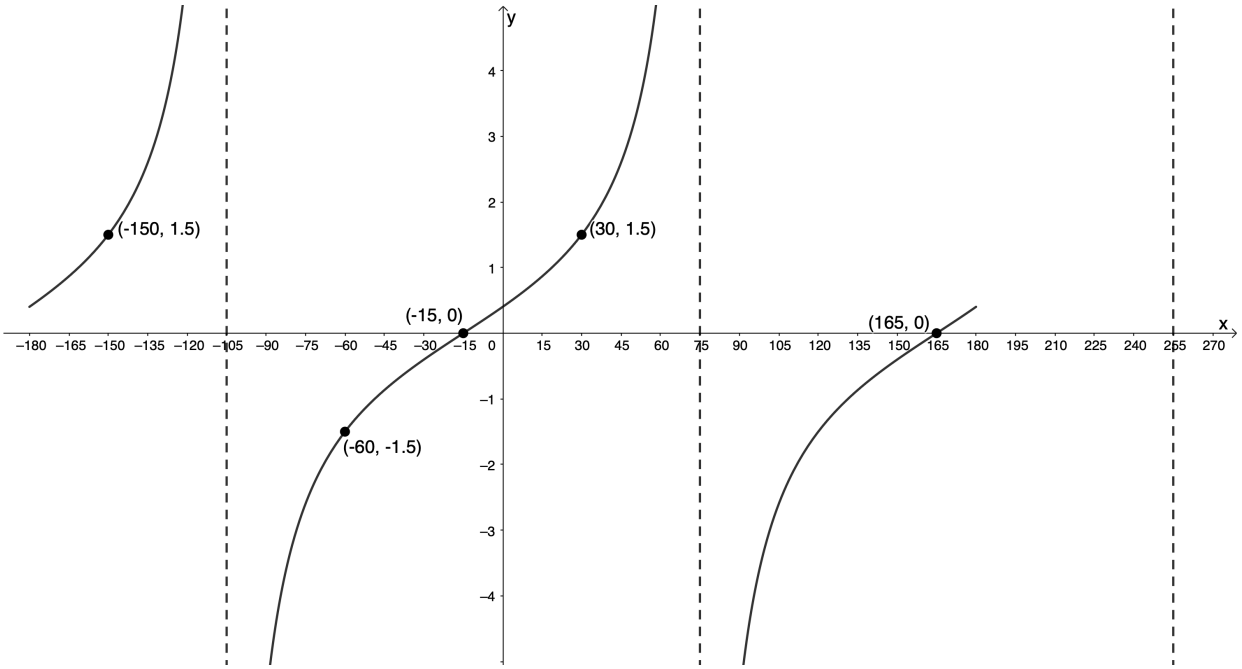
The full solutions are at the end of the unit.

Find the equation of a tangent function of the form $y = a\tan(x + p)$

By examining the amplitude and the horizontal shift of $y = a\tan(x + p)$, we can determine the values of a and p.

Example 12.4

The graph below is a function of the form $y = a\tan(x + p)$. Determine the values of a and p.



Solution

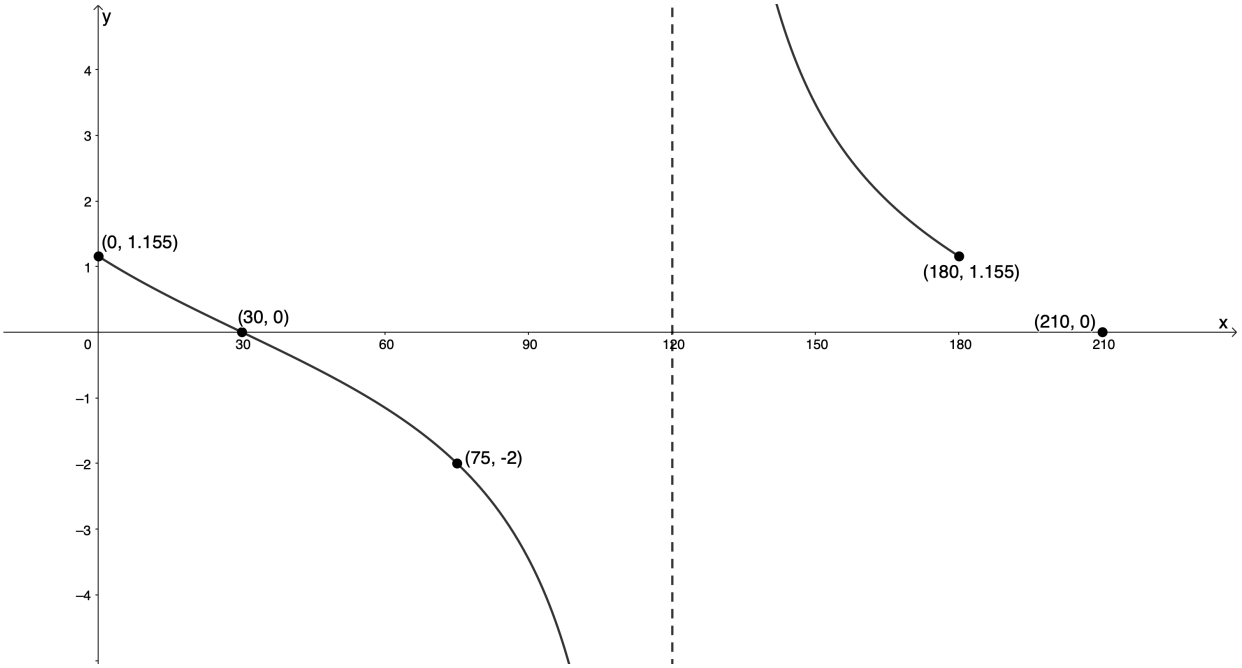
We are told that the function is of the form $y = a\tan(x + p)$.

We know that the function $y = \tan x$ has a vertical asymptote at $x = 90°$. This graph has an asymptote at $x = 75°$. Therefore, the graph has been shifted $15°$ to the left and $p = 15°$. The 'anchor point' that is normally at $(45°, 1)$ is now at $(30°, 1.5)$. This confirms the fact that the graph has been shifted $15°$ to the left but also tells us that $a = 1.5$.

Therefore, the function is $y = 1.5\tan(x + 15°)$.

Example 12.5

The graph below is a function of the form $y = a\tan(x + p)$. Determine the values of a and p.



Solution

We are told that the function is of the form $y = a\tan(x + p)$.

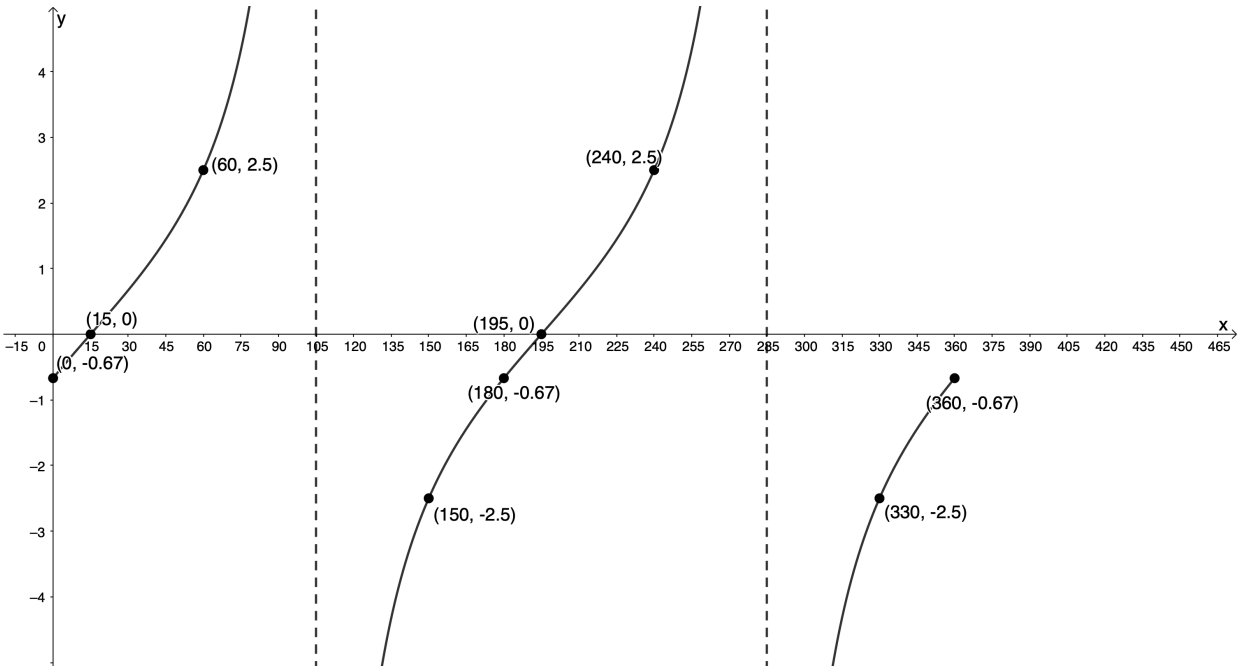
The asymptote that is normally at $x = 90^\circ$ is at $x = 120^\circ$. The graph has been shifted 30° to the right. Therefore, $p = -30^\circ$.

The anchor point that was at $(45^\circ, 1)$ is now at $(75^\circ, -2)$. In addition, the graph decreases to negative infinity as the function approaches the asymptote from the left. Therefore, $a = -2$.

The function is $y = -2\tan(x - 30^\circ)$.

Exercise 12.3

Given the graph of the form $y = a\tan(x + p)$:



1. Determine the values of a and p.
2. State the domain and range of the function.

The full solutions are at the end of the unit.

Summary

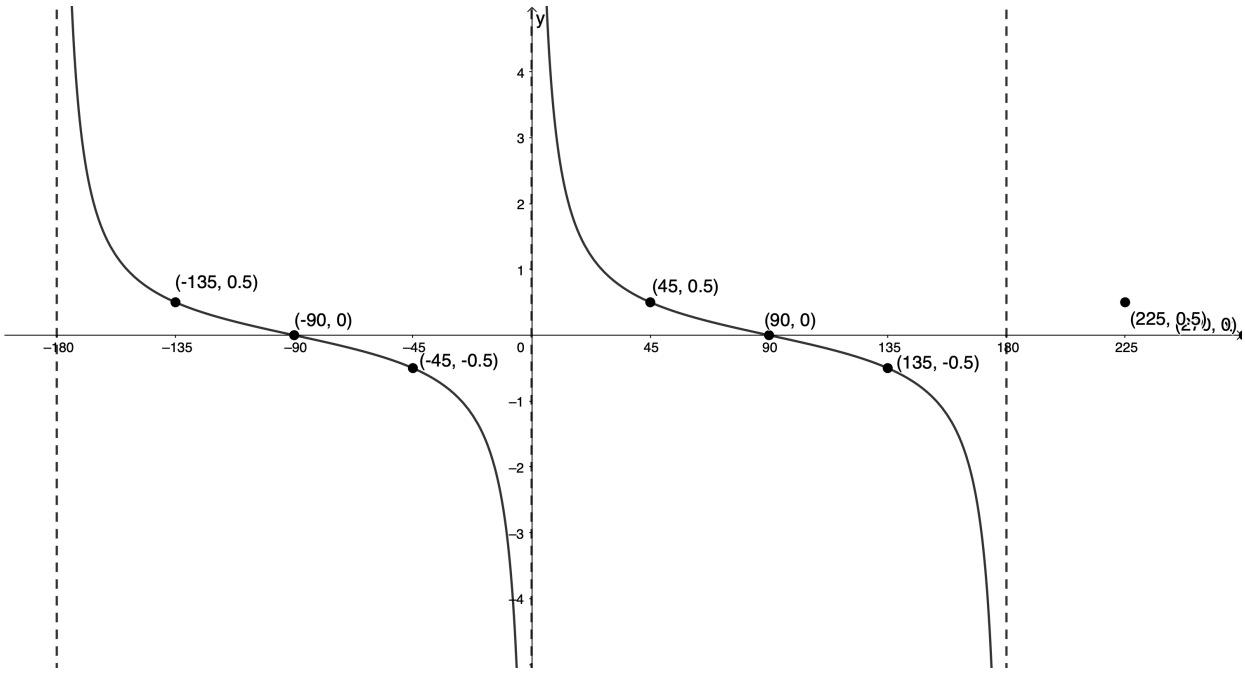
In this unit you have learnt the following:

- The effects of a and p on the tangent graph of the form $y = a\tan(x + p)$.
- How to sketch functions of the form $y = a\tan(x + p)$.
- How to find the values of a and p from a given tangent graph of the form $y = a\tan(x + p)$.

Unit 12: Assessment

Suggested time to complete: 45 minutes

1. Sketch the following functions for the given intervals:
   a. $2y = -\tan(x - 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$
   b. $g(x) = -3\tan(45^\circ - x)$ for $-180^\circ \leq x \leq 180^\circ$
2. From the graph below of the form $y = a\tan(x + p)$, determine the values of a and p.



The full solutions are at the end of the unit.

Unit 12: Solutions

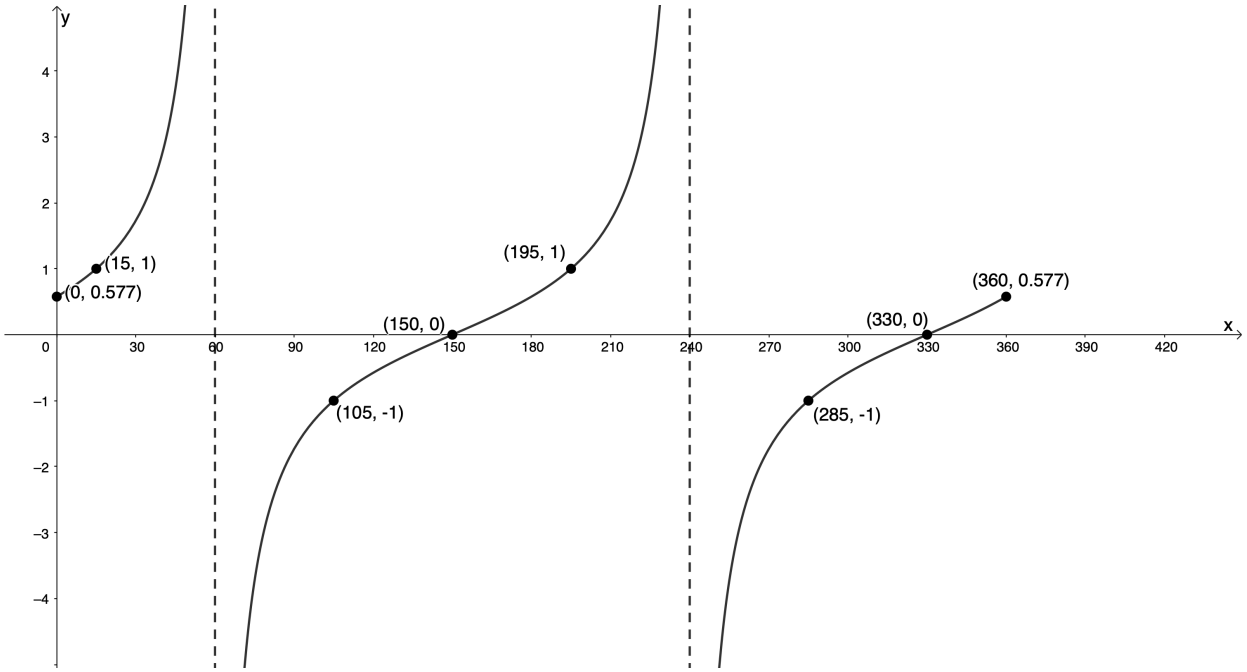
Exercise 12.1

1. $f(x) = \tan(x + 30^\circ)$ for $0^\circ \leq x \leq 360^\circ$
   $p = 30^\circ$. Therefore, the graph will be shifted 30° to the left.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan(x + 30^\circ)$	$(-30^\circ, 0)$	$(15^\circ, 1)$	$x = 60^\circ$ asymptote	$(105^\circ, -1)$	$(150^\circ, 0)$

   y-intercept (let $x = 0$):
   $f(0) = \tan(0^\circ + 30^\circ)$
   $\therefore f(0) = \dfrac{1}{\sqrt{3}} = 0.577$

   Because the period of the function is 180° we know that the function value at 180° will also be 0.577.

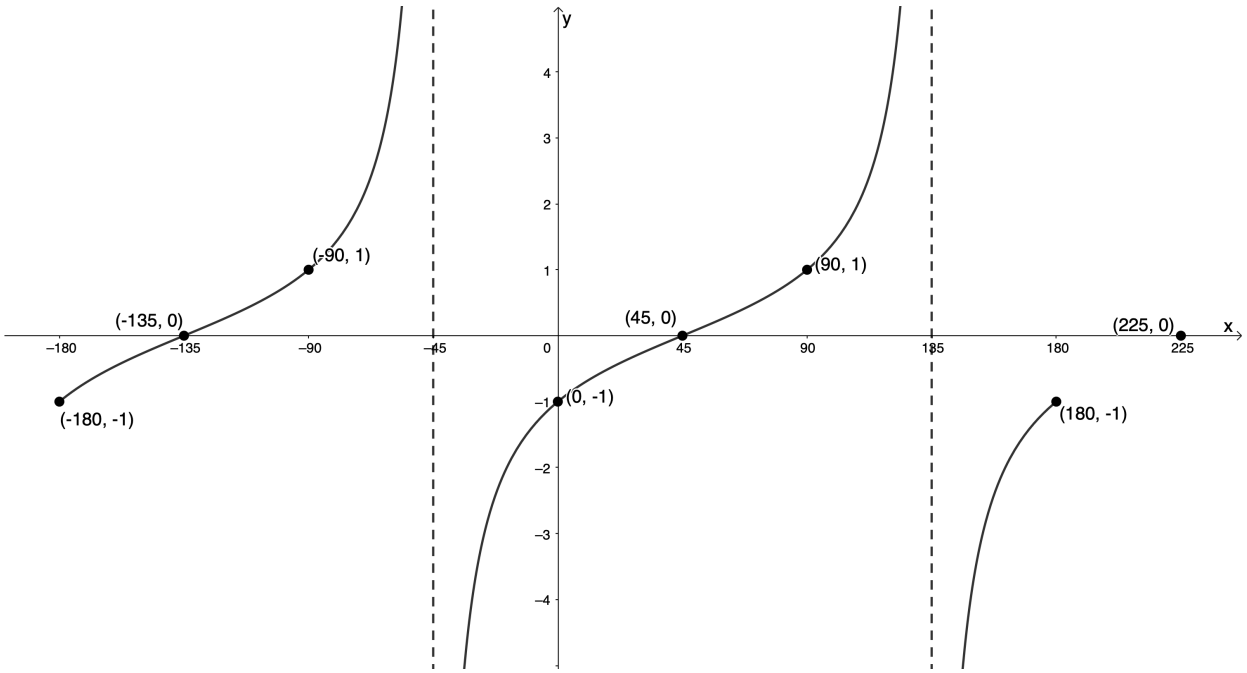


2. $g(x) = \tan(x - 45^\circ)$ for $-180^\circ \leq x \leq 180^\circ$
   $p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan(x - 45^\circ)$	$(45^\circ, 0)$	$(90^\circ, 1)$	$x = 135^\circ$ asymptote	$(180^\circ, -1)$	$(225^\circ, 0)$

y-intercept (let $x = 0$):
$g(0) = \tan(0^\circ - 45^\circ)$
$\therefore g(0) = -1$

Because the period of the function is 180° we know that the function values at -180° and 180° will also be 1.

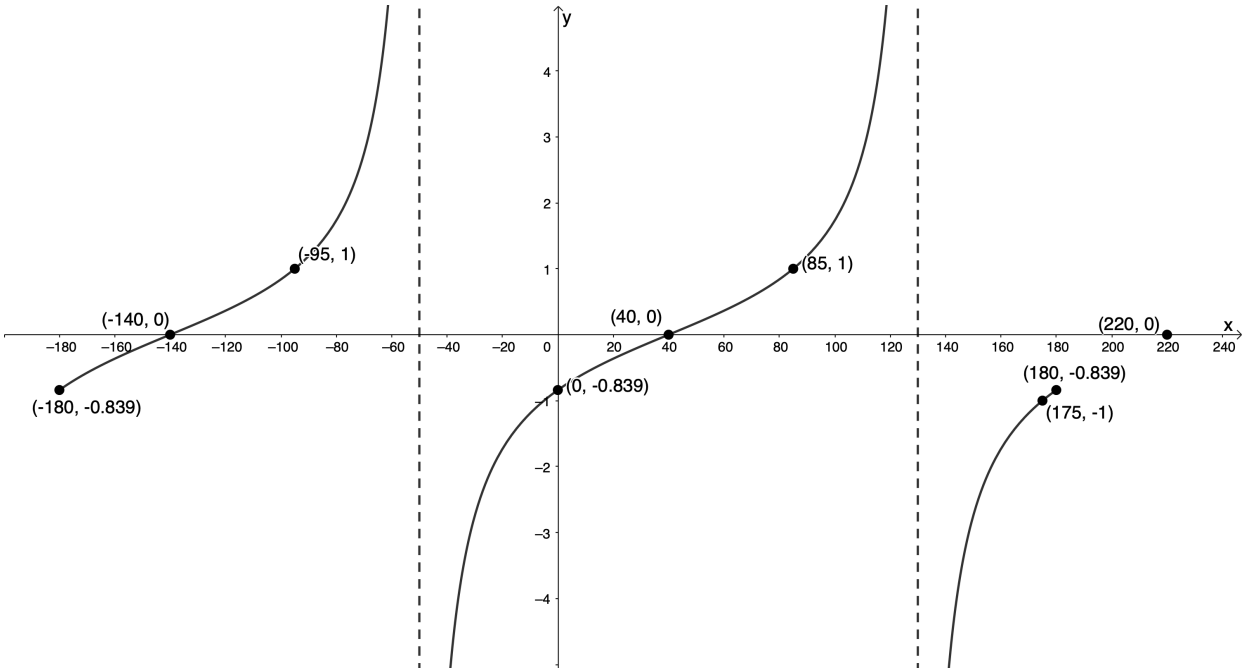


3. $h(x) = \tan(x - 40^\circ)$ for $-180^\circ \leq x \leq 180^\circ$
   $p = -40^\circ$. Therefore, the graph will be shifted 40° to the right.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\tan(x - 40^\circ)$	$(40^\circ, 0)$	$(85^\circ, 1)$	$x = 130^\circ$ asymptote	$(175^\circ, -1)$	$(220^\circ, 0)$

y-intercept (let $x = 0$):
$h(0) = \tan(0^\circ - 40^\circ)$
$\therefore h(0) = -0.839$

Because the period of the function is 180° we know that the function values at -180° and 180° will also be -0.839.



Back to Exercise 12.1

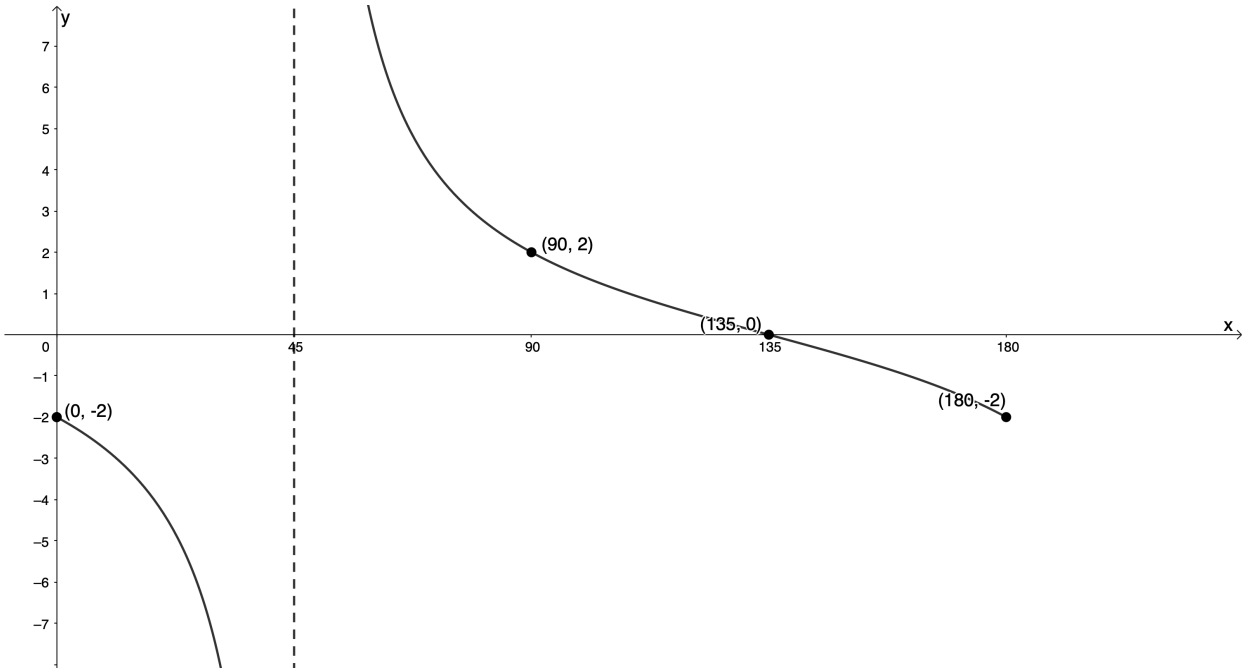
Exercise 12.2

1. $y = -2\tan(x + 45^\circ)$ for $0^\circ \leq x \leq 180^\circ$
   $a = -2$. Therefore, the graph be reflected about the x-axis.
   $p = 45^\circ$. Therefore, the graph will be shifted 45° to the left.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$-2\tan(x + 45^\circ)$	$(-45^\circ, 0)$	$(0^\circ, -2)$	$x = 45^\circ$ asymptote	$(90^\circ, 2)$	$(135^\circ, 0)$

y-intercept (let $x = 0$):
$y = -2\tan(0^\circ + 45^\circ)$
$\therefore y = -2 \times 1$
$= -2$

Because the period of the function is 180° we know that the function value at 180° will also be -2.



y-intercept: $(0^\circ, -2)$
x-intercepts: $(135^\circ, 0)$
Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 180^\circ; x \neq 45^\circ\}$
Range: $\{y | y \in \mathbb{R}\}$

2. $2y = \tan(x - 45^\circ)$ for $0^\circ \leq x \leq 360^\circ$

$2y = \tan(x - 45^\circ)$

$\therefore y = \frac{1}{2}\tan(x - 45^\circ)$

$a = \frac{1}{2}$. The graph will not be reflected about the x-axis.

$p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

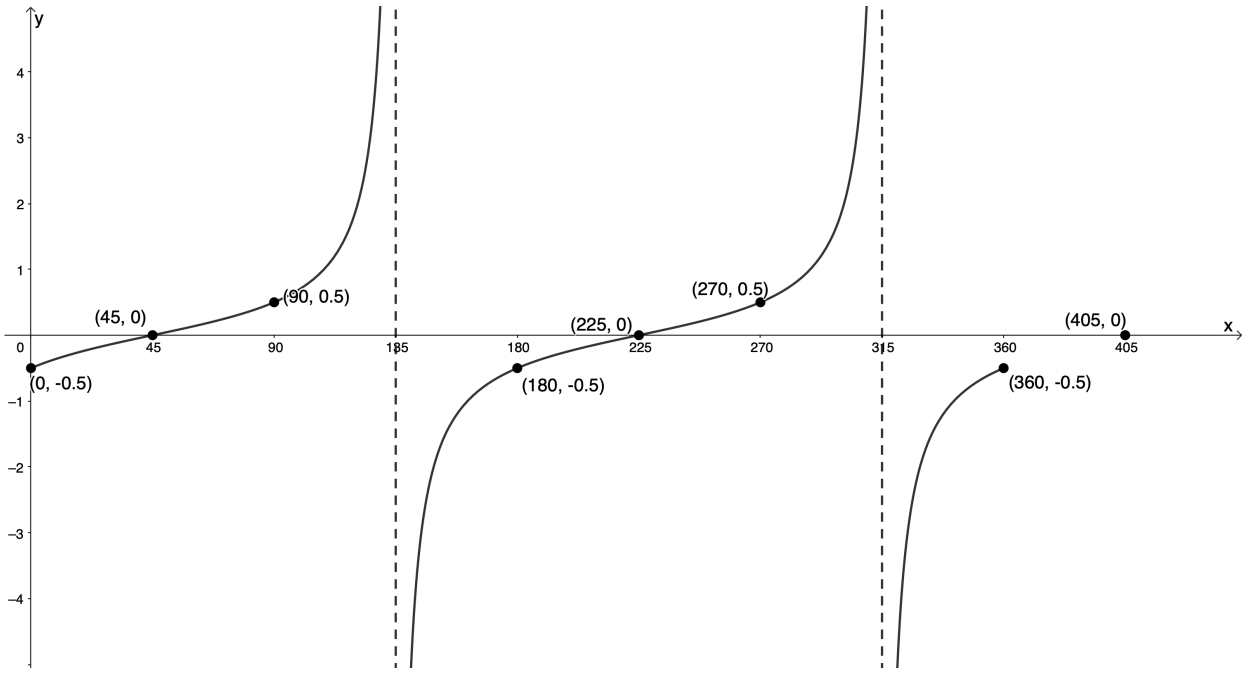
$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$\frac{1}{2}\tan(x - 45^\circ)$	$(45^\circ, 0)$	$(90^\circ, \frac{1}{2})$	$x = 135^\circ$ asymptote	$(180^\circ, -\frac{1}{2})$	$(225^\circ, 0)$

y-intercept (let $x = 0$):

$y = \frac{1}{2}\tan(0^\circ - 45^\circ)$

$\therefore y = \frac{1}{2} \times (-1) = -\frac{1}{2}$

Because the period of the function is 180° we know that the function value at 180° and 360° will also be $-\frac{1}{2}$.



y-intercept: $(0^\circ, -\frac{1}{2})$

x-intercepts: $(45^\circ, 0)$, $(225^\circ, 0)$

Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ; x \neq 135^\circ \text{ or } 315^\circ\}$

Range: $\{y | y \in \mathbb{R}\}$

3. $\frac{1}{3}y = \tan(60^\circ - x)$ for $-180^\circ \leq x \leq 180^\circ$

$$\frac{1}{3}y = \tan(60^\circ - x)$$
$$\therefore y = 3\tan(60^\circ - x)$$
$$= 3\tan(-x + 60^\circ)$$
$$= 3\tan(-(x - 60^\circ))$$
$$= -3\tan(x - 60^\circ)$$

$a = -3$. Therefore, the graph will be reflected about the x-axis.

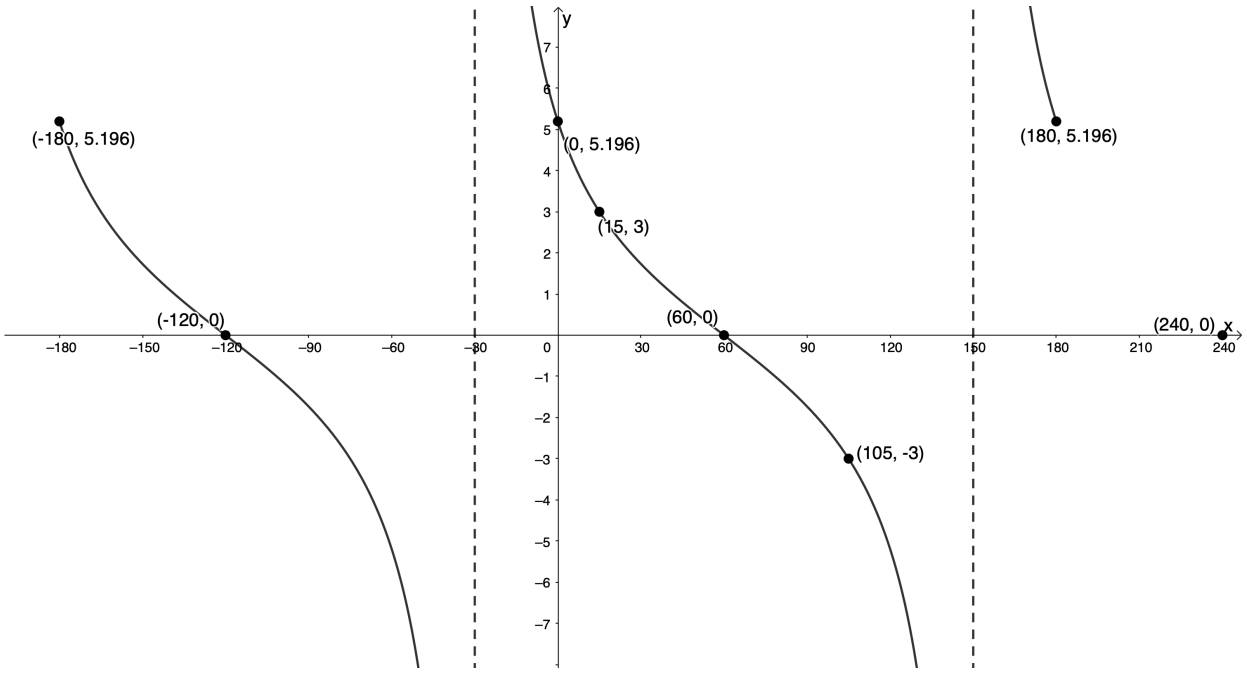
$p = -60^\circ$. Therefore, the graph will be shifted 60° to the right.

$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$-3\tan(x - 60^\circ)$	$(60^\circ, 0)$	$(105^\circ, -3)$	$x = 150^\circ$ asymptote	$(195^\circ, 3)$	$(240^\circ, 0)$

y-intercept (let $x = 0$):

$$y = -3\tan(0^\circ - 60^\circ)$$
$$\therefore y = -3 \times -\frac{\sqrt{3}}{1} = 3\sqrt{3}$$
$$= 5.196$$

Because the period of the function is 180° we know that the function values at -180° and 180° will also be 5.196.



y-intercept: $(0^\circ, 5.196)$ or $(0^\circ, 3\sqrt{3})$
x-intercepts: $(-120^\circ, 0)$, $(60^\circ, 0)$
Domain: $\{x | x \in \mathbb{R},\ -180^\circ \leq x \leq 180^\circ; x \neq -30^\circ; x \neq 150^\circ\}$
Range: $\{y | y \in \mathbb{R}\}$

Back to Exercise 12.2

Exercise 12.3

1. The asymptote is at $x = 105^\circ = (90^\circ + 15^\circ)$. Therefore, the graph has been shifted 15° to the right and $p = -15^\circ$.
   $(45^\circ, 1)$ has been transformed to $(60^\circ, 2.5)$. Therefore, $a = 2.5$.
   $y = 2.5\tan(x - 15^\circ)$
2. Domain: $\{x | x \in \mathbb{R},\ 0^\circ \leq x \leq 360^\circ; x \neq 105^\circ, 285^\circ\}$
   Range: $\{y | y \in \mathbb{R}\}$

Back to Exercise 12.3

Unit 12: Assessment

1.

a. $2y = -\tan(x - 60^\circ)$ for $0^\circ \leq x \leq 360^\circ$

$$2y = -\tan(x - 60^\circ)$$

$$\therefore y = -\frac{1}{2}\tan(x - 60^\circ)$$

$a = -\frac{1}{2}$. Therefore, the graph will be reflected about the x-axis.

$p = -60^\circ$. Therefore, the graph will be shifted 60° to the right.

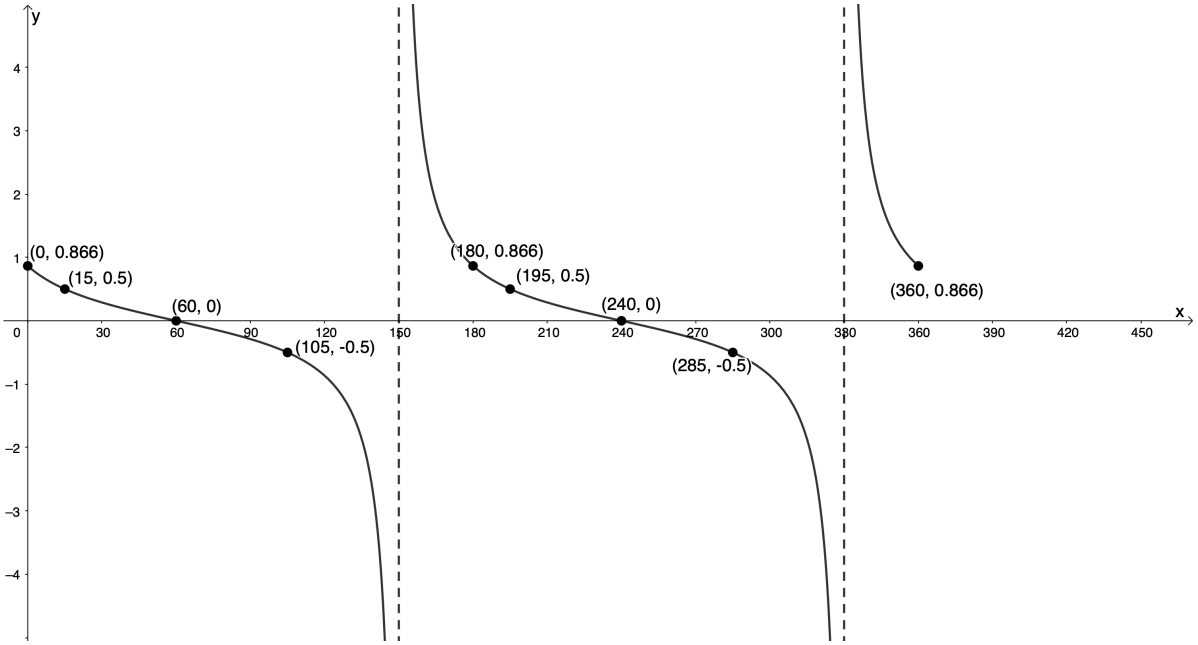
$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$-\frac{1}{2}\tan(x - 60^\circ)$	$(60^\circ, 0)$	$(105^\circ, -\frac{1}{2})$	$x = 150^\circ$ asymptote	$(195^\circ, \frac{1}{2})$	$(240^\circ, 0)$

y-intercept (let $x = 0$):

$$y = -\frac{1}{2}\tan(0^\circ - 60^\circ)$$

$$\therefore y = -\frac{1}{2} \times -\frac{\sqrt{3}}{1} = \frac{\sqrt{3}}{2} = 0.866$$

Because the period of the function is 180° we know that the function value at 180° and 360° will also be 0.866.



b. $g(x) = -3\tan(45^\circ - x)$ for $-180^\circ \leq x \leq 180^\circ$

$$\begin{aligned} g(x) &= -3\tan(45^\circ - x) \\ &= -3\tan(-x + 45^\circ) \\ &= -3\tan(-(x - 45^\circ)) \\ &= 3\tan(x - 45^\circ) \end{aligned}$$

$a = 3$. Therefore, the graph will not be reflected about the x-axis.

$p = -45^\circ$. Therefore, the graph will be shifted 45° to the right.

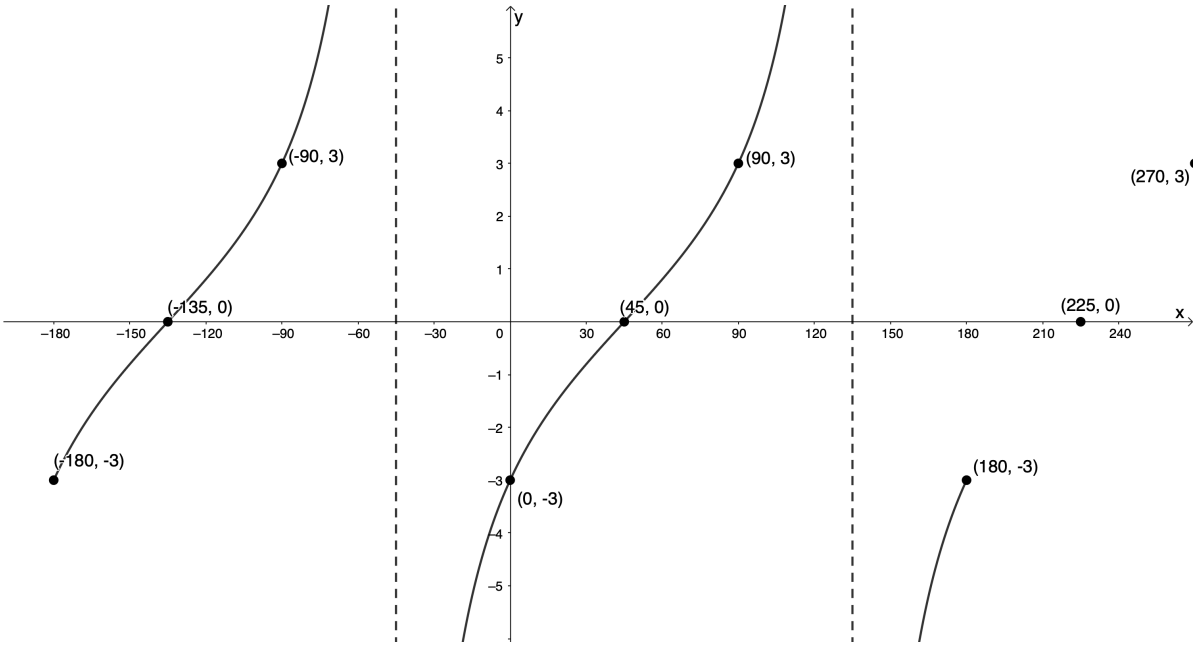
$\tan x$	$(0^\circ, 0)$	$(45^\circ, 1)$	$x = 90^\circ$ asymptote	$(135^\circ, -1)$	$(180^\circ, 0)$
$3\tan(x - 45^\circ)$	$(45^\circ, 0)$	$(90^\circ, 3)$	$x = 135^\circ$ asymptote	$(180^\circ, -3)$	$(225^\circ, 0)$

y-intercept (let $x = 0$):

$$g(0) = 3\tan(0^\circ - 45^\circ)$$

$$\therefore y = 3 \times -1 = -3$$

Because the period of the function is 180° we know that the function values at -180° and 180° will also be -3.



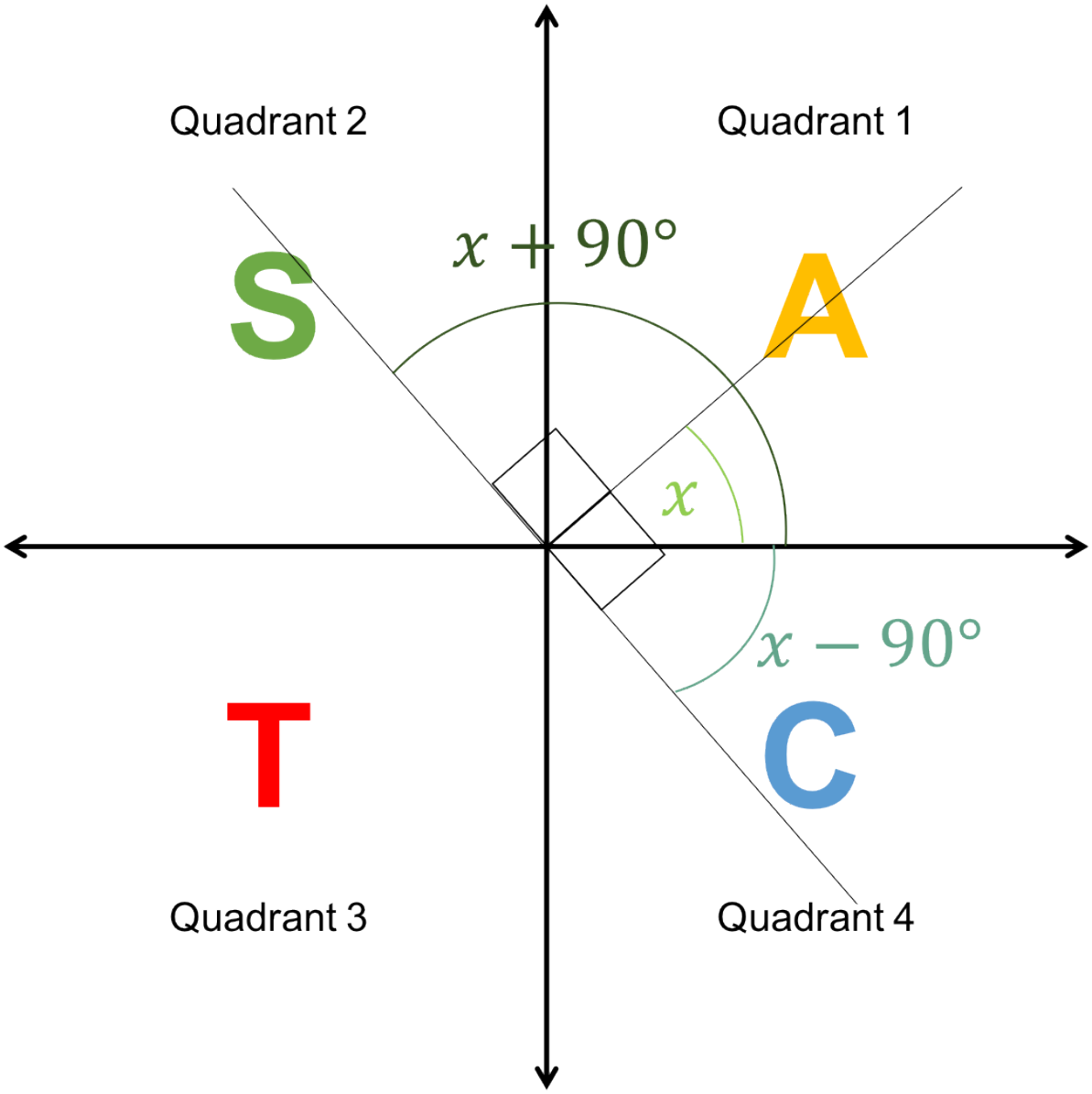
2. The function is of the form $y = a\tan(x + p)$. The asymptote $x = 90^\circ$ is now $x = 180^\circ$. Therefore, the graph has been shifted 90° to the right.

   The function decreases to negative infinity when approaching the asymptote from the left. This means that the function has been reflected about the x-axis and $a < 0$.

   Also, the 'anchor point' that was at $(45^\circ, 1)$ is now at $(135^\circ, -0.5)$. Therefore, $a = -\frac{1}{2}$ and the function is $y = -\frac{1}{2}\tan(x - 90^\circ)$.

   However, we can also say that the asymptote $x = 270^\circ$ is now $x = 180^\circ$ and therefore, the graph has been shifted 90° to the left. Hence $y = -\frac{1}{2}\tan(x + 90^\circ)$, which is another possible equation. This means that $\tan(x + 90^\circ) = \tan(x - 90^\circ)$.

   If we assume $0^\circ < x < 90^\circ$, we can see why this is the case by looking at the CAST diagram. Tangent is negative in the second and fourth quadrants and $x - 90^\circ$ and $x + 90^\circ$ are the same size angles.



Back to Unit 12: Assessment

Media Attributions

- takenote © DHET is licensed under a CC BY (Attribution) license
- example12.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example12.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example12.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example12.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example12.5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.1A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.1A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.1A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.2A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise12.2A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- CAST © DHET is licensed under a CC BY (Attribution) license

SUBJECT OUTCOME VI

FUNCTIONS AND ALGEBRA: USE MATHEMATICAL MODELS TO INVESTIGATE LINEAR PROGRAMMING PROBLEMS

Subject outcome

Subject outcome 2.4: Use Mathematical models to investigate linear programming problems

Learning outcomes

- Solve linear programming problems by optimising a function in two variables, subject to one or more linear constraints, by numerical search along the boundary of the feasible region.
  Note: *Explicit constraints will be given in all examples.*
  Method:
  - Sketch the given functions/constraints
  - Determine and shade the feasible region
  - Complete a boundary search to find the vertices of the feasible region
  - Optimise the maximum or minimum from the given objective function.

Unit 1 outcomes

By the end of this unit you will be able to:

- Sketch given constraints.
- Find the feasible region and objective function.
- Use a boundary search to find the vertices of the feasible region.
- Optimise the objective function.

Unit 1: Linear programming

DYLAN BUSA



Unit outcomes

By the end of this unit you will be able to:

- Sketch given constraints.
- Find the feasible region and objective function.
- Use a boundary search to find the vertices of the feasible region.
- Optimise the objective function.

What you should know

Before you start this unit, make sure you can:

- Simplify algebraic expressions including those with fractions. Refer to level 2 subject outcome 2.2 unit 1 and unit 3 if you need help with this.
- Solve linear equations. Refer to level 2 subject outcomes 2.3 unit 1 if you need help with this.
- Solve linear inequalities. Refer to level 3 subject outcome 2.3 unit 3 if you need help with this.
- Sketch graphs of linear functions. Refer to level 2 subject outcome 2.1 unit 1 if you need help with this.

Introduction

Qhubeka is a wonderful organisation that donates bicycles to children in rural areas to make things such as getting to and from school easier and quicker.



Figure 1: A Qhubeka bicycle

Qhubeka makes all the bicycles it donates, thereby also creating employment. Suppose it makes two types of bicycles – the Buffalo and the Cheetah. The Buffalo is rugged and designed for areas where the roads aren't so good. The Cheetah is designed for areas where there are at least basic roads.

Now suppose that each day Qhubeka can produce at most seven Buffalos and six Cheetahs. It takes three technicians to build a Buffalo and two technicians to build a Cheetah. There is a total of 18 technicians at the organisation.

Each Buffalo costs $\mathrm{R}2\ 400$ to make and each Cheetah costs $\mathrm{R}1\ 200$ to make. If every bicycle that Qhubeka makes is needed by a child, how many of each type should Qhubeka make in order to minimise costs?

This is an example of an optimisation problem. There are several constraints that all need to be taken into account to find an optimal solution, in this case the lowest cost.

Another example of an optimisation problem would be a farmer working out how much of two different crops to plant, with different input costs and different market values, in order to maximise his profit. In all cases, we want to calculate the maximum or minimum in a specific situation.

Solve optimisation problems graphically

The easiest and most efficient way to solve these kinds of optimisation problems is graphically using a technique called linear programming. **Linear programming** is a graphical modelling technique in which the optimal solution of a linear function is found by applying various other constraints. Linear programming is, therefore, a special case of the broader area of mathematical programming and is most often used to guide decision making in business, engineering, social sciences, and physical sciences.

A detailed worked example of the solution to the problem Qhubeka faces is the best way to demonstrate how linear programming works.



Example 1.1

This is what we know:

- Qhubeka makes two types of bicycles – the Buffalo and the Cheetah.
- Each day Qhubeka can produce at most seven Buffalos and four Cheetahs.
- Each Buffalo needs three technicians to build it.
- Each Cheetah needs two technicians.
- There is a total of 18 technicians.
- Each Buffalo costs $\text{R}2\ 400$ to make and each Cheetah costs $\text{R}1\ 200$ to make.

1. If every bicycle that Qhubeka makes is needed by a child, how many of each type should Qhubeka make per day in order to minimise costs?
2. What is this minimum cost?

Solution

There are five basic steps to using linear programming to solve optimisation problems.

Step 1: Assign variables
In this problem, there are two types of bicycles and we want to find out how many of each kind should be produced. We need to assign a variable to each type of bicycle. You can assign any variables you like but, because we will be sketching linear functions, it is often simplest to just use x and y.

Let the number of Buffalo bicycles be x.
Let the number of Cheetah bicycles be y.

In both cases, the values of x and y are limited to natural numbers. We cannot have a fraction of a bicycle. Therefore $\{x|x \in \mathbb{N}\}$ and $\{y|y \in \mathbb{N}\}$.

Step 2: Express the constraints
In all cases a set of constraints is given to you. The first constraint we are given is that each day at most seven Buffalo bicycles can be built. Therefore, $x \leq 7$.

We are also told that at most four Cheetah bicycles can be built. Therefore, $y \leq 4$.

Each Buffalo needs three technicians and each Cheetah needs two technicians. But there are only 18 technicians. Therefore, we can say that $3x + 2y \leq 18$ where $3x$ is the total number of technicians needed to build x Buffalos and $2y$ is the total number of technicians needed to build y Cheetahs.

Step 3: Determine the objective function
The **objective function** is the linear function we are trying to find a maximum or minimum solution for. In our case, we need to minimise cost. We are told that each Buffalo costs $\text{R}2\ 400$ to make and each Cheetah costs $\text{R}1\ 200$ to make. Therefore, $2\ 400x + 1\ 200y = C$ where C is the total cost of producing x Buffalo bicycles and y Cheetah bicycles.

Step 4: Graph the constraints
It is time to start graphing our constraints so that we can optimise the objective function. Here are our constraints:

- $x \leq 7$
- $y \leq 4$
- $3x + 2y \leq 18$

Let's start with $x \leq 7$. If we plot the graph of $x = 7$ we get a vertical line that cuts the x-axis at 7. But our graph needs to show $x \leq 7$. Therefore, this is not a line but a region (see Figure 2). Remember that $x \in \mathbb{N}$ so we are only including natural numbers between zero and seven.

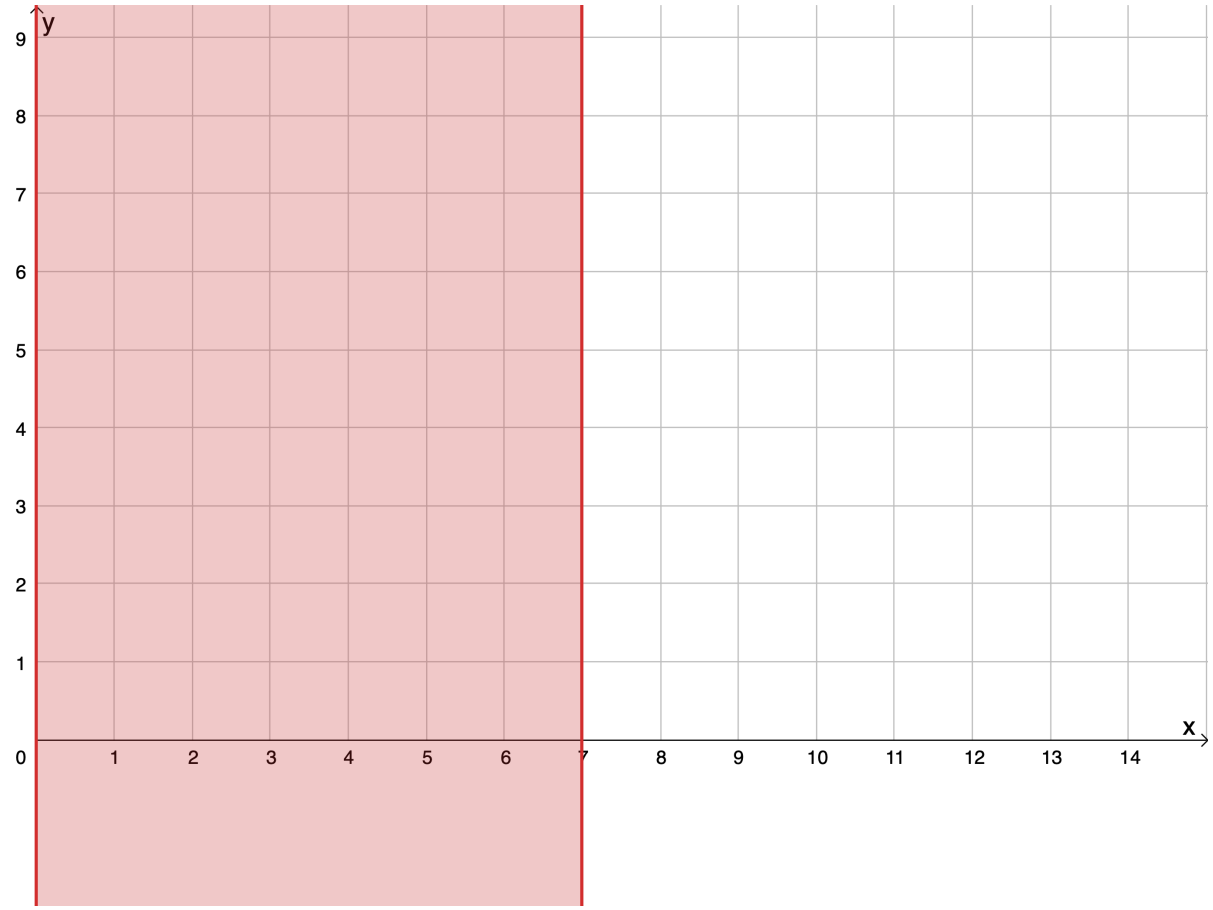


Figure 2: The region represented by $x \leq 7, x \in \mathbb{N}$

Next, we graph the next constraint, $y \leq 4$ (see Figure 3). Again, $y \in \mathbb{N}$ so we are only including natural numbers between zero and four.

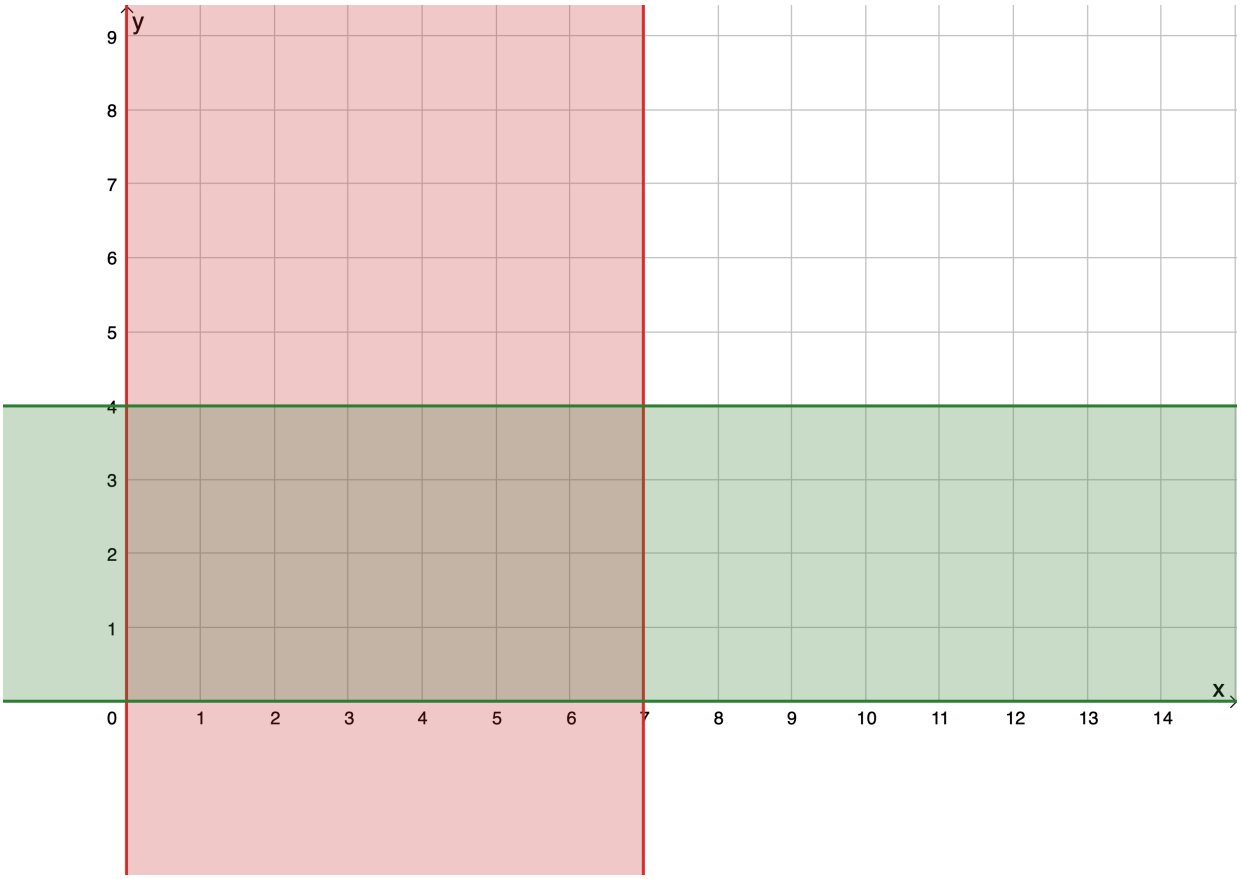


Figure 3: The region represented by $x \leq 7, x \in \mathbb{N}$ *and* $y \leq 7, y \in \mathbb{N}$

The overlapping region is what we are interested in because this represents all the values for x and y that would satisfy both constraints at the same time.

But we have another constraint: $3x + 2y \leq 18$.

$$\begin{aligned} 3x + 2y &\leq 18 \\ \therefore 2y &\leq -3x + 18 \\ \therefore y &\leq -\frac{3}{2}x + 9 \end{aligned}$$

First, we sketch the line $y = -\frac{3}{2}x + 9$ and then we shade the region below this line because our constraint is $y \leq -\frac{3}{2}x + 9$ (see Figure 4). If the constraint was $y \geq -\frac{3}{2}x + 9$, we would shade the region above the line.

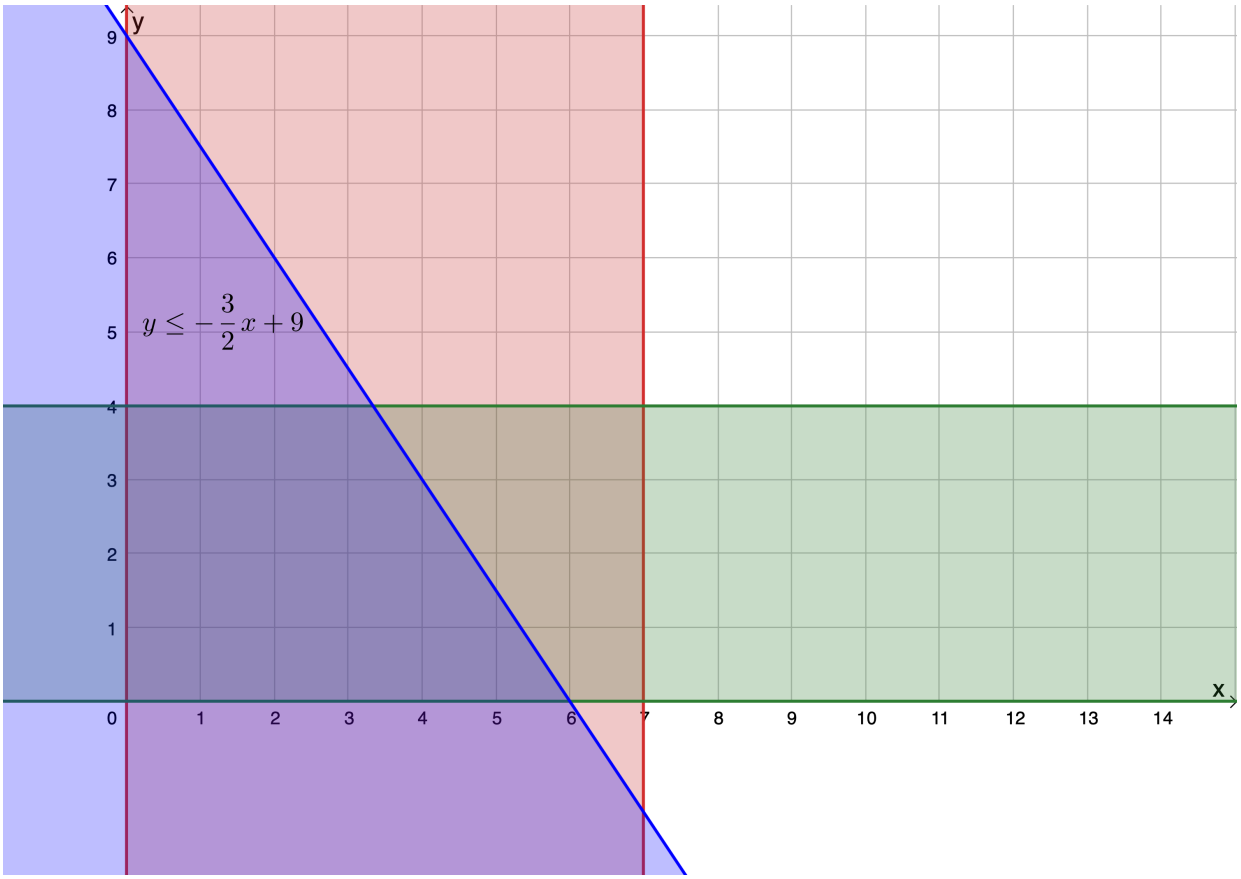


Figure 4: The region represented by $x \leq 7, x \in \mathbb{N}$, $y \leq 7, y \in \mathbb{N}$ *and* $3x + 2y \leq 18, x \in \mathbb{N}, y \in \mathbb{N}$

The region where all three separate regions overlap is called the **feasible region**. It is in this region that the solution to our problem lies.

We could explicitly show that we are only interested in the natural numbers by plotting these points within the feasible region (see Figure 5).

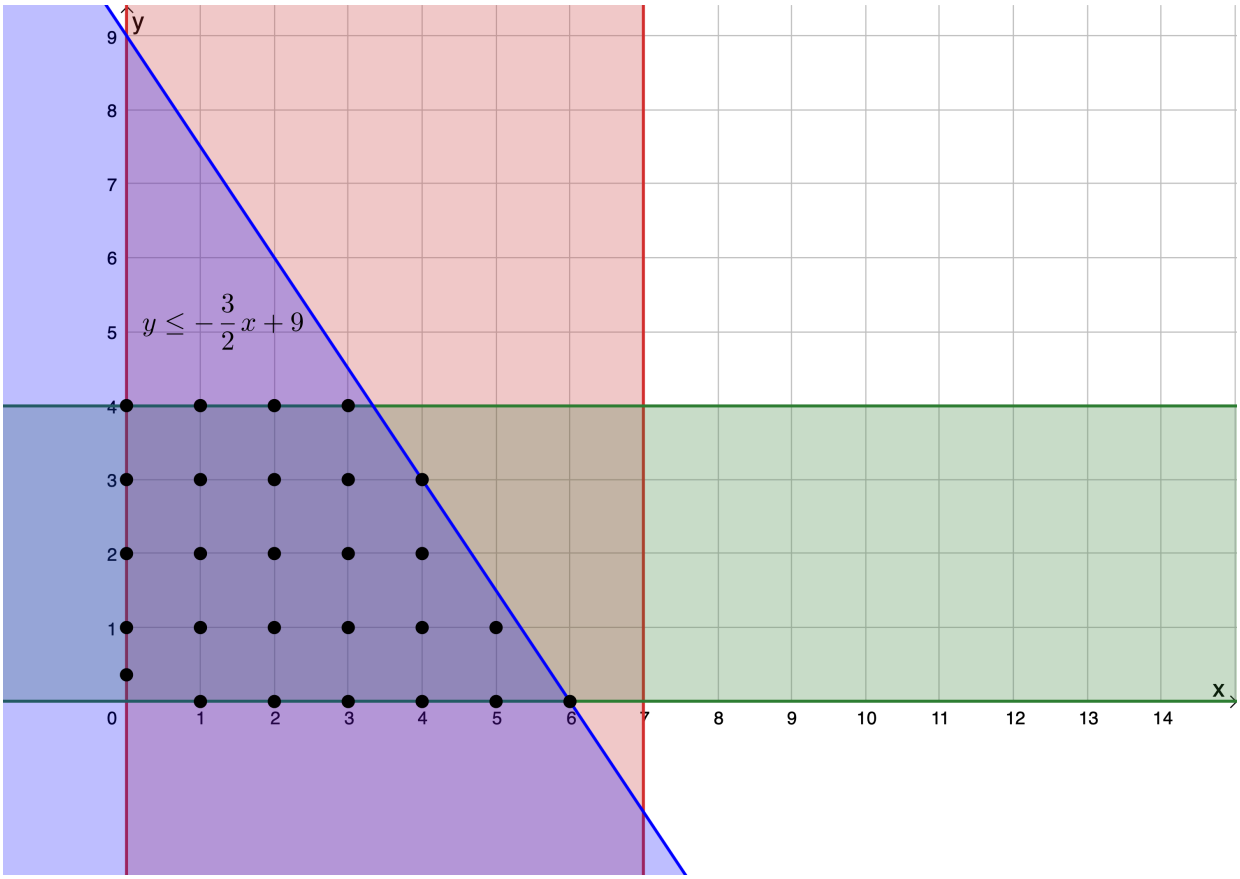


Figure 5: The explicit points within the feasible region

Step 5: Graph the objective function
Our objective function is $2\ 400x + 1\ 200y = C$.

$$2\ 400x + 1\ 200y = C$$
$$\therefore 1\ 200y = -2\ 400x + C$$
$$\therefore y = \frac{-2\ 400}{1\ 200}x + \frac{C}{1\ 200}$$
$$\therefore y = -2x + \frac{C}{1\ 200}$$

Now we cannot really graph this linear function because we don't yet know what the value of C is. But remember, C is what we are trying to minimise, for a maximum number of bicycles produced. In other words, we are trying to find the lowest value of C that will make our objective function pass through one of the possible solution points with the largest coordinates within the feasible region.

So, we start with any line with a gradient of -2 and move this line up and down to find the smallest y-intercept value that still lets the objective function pass through one of the points in the feasible region with the largest possible x and y coordinates. The objective function must pass through $(3, 4)$ for the y-intercept to be a minimum (10), and x and y to be maximum. The solution is shown below in Figure 6.

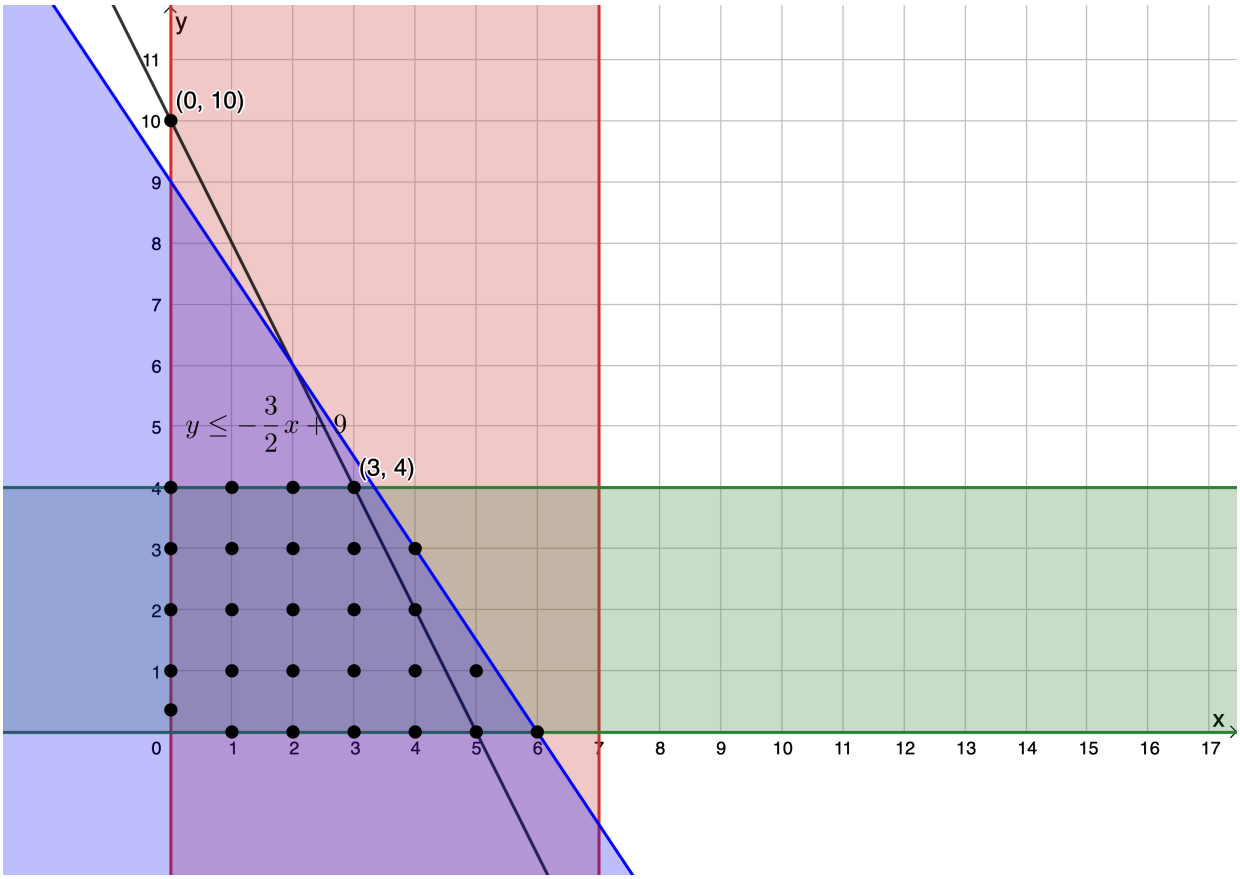


Figure 6: Solution to minimise the objective function for maximum production

Step 6: Answer the questions
We are now in a position to answer the questions.

1. Qhubeka must make three Buffalo bicycles and four Cheetah bicycles per day to minimise costs.
2. The minimum solution to the objective function results in a y-intercept of $(0, 10)$. The objective function is $y = -2x + \frac{C}{1\ 200}$. Therefore:

   $\frac{C}{1\ 200} = 10$
   $\therefore C = 12\ 000$

   The minimum cost is $\text{R}12\ 000$.

Note

If you have an internet connection, visit a simulation of this linear programming solution.



There is a slider to change the value of C in order to find the minimum or maximum solution to the objective function.

Let's look at another example.

Example 1.2

A TVET college class wants to raise funds to help buy new workshop tools. To raise money, they decide to host a cake sale and sell two different types of cakes – chocolate cake and carrot cake.

The class estimates that they can sell at most 60 chocolate cakes and 50 carrot cakes.

Each chocolate cake takes $50\ \text{min}$ to bake and each carrot cake takes $80\ \text{min}$ to bake. In total, the class calculates that they have $4\ 400\,\text{min}$ to bake all the cakes.

Each chocolate cake costs $\text{R}30$ to make and each carrot cake costs $\text{R}40$ to make and they have a total of $\text{R}2\ 400$ to spend on ingredients.

They can sell each chocolate cake for $\text{R}30$ profit and each carrot cake for $\text{R}60$ profit.

1. Assuming they can sell all the cakes they bake, how many of each type of cake should they bake to maximise their profit?
2. What will this maximum profit be?

Solution

Step 1: Assign variables
Let the number of chocolate cakes be x.
Let the number of carrot cakes be y.

We cannot have fractions of cakes or negative numbers of cakes. Therefore $\{x|x \in \mathbb{N}\}$ and $\{y|y \in \mathbb{N}\}$.

Step 2: Express the constraints

- $x \leq 60$ (at most 60 chocolate cakes can be sold)
- $y \leq 50$ (at most 50 carrot cakes can be sold)
- $50x + 80y \leq 4\ 400$ (each chocolate cake takes $50\ \text{min}$ and each carrot cake takes $80\ \text{min}$ to bake and they have a total of $4\ 400\ \text{min}$ to bake).
- $30x + 40y = 2\ 400$ (each chocolate cake costs $\text{R}30$ and each carrot cake costs $\text{R}40$ and they have a total of $\text{R}2\ 400$ to spend)

Step 3: Determine the objective function
$30x + 60y = P$, where P is profit.

Step 4: Graph the constraints

$$50x + 80y \leq 4\ 400$$
$$\therefore 80y \leq -50x + 4\ 400$$
$$\therefore y \leq -\frac{50}{80}x + \frac{4\ 400}{80}$$
$$\therefore y \leq -\frac{5}{8}x + 55$$

$$30x + 40y \leq 2\ 400$$
$$\therefore 40y \leq -30x + 2\ 400$$
$$\therefore y \leq -\frac{30}{40}x + \frac{2\ 400}{40}$$
$$\therefore y \leq -\frac{3}{4}x + 60$$

The graphed constraints are shown in Figure 7.

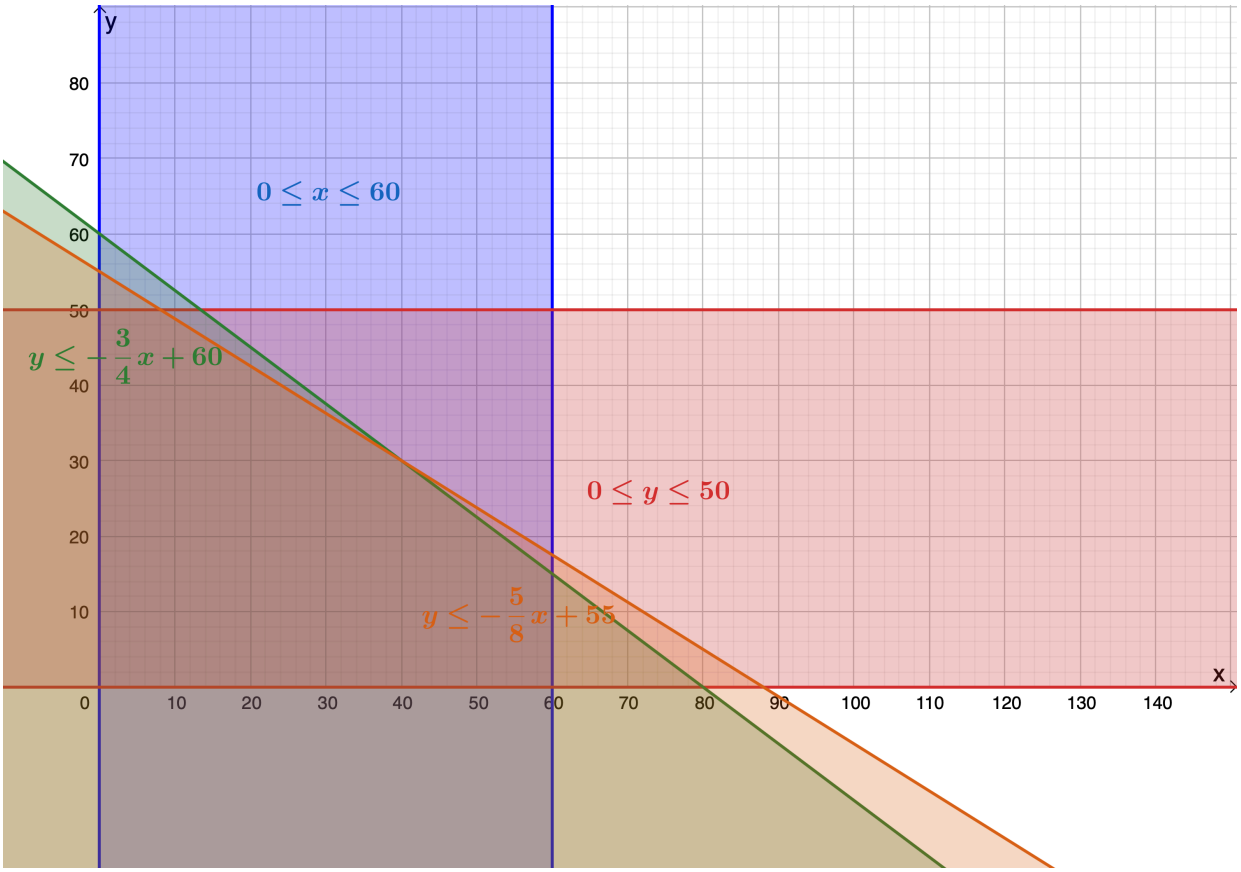


Figure 7: The graphed constraints of $x \leq 60, x \in \mathbb{N}$, $y \leq 55, y \in \mathbb{N}$, $y \leq -\frac{5}{8}x + 55$ *and* $y \leq -\frac{3}{4}x + 60$

Step 5: Graph the objective function

$$30x + 60y = P$$
$$\therefore 60y = -30x + p$$
$$\therefore y = -\frac{30}{60}x + \frac{P}{60}$$
$$\therefore y = -\frac{1}{2}x + \frac{P}{60}$$

An instance of the objective function is graphed in Figure 8.

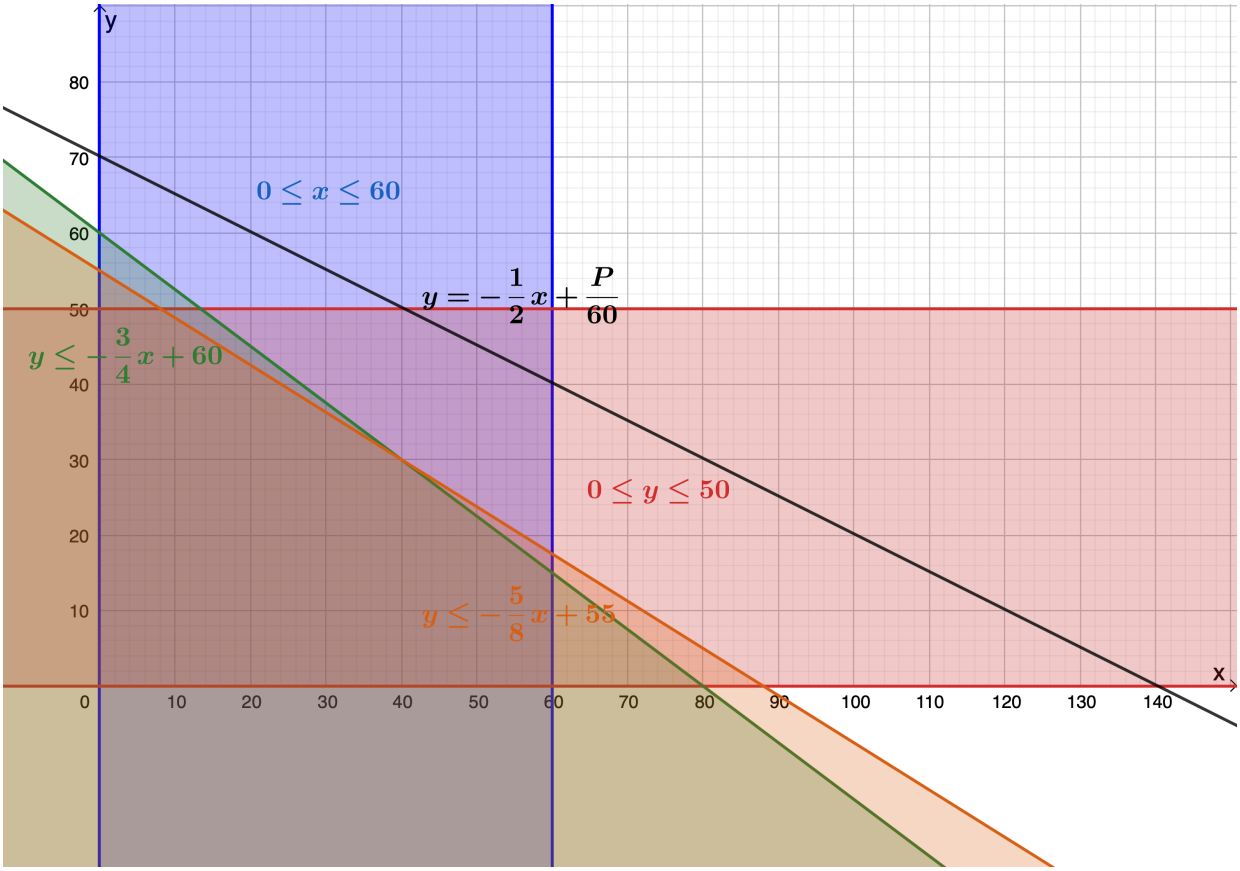


Figure 8: The graph of the objective function

In this case, we want to maximise the objective function, in other words, we want to find the **maximum** y-intercept that the graph can have while still passing through one of the possible solution points in the feasible region. From the graph of the feasible region, we can see that this point is $(40, 30)$. When the objective function is made to pass through this point its y-intercept is 50 (see Figure 9).

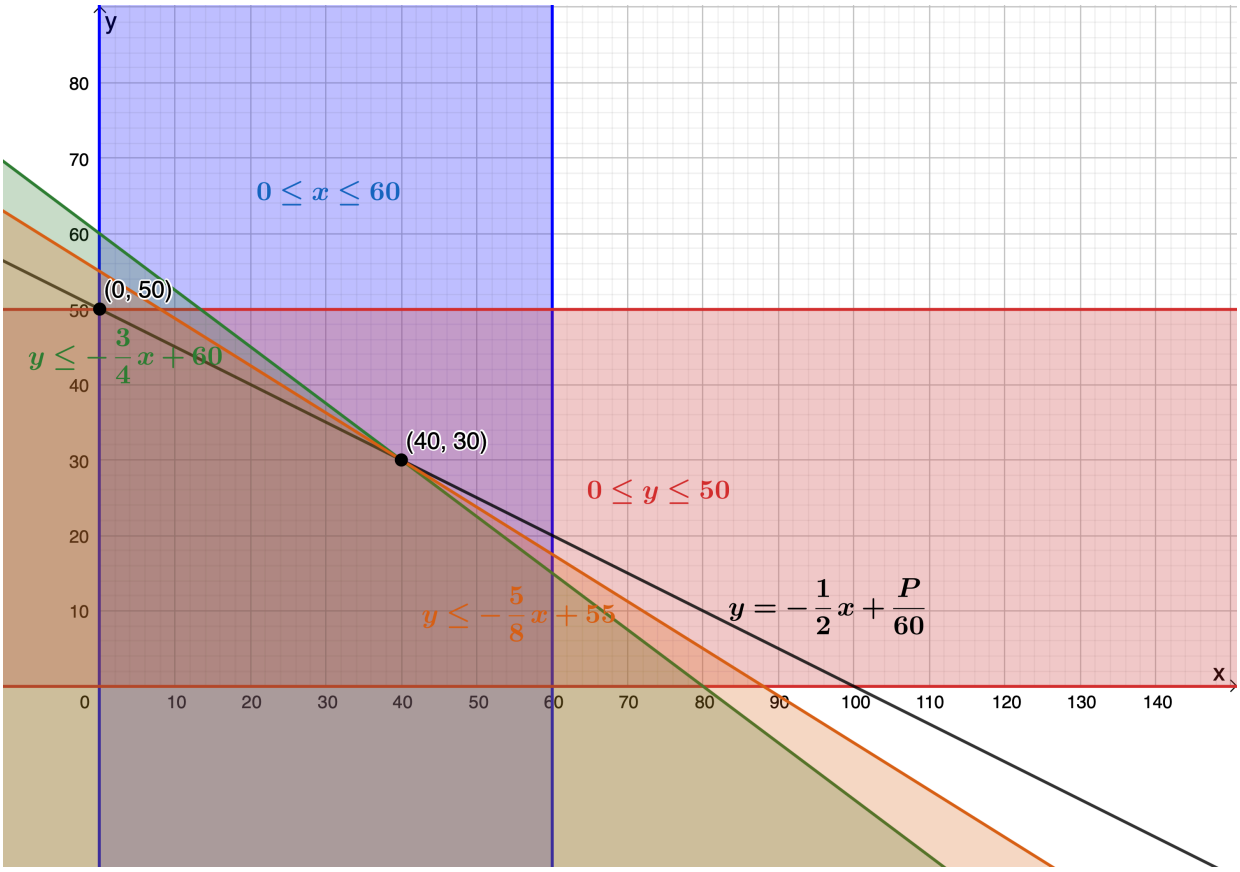


Figure 9: The solution to maximise the objective function

Step 6: Answer the questions

1. The class should bake 40 chocolate cakes and 30 carrot cakes to maximise their profit.
2. $y = -\frac{1}{2}x + \frac{P}{60}$

   $\frac{P}{60} = 50$

   $\therefore P = 3\ 000$

   The maximum profit they can make is $\mathrm{R}3\ 000$.

Exercise 1.1

Question taken from Question 2.3 of NC(V) Mathematics Paper 1 March 2014

A clothing company manufactures yellow T-shirts (x) and black trousers (y) for a day-care school. The following system of inequalities are obtained:

$x \geq 200$

$x + y \leq 600$

$x + 2y \leq 900$

$50x + 100y \leq 45\ 000$

1. Sketch the graphs of the given constraints.
2. Indicate the feasible region.
3. Determine the number of t-shirts and pairs of trousers that will yield a maximum profit if $P = 30x + 40y$.
4. Determine the number of t-shirts and pairs of trousers that will yield a minimum profit if $P = 30x + 40y$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to express given constraints using variables.
- How to plot these constraints on the Cartesian plane.
- How to determine the feasible region.
- How to determine the objective function.
- How to maximise or minimise the objective function.

Unit 1: Assessment

Suggested time to complete: 15 minutes

Question 1 is based on Question 2.5 from NC(V) Mathematics Paper 1 October 2012

1. An entrepreneur sells apples and bananas at the fruit stand. Suppose x apples and y bananas are sold every hour, and that the following constraints are applicable:
   $10 \leq x \leq 40$
   $x + y \leq 60$
   $15 \leq y \leq 45$
   $2y + x \geq 60$
   a. Sketch the graph with the given constraints.
   b. Determine the feasible region.
   c. Determine how many apples and bananas should be sold every hour so that the maximum profit can be made if $P = 3x + 5y$.

Question 2 is taken from Question 3.3 from NC(V) Mathematics Paper 1 October 2014

2. Lesedi is a small company that makes two types of cards, type A (x) and type B (y). With the available labour and materials, the following are the constraints inequalities:
   $x + y \leq 200$
   $x \geq 40$
   $y \geq 10$
   $x \leq 150$
   $y \leq 120$
   a. Sketch the graph of the given constraints.

b. Determine the feasible region.

c. Find the values of x and y which will maximise the objective equation $P = 5x + 10y$.

d. Find the values of x and y which will minimise the objective equation $P = 5x + 10y$ and state the value of P when this function is minimised.

The full solutions are at the end of the unit.

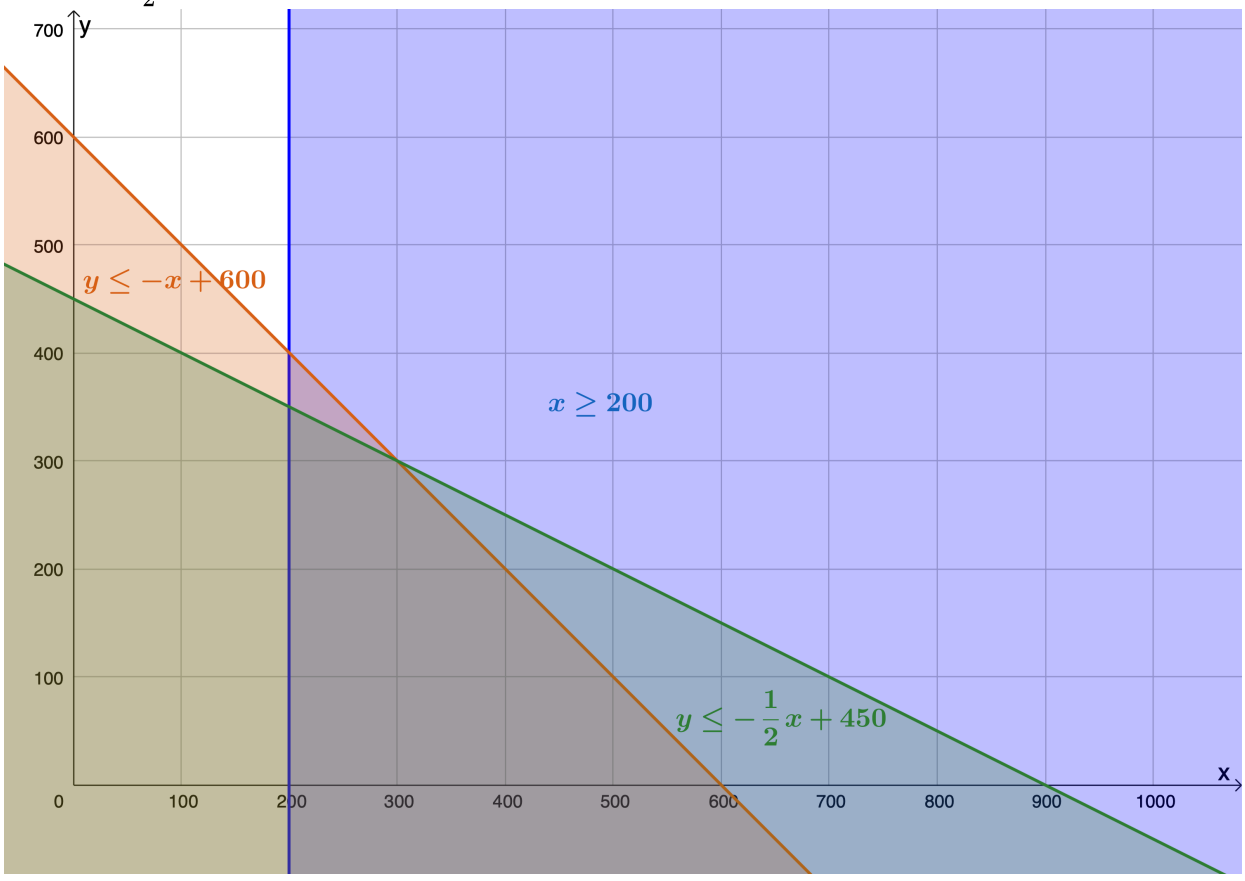
Unit 1: Solutions

Exercise 1.1

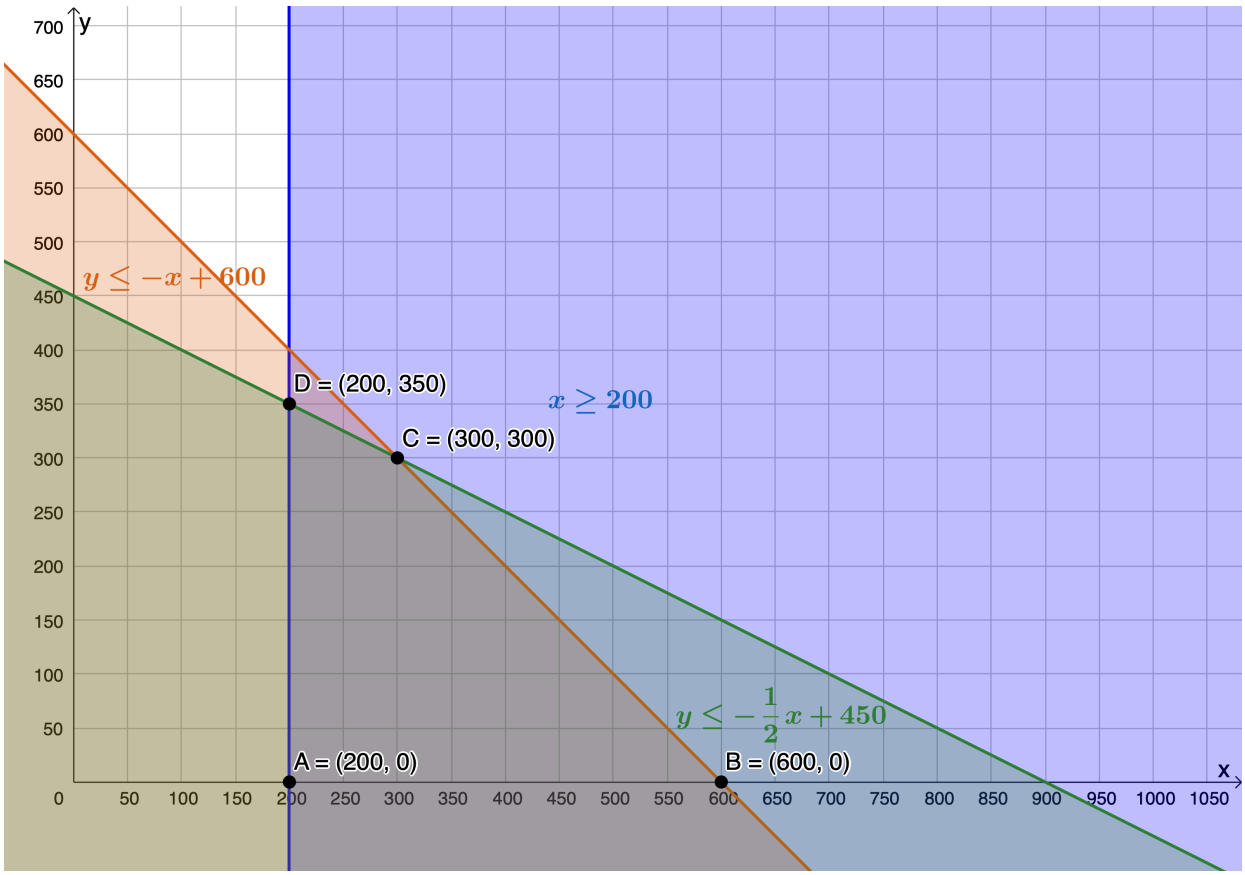
1. $x \in \mathbb{N}$ and $y \in \mathbb{N}$
   Constraints are:
   $x \geq 200$

   $x + y \leq 600$
   $\therefore y \leq -x + 600$

   $x + 2y \leq 900$
   $\therefore 2y \leq -x + 900$
   $\therefore y \leq -\frac{1}{2}x + 450$



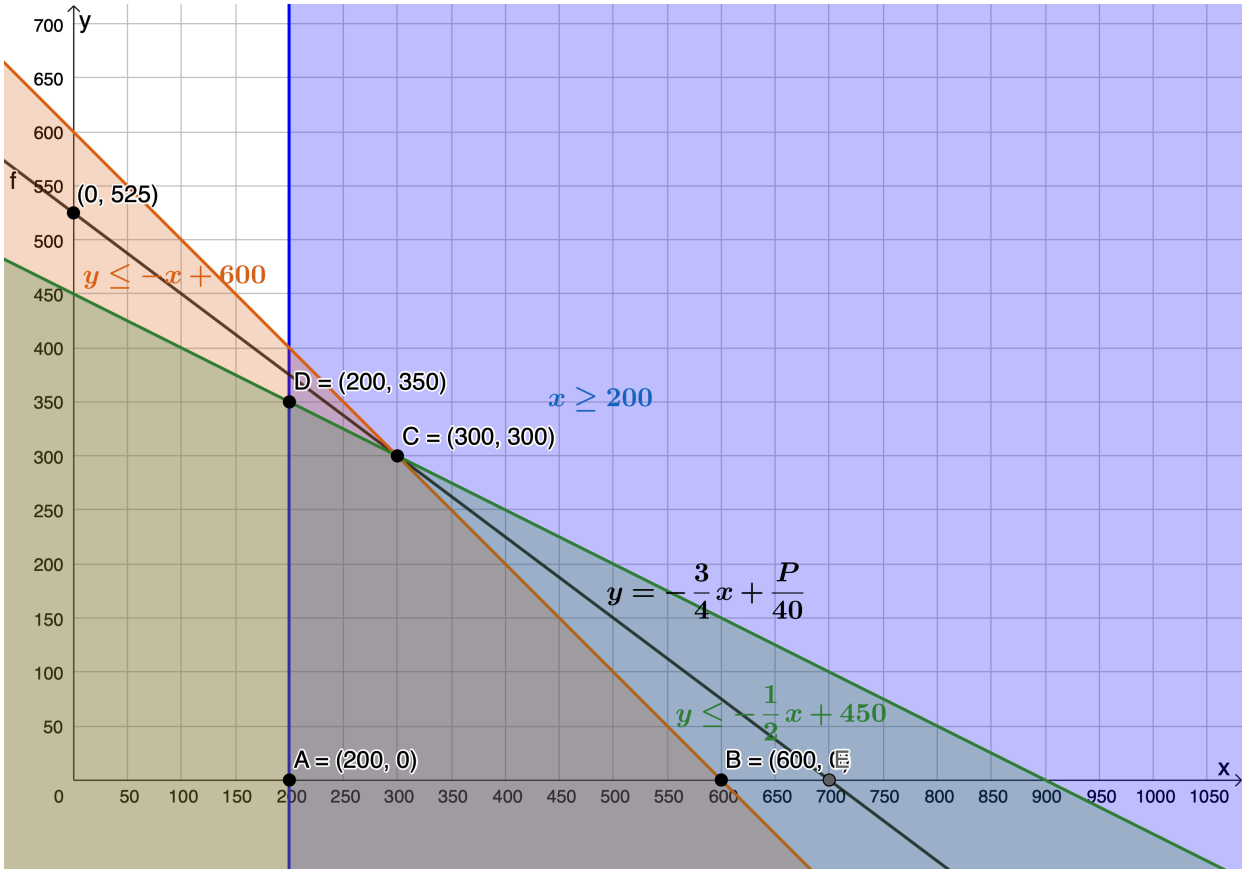
2. The feasible region is the region bound by points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D.



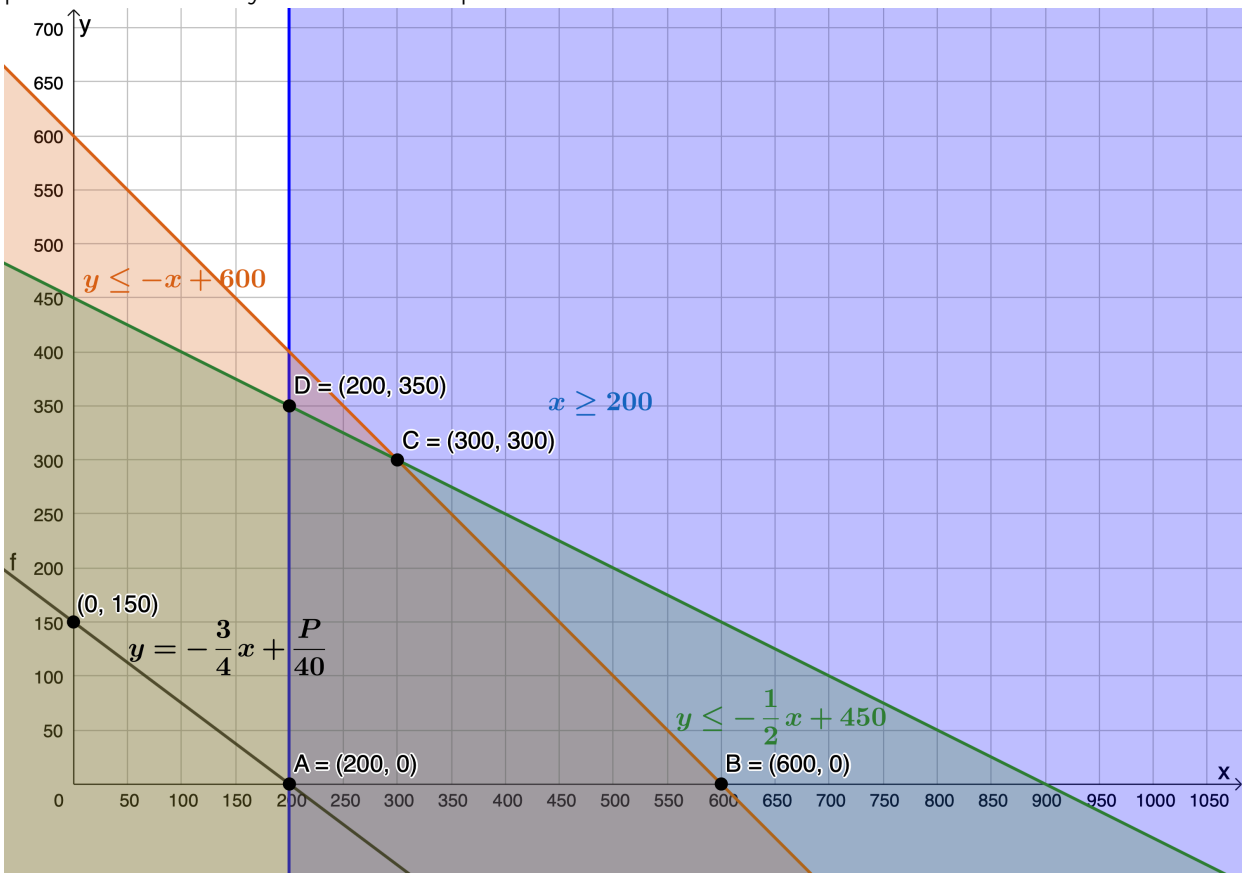
3.

$$P = 30x + 40y$$
$$\therefore 40y = -30x + P$$
$$\therefore y = -\frac{3}{4}x + \frac{P}{40}$$

The objective function is maximised when passing through $C(300, 300)$. Therefore, 300 t-shirts and 300 pairs of trousers will yield a maximum profit.



4. The objective function is minimised when passing through $\mathrm{A}(200, 0)$. Therefore, 200 t-shirts and zero pairs of trousers will yield a minimum profit.

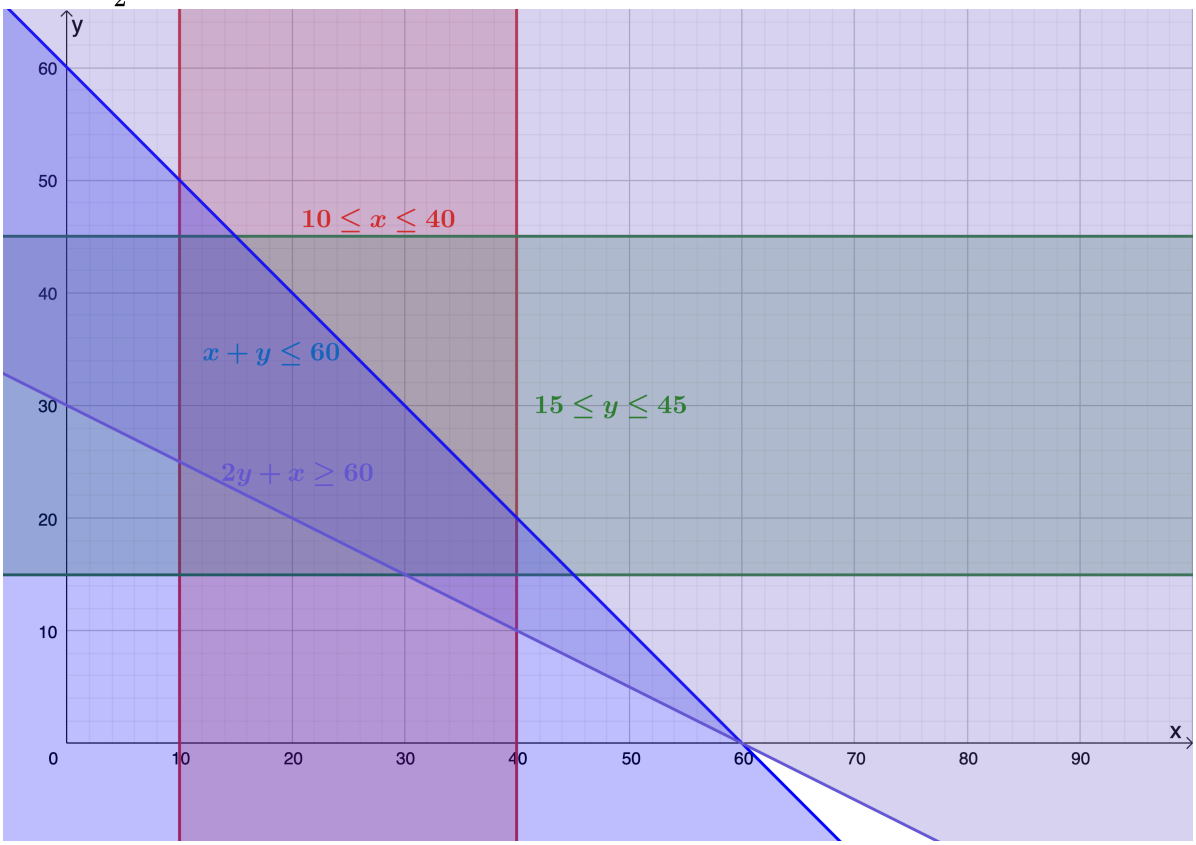


Back to Exercise 1.1

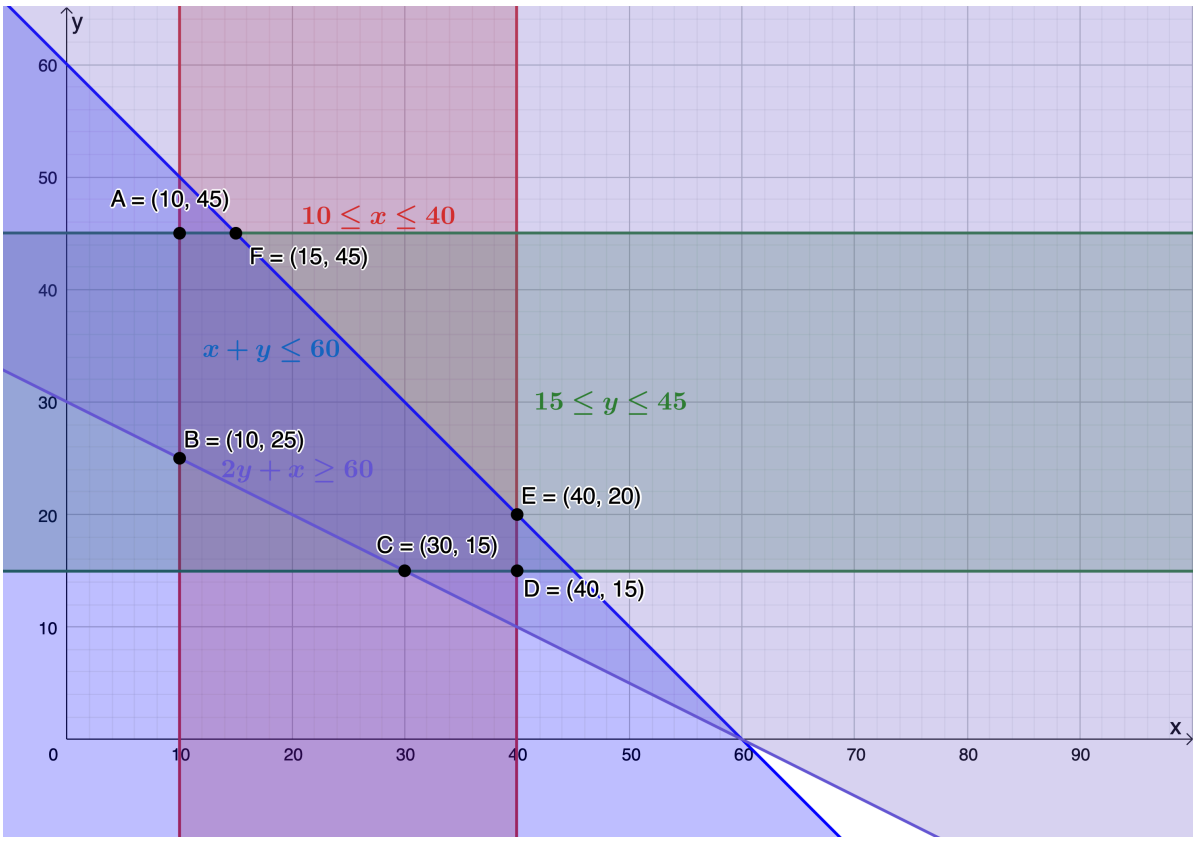
Unit 1: Assessment

1.

a. $x \in \mathbb{N}$ and $y \in \mathbb{N}$
Constraints are:
$10 \leq x \leq 40$
$x + y \leq 60$
$\therefore y \leq -x + 60$
$15 \leq y \leq 45$
$2y + x \geq 60$
$\therefore y \geq -\frac{1}{2}x + 30$ Remember that because $y \geq$ you need to take the region above the line



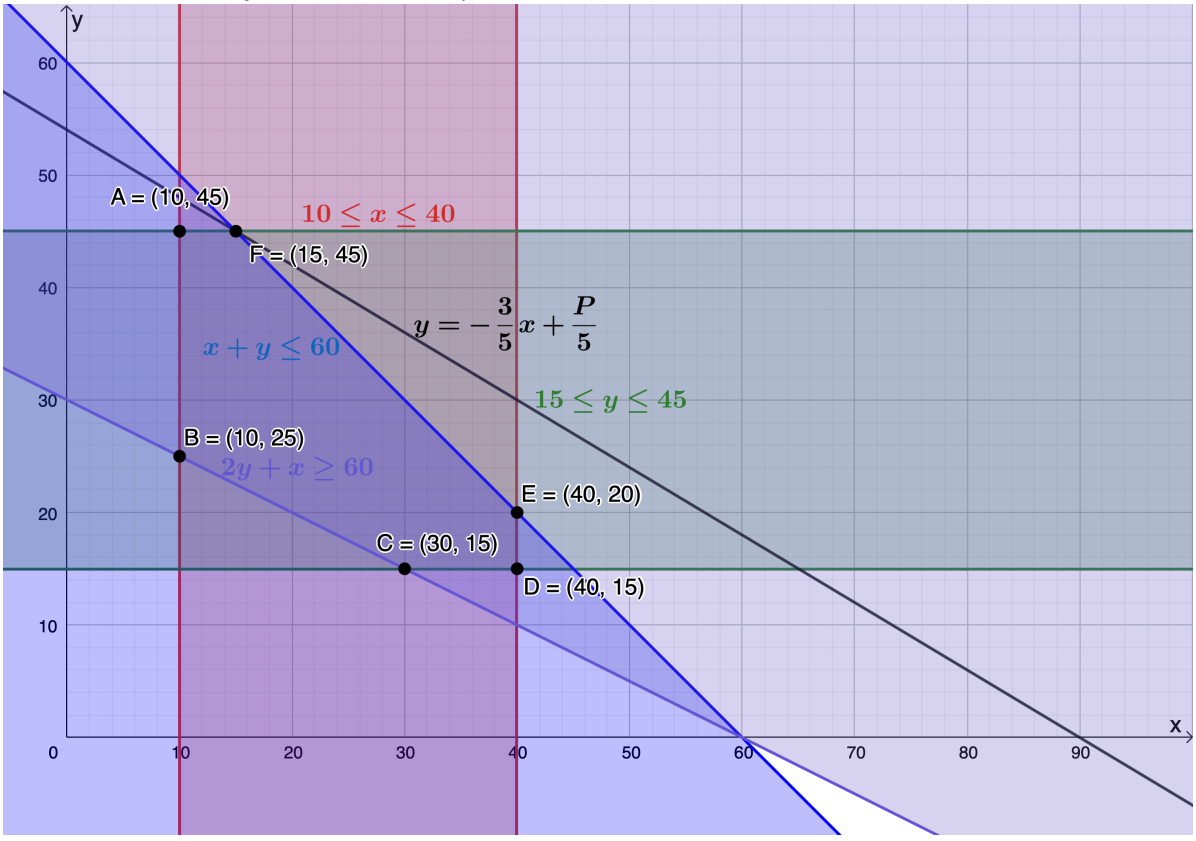
b. The feasible region is the region bound by points $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ and F.



c.

$$P = 3x + 5y$$
$$\therefore 5y = -3x + P$$
$$\therefore y = -\frac{3}{5}x + \frac{P}{5}$$

The objective function is maximised when passing through $\mathrm{F}(15, 45)$. Therefore, selling 15 apples and 45 bananas will yield a maximum profit.



2.

a. $x \in \mathbb{N}$ and $y \in \mathbb{N}$

Constraints are:

$x + y \leq 200$

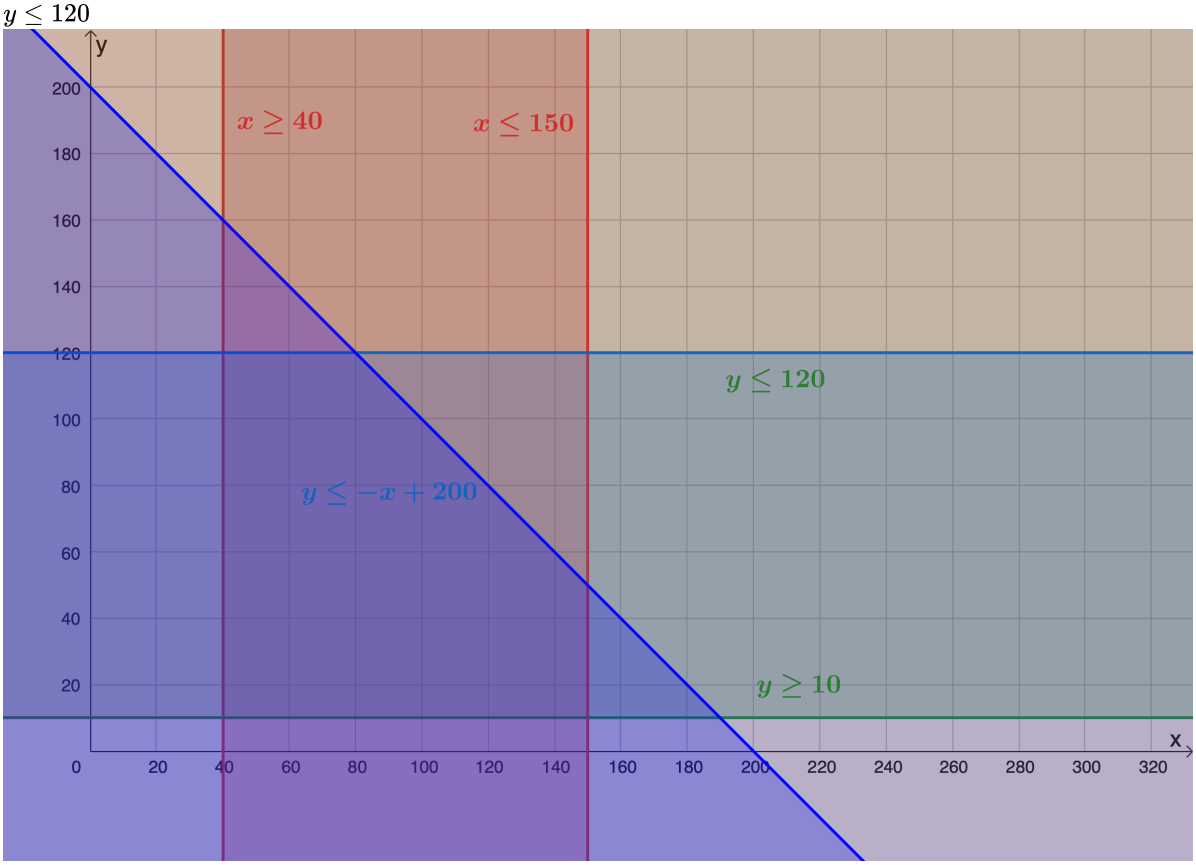
$\therefore y \leq -x + 200$

$x \geq 40$

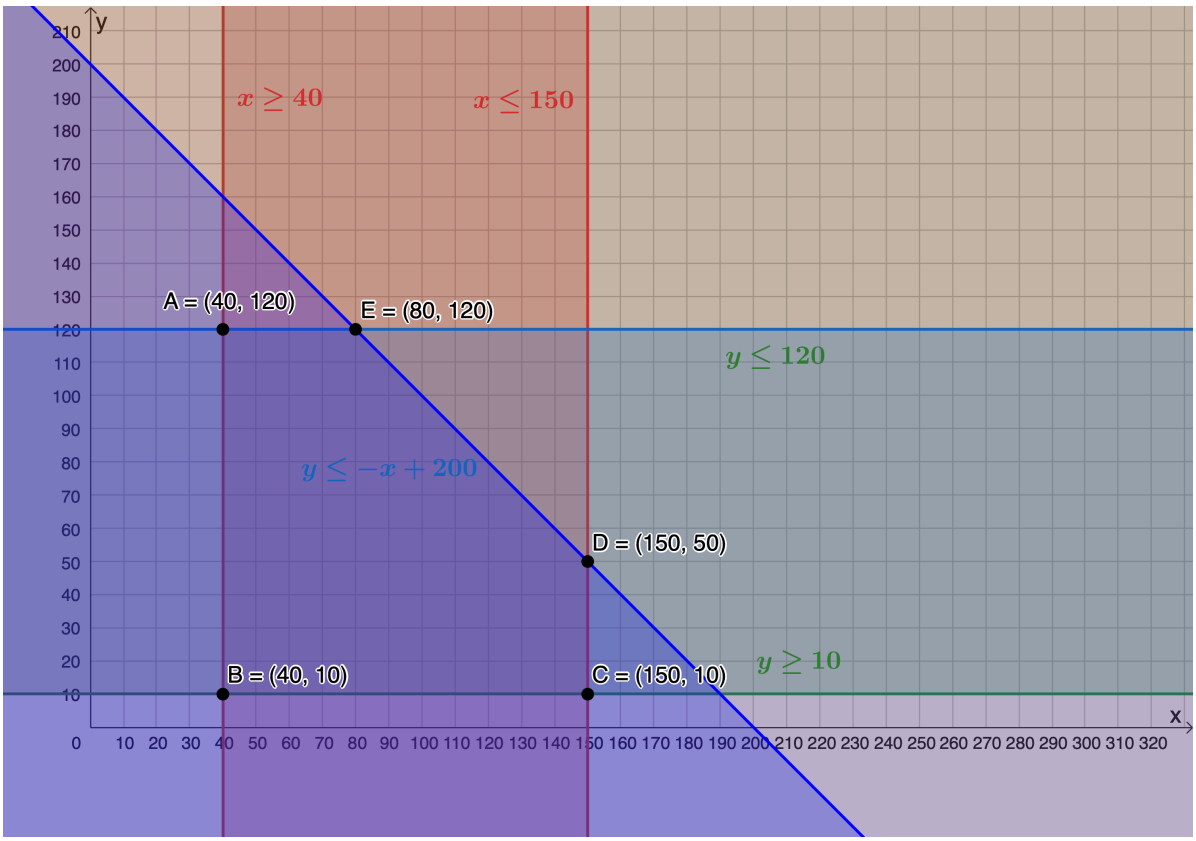
$y \geq 10$

$x \leq 150$

$y \leq 120$



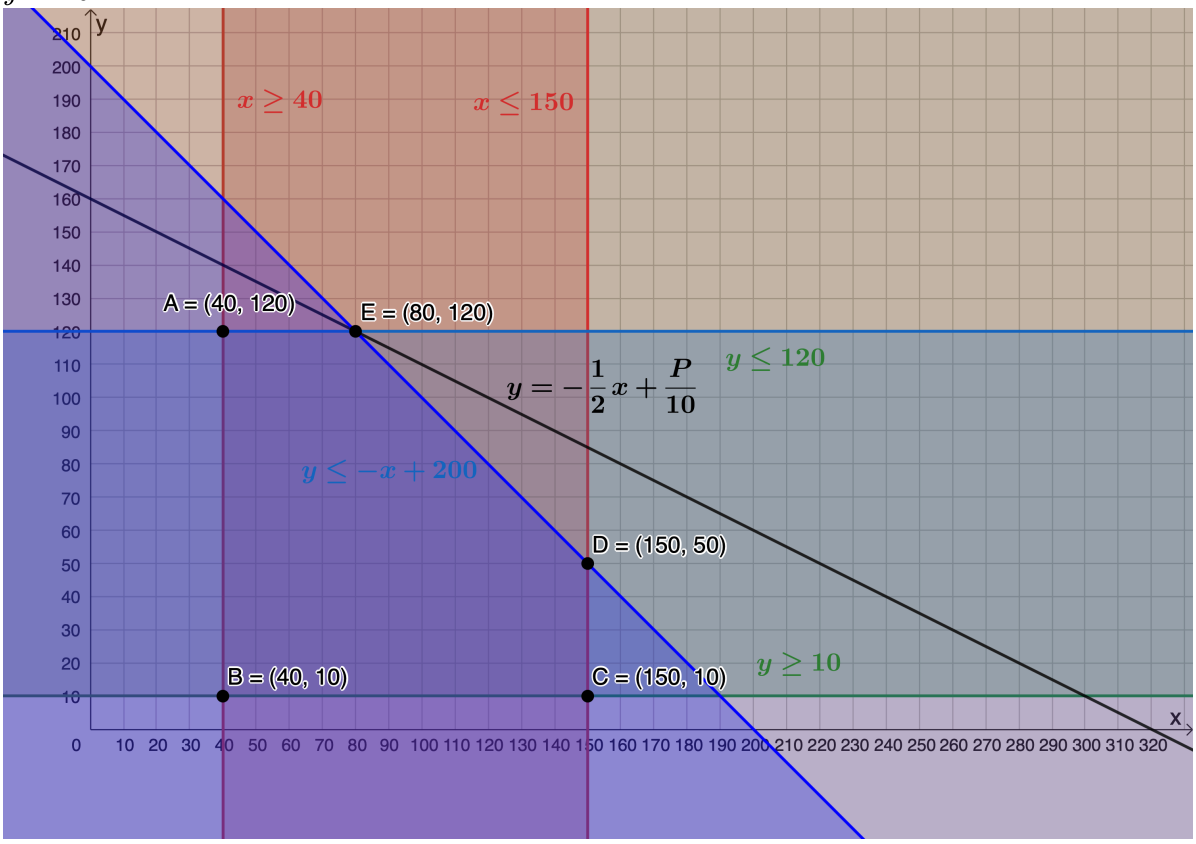
b. The feasible region is the region bound by points $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and E.



c.

$$P = 5x + 10y$$
$$\therefore 10y = -5x + P$$
$$\therefore y = -\frac{1}{2}x + \frac{P}{10}$$

The objective function is maximised when passing through $\mathrm{E}(80, 120)$. Therefore, $x = 80$ and $y = 120$.

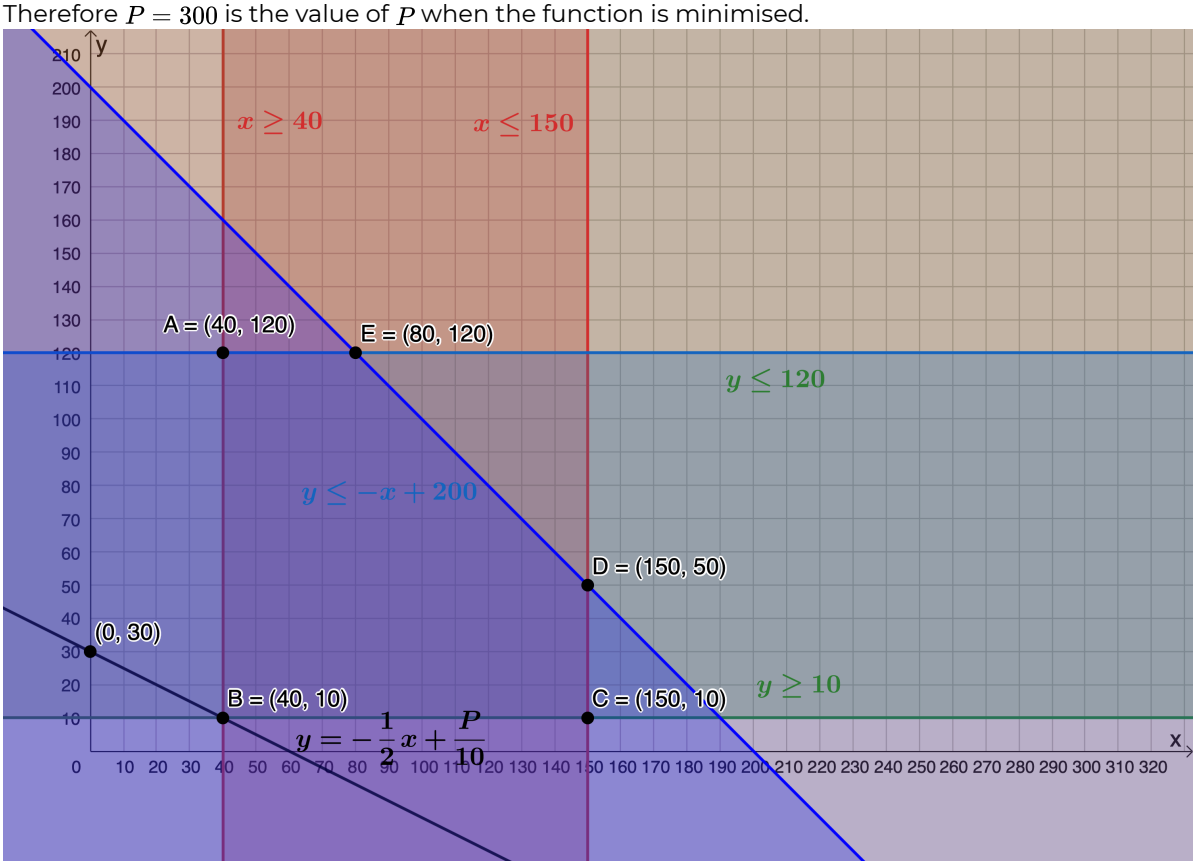


d. The objective function is minimised when passing through $\mathrm{B}(40, 10)$. Therefore, $x = 40$ and $y = 10$. When minimised, the objective function has a y-intercept of $(0, 30)$.

$$\frac{P}{10} = 30$$

$$\therefore P = 300$$

Therefore $P = 300$ is the value of P when the function is minimised.



Back to Unit 1: Assessment

Media Attributions

- figure1 © Qhubeka is licensed under a All Rights Reserved license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure6 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure7 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure8 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure9 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise1.1A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise1.1A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise1.1A3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise1.1A4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA1c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2b © Geogebra is licensed under a CC BY (Attribution) license
- assessmentA2c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA2d © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

SUBJECT OUTCOME VII

FUNCTIONS AND ALGEBRA: INVESTIGATE AND USE INSTANTANEOUS RATE OF CHANGE

Subject outcome

Subject outcome 2.5: Investigate and use instantaneous rate of change

Learning outcomes

- Determine limits of functions intuitively.
- Distinguish between the value of a function at a particular point and the limit of that function at that point.
- Establish the derivatives of the following functions from first principles:
  $f(x) = b, \quad f(x) = x, \quad f(x) = ax + b, \quad f(x) = x^2, \quad f(x) = ax^2 + b$
- Find the derivative of functions in the form:
  - $f(x) = ax^n$ or $y = ax^n, f'(x) = n.ax^{(n-1)}$ or $\frac{dy}{dx} = n.ax^{(n-1)}$ Examples to include:
    $4x^2; \ \frac{3}{x^{-3}}; \frac{2}{\sqrt[3]{x^2}}; \frac{5}{3x^2}$ (all examples within this range).
- Use the constant, sum and/or difference rule by first simplifying the expression.
  - If $y = f(x) = a$ and $a$is a constant function then: $\frac{dy}{dx} = f'(x) = 0$
  - If $y = \mathrm{k}f(x)$ then $\frac{d}{dx}[\mathrm{k}f(x)] = \mathrm{k}\frac{d}{dx}[f(x)]$
  - If $y = f(x) \pm g(x)$ then $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$.
- Solve maxima and minima problems about real life situations from given equations.

Unit 1 outcomes

By the end of this unit you will be able to:

- Determine limits of functions intuitively.
- Distinguish between the value of a function at a particular point and the limit of the function at that point.

Unit 2 outcomes

By the end of this unit you will be able to:

- Develop an intuitive understanding of the gradient at a point.
- Differentiate using first principles.

Unit 3 outcomes

By the end of this unit you will be able to:

- Apply the constant, sum and difference rules for differentiation.
- Use notation appropriately.

Unit 4 outcomes

By the end of this unit you will be able to:

- Use differentiation to find optimal solutions.

Unit 1: Limits of a function

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Determine limits of functions intuitively.
- Distinguish between the value of a function at a particular point and the limit of the function at that point.

What you should know

Before you start this unit, make sure you can:

- Solve algebraic equations and inequalities. If you need to revise this topic then go over level 3 subject outcome 2.3.
- Sketch and interpret functions. Revise this topic in level 3 subject outcome 2.1.

Introduction

The concept of a limit is essential to all of calculus. Two key ideas led to the development of calculus: 1) the tangent problem, or how to determine the slope of a tangent to a curve at a point and 2) the area problem, or how to determine the area under a curve.

Calculus consists of two related topics: differential calculus and integral calculus. The tangent problem resulted in the formation of differential calculus and the area problem resulted in integral calculus. You will learn about integral calculus and the tangent to a curve in Maths level 4. In level 3 we focus on basic differential calculus and using it to solve optimisation problems.

Limits

Early mathematicians used a limiting process to get better and better approximations of areas of circles. The formal definition of a limit—as we know and apply it today—did not appear until the late 19th century. We begin our quest to understand limits by using an intuitive approach. Once we have a basic understanding of limits, we will look at the formal definition of a limit.

Activity 1.1: Finding limits

Time required: 20 minutes

What you need:

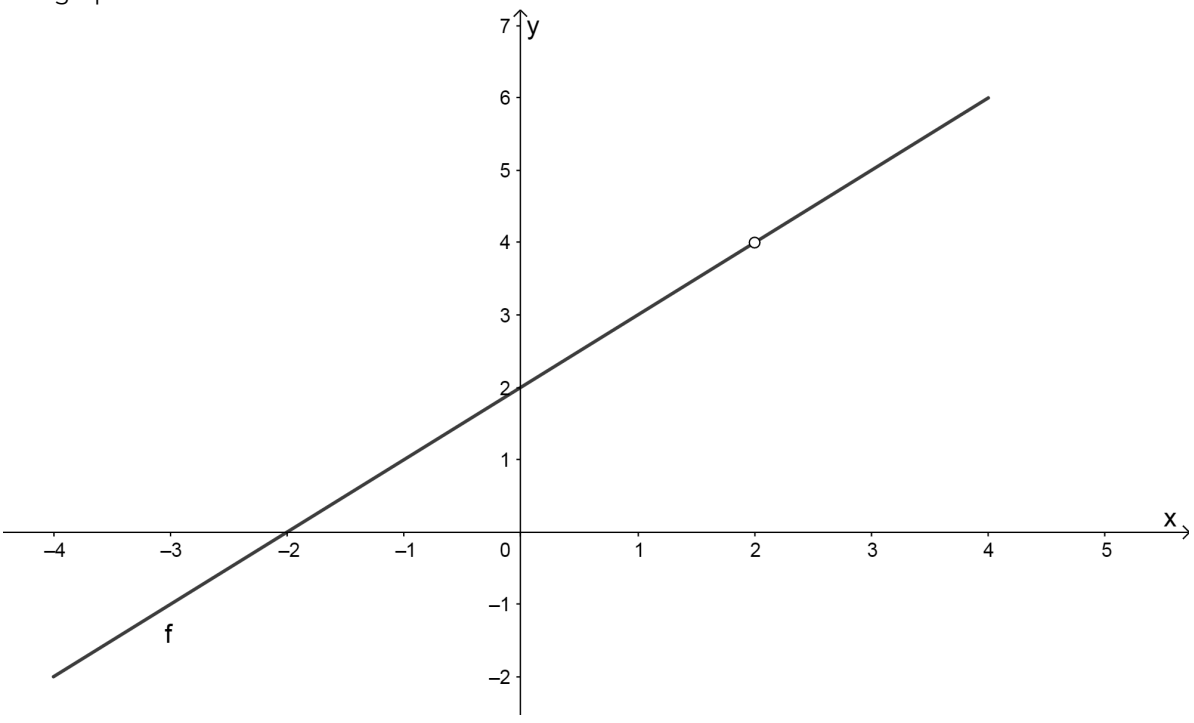
- pen and paper

What to do:

1. Consider the function $y = \dfrac{x^2 - 4}{x - 2}$. Write down the domain of the function.
2. Explain what happens when $x = 2$.
3. Copy and complete the table below to see what happens to the function values as x gets closer to 2 from the left (values smaller than 2) and the right (values greater than 2).

x	$y = \dfrac{x^2 - 4}{x - 2}$
1.9	
1.99	
1.999	
1.9999	
2.0001	
2.001	
2.01	
2.1	

4. What happens to the y-values as x gets closer to 2? Will the function value ever be equal to 4?
5. The graph of the function is shown below.



Have a look at the behaviour of the graph at and around $x = 2$. Describe what happens to the

graph as x approaches 2.

What did you find?

1. We can factorise $y = \dfrac{x^2 - 4}{x - 2}$ as $y = \dfrac{(x - 2)(x + 2)}{(x - 2)}$ before cancelling the common factor of $x - 2$. We have to note the restriction on the denominator$x \neq 2$. The domain is $\{x \in \mathbb{R}, x \neq 2\}$.
2. If $x = 2$, then the denominator becomes 0 and the function is not defined. This means that the domain of the function does not include $x = 2$.
3.

x	$y = \dfrac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
2.0001	4.0001
2.001	4.001
2.01	4.01
2.1	4.1

4. From the table we can see that as x gets closer to 2, the y-values get closer and closer to 4. The function will never have a value of 4 as we can find no x-value that will make the y-value 4.
5. We can see that the graph has a hole at $x = 2$, which means that the function is discontinuous. We know that the function is undefined at $x = 2$ but if we make this statement and no other, we will give a very incomplete picture of how it behaves as x gets closer to 2. As the values of x approach 2 from the left and right, the values of y approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4.

Notation

When a function approaches the same y-value from the left and from the right, we say that the limit exists. To express the limit of a function we use the following notation: $\lim\limits_{x \to a} f(x) = L$.

If all values of a function $f(x)$ approach a real number L as the values of $x (x \neq a)$ approach the number a, then we say that the limit of $f(x)$ exists as x approaches a and is equal to L. Symbolically, we express this idea as $\lim\limits_{x \to a} f(x) = L$.

For the function in Activity 1.1, $y = \dfrac{x^2 - 4}{x - 2}$, we write $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$. This is read as the limit of $\dfrac{x^2 - 4}{x - 2}$ as x tends to 2 is equal to 4. Again, this does not mean that the function value is 4 just that the function approaches 4 as x tends to 2.

Example 1.1

Determine $\lim\limits_{t\to 6} \dfrac{t^2-36}{t-6}$

Solution

Simplify the expression:

$\lim\limits_{t\to 6} \dfrac{t^2-36}{t-6} = \lim\limits_{t\to 6} \dfrac{(t-6)(t+6)}{t-6}$ Factorise the numerator.
As $t \to 6$, the denominator $t-6 \to 0$, therefore the expression is not defined for $t=6$ since division by zero is undefined.

Cancel all common terms:
$\lim\limits_{t\to 6} \dfrac{(t-6)(t+6)}{t-6} = \lim\limits_{t\to 6} (t+6)$

Calculate the limit by substituting the value of t:
$\lim\limits_{t\to 6} (t+6) = 12$

This means the $\lim\limits_{t\to 6} \dfrac{t^2-36}{t-6} \to 12$ as $t \to 6$, it does not mean that the function value is 12 when $t=6$, the function is undefined when $t=6$.

Exercise 1.1

1. If $f(x) = \dfrac{x^2+4x-12}{x+6}$
   a. Determine $\lim\limits_{x\to -6} f(x)$
   b. Illustrate the answer graphically.
2. Calculate:
   a. $\lim\limits_{x\to 1} 10$
   b. $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

The full solutions are at the end of the unit.

One sided limits

It is possible to have a situation where a function tends to a different limit depending on whether x approaches from the left or the right.
For example, have a look at the following graph:

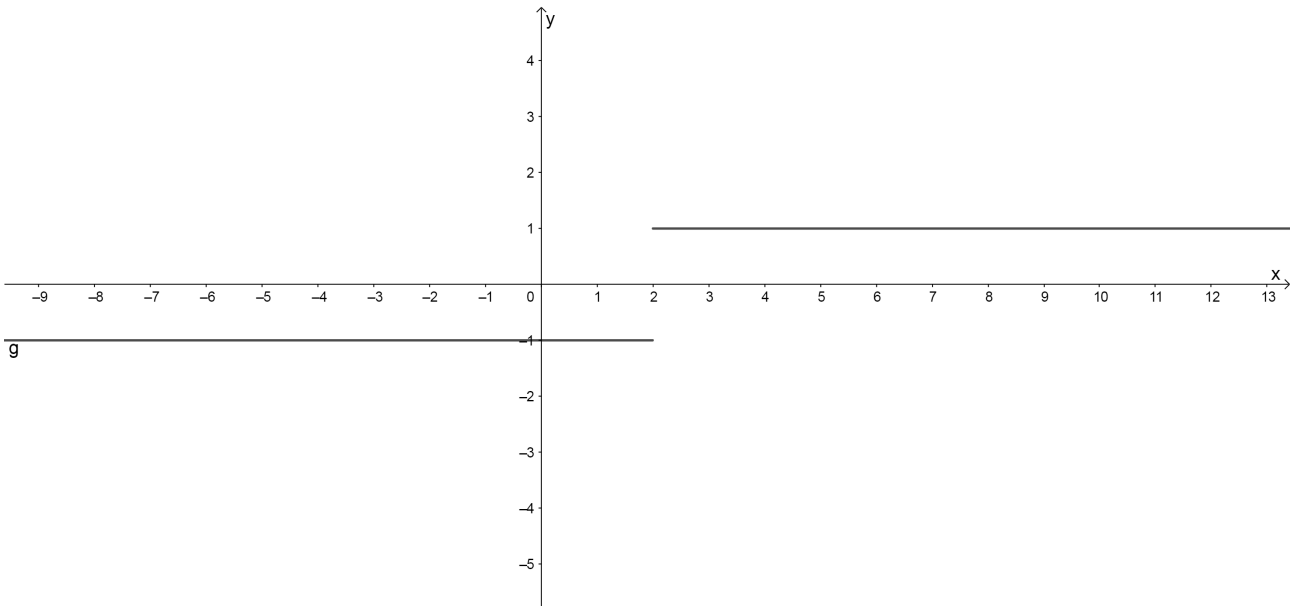


Figure 1: One sided limits

As x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x tends to 2 from the left is −1.

Using notation, we express this idea as $\lim_{x \to 2^-} g(x) = 1$. Where, $x \to 2^-$ means x approaches 2 from the left.

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Using limit notation we write this as $\lim_{x \to 2^+} g(x) = 1$. Where, $x \to 2^+$ means x approaches 2 from the right.

Therefore, since $g(x)$ does not approach the same value from both sides, we can conclude that the limit as x tends to 2 does not exist. We can conclude that it is only when a function approaches the same y-value from the left and from the right that the limit exists.

Note

If you would like to learn more about limits you can watch this short video, “Introduction to limits”.

Introduction to limits (Duration: 04.16)

Limits approaching infinity

Evaluating the limit of a function at a point, or from the left and right of a point, helps us to describe the behaviour of a function around a given value. We can also describe the behaviour of functions as $x \to \infty$.

Limits at infinity are used to describe the behaviour of functions as the independent variable increases or decreases without bound. But, what does infinity really mean?

What is the value of $\frac{1}{\infty}$? No one really knows as infinity is not an actual number but an idea. Saying $\frac{1}{\infty}$ is similar to saying $\frac{1}{\text{space}}$ or $\frac{1}{\text{hair colour}}$. In maths $\frac{1}{\infty}$ is undefined, there is no answer that makes sense.

We cannot reach infinity but we can approach it, using the concept of a limit.

Have a look at the function and table shown in Figure 2.

x	$\frac{1}{x}$
1	1
10	0.1
100	0.01
1 000	0.001
10 000	0.00001

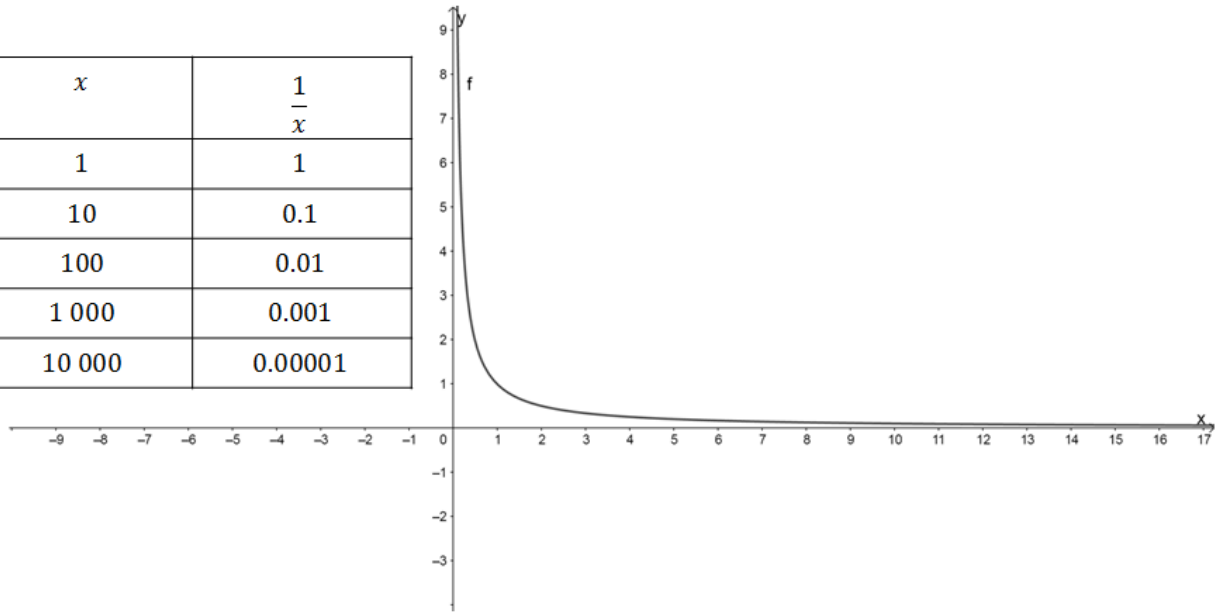


Figure 2: The limit at infinity of $\frac{1}{x}$

We can see that as the value of x gets bigger, the function $\frac{1}{x}$ gets closer and closer to 0. But, as we discussed previously, we cannot say that $\frac{1}{x}$ is ever equal to zero, no matter how big x gets. So we use the limit to show what is going on as x gets bigger or tends to ∞.

To show that $\frac{1}{x} \to 0$ as $x \to \infty$ we write this as $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

Example 1.2

Find the following:

1. $\lim\limits_{n\to\infty} 2n$
2. $\lim\limits_{x\to\infty} (2x^2 - x)$
3. $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{3x^2-1}$
4. $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{3x-1}$
5. $\lim\limits_{x\to\infty} \dfrac{2x^2+x}{3x-2}$

Solutions

1. As $n \to \infty$, $2n$ also approaches ∞.
   $\therefore \lim\limits_{n\to\infty} 2n = \infty$
   Remember when you see "$= \infty$" we cannot actually get to infinity, but using limits we can say the limit is infinity. This is just telling us that the function goes on forever, without limit, as the independent variable gets bigger and bigger in numerical value.
2. As $x \to \infty$, $2x^2 \to +\infty$
   As $x \to \infty$, $-x \to -\infty$
   But x^2 grows more rapidly than x, so $(2x^2 - x)$ will head towards $+$infinity.
   $\therefore \lim\limits_{x\to\infty} (2x^2 - x) = \infty$
3. When we deal with rational functions such as $\dfrac{g(x)}{h(x)}$ we must compare the degree (highest powers) of the functions.
   If the degree of $g(x) = h(x)$, then divide the coefficients of the terms with the highest powers. This is the case in this example.

   We see both functions have the same degree of 2; the highest power of$x$. The coefficient of x^2 in the numerator is 2. The coefficient of x^2 in the denominator is 3.

   $\therefore \lim\limits_{x\to\infty} \dfrac{2x^2+1}{3x^2-1} = \dfrac{2}{3}$
4. If the degree of $g(x) < h(x)$, then the limit is 0.
   $\lim\limits_{x\to\infty} \dfrac{2x+1}{3x^2-1} = 0$
5. If the degree of $g(x) > h(x)$, then the limit will either be positive infinity or negative infinity. We have to look at the signs of the terms with the largest exponents in both the numerator and denominator.
   $\lim\limits_{x\to\infty} \dfrac{2x^2+x}{3x-2} = +\infty$
   This is because both x^2 in the numerator and $3x$ in the denominator are positive.
   While $\lim\limits_{x\to\infty} \dfrac{4x^2-5x^3}{3x-2} = -\infty$ because $\dfrac{-5}{3}$ is negative.



Exercise 1.2

Calculate the following:

1. $\lim\limits_{x \to \infty} \dfrac{2}{x^2}$
2. $\lim\limits_{x \to \infty} (x^3 + 1)$
3. $\lim\limits_{x \to \infty} \dfrac{6x^2 - x^3}{2x - 1}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to evaluate limits of a function at a point by factorising.
- How to use the limit from the left and right to determine if the limit of a function exists.
- How to describe a graph by looking at how the function values behave as the independent variable approaches a point that is undefined.
- How to evaluate limits at infinity.

Unit 1: Assessment

Suggested time to complete: 15 minutes

Calculate the following:

1. $\lim\limits_{y \to 1} \dfrac{y - 1}{y + 1}$
2. $\lim\limits_{h \to 0} \dfrac{2h + h^2}{h}$
3. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 - 3x}$
4. $\lim\limits_{h \to 0} \dfrac{(4 + h)^2 - 16}{h}$
5. $\lim\limits_{x \to \infty} \dfrac{2x^2 + 1}{x^2 + 1}$

The full solutions are at the end of the unit.

Unit 1: Solutions

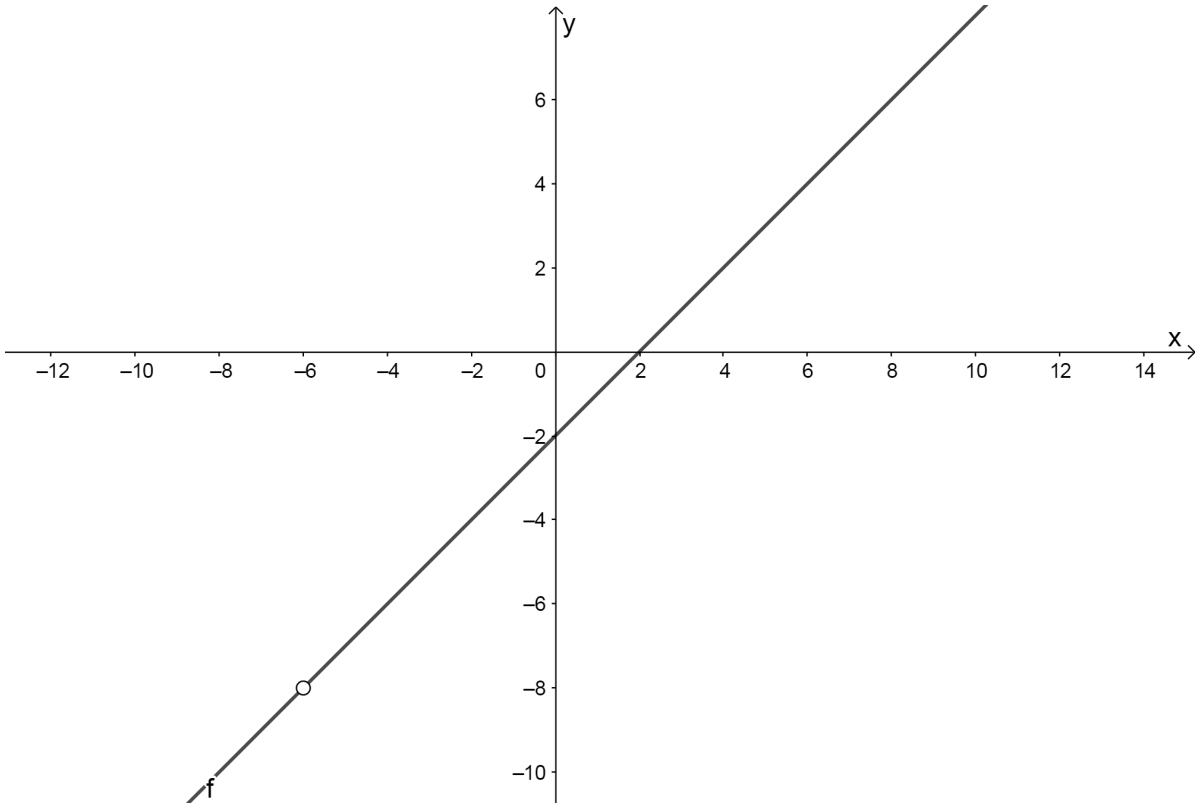
Exercise 1.1

1. If $f(x) = \dfrac{x^2 + 4x - 12}{x + 6}$

   a.

$$\begin{aligned}\lim_{x \to -6} \frac{x^2 + 4x - 12}{x + 6} &= \lim_{x \to -6} \frac{(x + 6)(x - 2)}{(x + 6)} \text{ Factorise and cancel first} \\ &= \lim_{x \to -6} (x - 2) \text{ Substitute } x \text{ with } -6 \\ &= -8\end{aligned}$$

   b.



2.

   a. We cannot simplify further and there are no terms to cancel.
   $\lim\limits_{x \to 1} 10 = 10$

   b.

$$\begin{aligned}\lim_{x \to 3} \frac{x^2 - 9}{x - 3} &= \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)} \\ &= \lim_{x \to 3} (x + 3) \\ &= 6\end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1. $\lim\limits_{x\to\infty} \dfrac{2}{x^2} = 0$
2. $\lim\limits_{x\to\infty} (x^3 + 1) = \infty$
3. $\lim\limits_{x\to\infty} \dfrac{6x - x^3}{2x^2 - 1} = -\infty$

Back to Exercise 1.2

Unit 1: Assessment

1. $$\lim_{y\to 1} \frac{y-1}{y+1} = 0$$
2. $$\begin{aligned}\lim_{h\to 0} \frac{2h + h^2}{h} &= \lim_{h\to 0} \frac{h(2+h)}{h} \\ &= \lim_{h\to 0} 2 + h \\ &= 2\end{aligned}$$
3. $$\begin{aligned}\lim_{x\to 3} \frac{x^2 - 9}{x^2 - 3x} &= \lim_{x\to 3} \frac{(x-3)(x+3)}{x(x-3)} \\ &= \lim_{x\to 3} \frac{(x+3)}{x} \\ &= \frac{6}{3} \\ &= 2\end{aligned}$$
4. $$\begin{aligned}\lim_{h\to 0} \frac{(4+h)^2 - 16}{h} &= \lim_{h\to 0} \frac{16 + 8h + h^2 - 16}{h} \\ &= \lim_{h\to 0} \frac{8h + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(8+h)}{h} \\ &= 8\end{aligned}$$
5. $$\lim_{x\to\infty} \frac{2x^2 + 1}{x^2 + 1} = 2$$

Back to Unit 1: Assessment

Media Attributions

- Unit 1 Finding limits_Activity © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Fig 1One sided limit © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Limits at infinity figure 2 and table © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Limts Exercise 1.1 Q1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 2: Find the derivative by first principles

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Develop an intuitive understanding of the gradient at a point.
- Differentiate using first principles.

What you should know

Before you start this unit, make sure you can:

- Find limits of a function. To revise finding limits of a function see unit 1 of this subject outcome.

Introduction

It is widely agreed that calculus was developed independently by Sir Isaac Newton and Gottfried Leibniz in the 17th century. Since neither of them published their work as soon as they discovered their results, there is no clear indication of who discovered calculus first.



Figure 1: Newton (left) and Leibniz (right) are credited with developing calculus

Calculus makes the study and comparison of infinitesimal (very, very, very small) changes of one quantity with respect to another possible. This has important applications in many sciences including engineering, biology, chemistry and physics.

Now that we have a basic understanding of a limit and have learnt how to compute limits (unit 1 of this subject outcome), we are ready to begin the study of calculus.

Gradient at a point on a curve

We saw in level 2 subject outcome 2.3 unit 1 that for linear graphs, it is easy to determine the slope or gradient of the graph by dividing the change in y-values by the change in x-values between the same two points.

Recall that for straight lines $m = \frac{y_2 - y_1}{x_2 - x_1}$ and that the gradient of a straight line is constant between any two points on the graph.

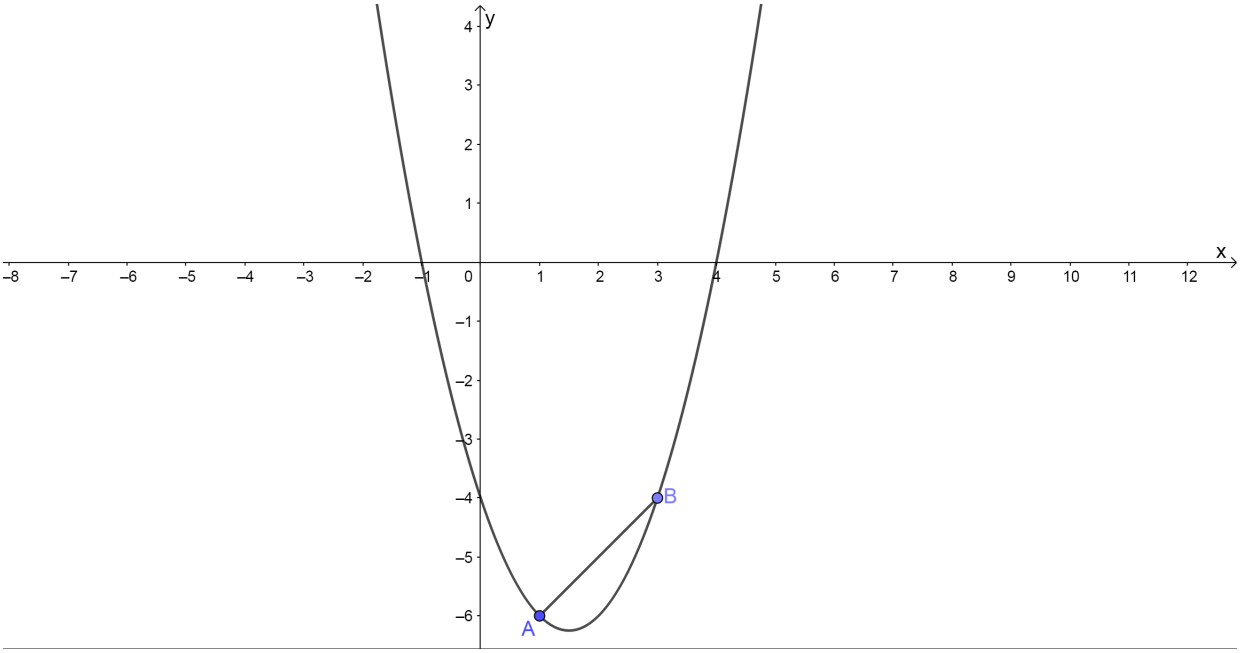
With curved graphs the gradient changes at every point on the curve so we need to work with the average gradient. The average gradient between two points is equal to the gradient of a straight line, called a secant, drawn between the two points. To find the average gradient between any two points on a curve we calculate the gradient of the secant line that passes through both points.

Example 2.1

Find the average gradient of the graph $f(x) = x^2 - 3x - 4$ between the points $x = 1$ and $x = 3$.

Solution

Let's draw the graph, plot the points and draw the secant between them so we can visualise what is being asked.



Since we can draw a straight line between the points A and B, we can find the average gradient using the gradient formula.

$$m_{\text{AB}} = \frac{y_2 - y_1}{x_2 - x_1}$$

To use this formula, we need to find the function values at $x = 1$ and $x = 3$.

$f(1) = (1)^2 - 3(1) - 4 = -6$ and $f(3) = (3)^2 - 3(3) - 4 = -4$

So we need to find the gradient between the points $(1, -6)$ and $(3, -4)$ to find the gradient of the graph $f(x) = x^2 - 3x - 4$ between the points $x = 1$ and $x = 3$.

$$\begin{aligned} m_{\text{AB}} &= \frac{-6 - (-4)}{1 - 3} \\ &= \frac{-2}{-2} \\ &= 1 \end{aligned}$$

In Example 2.1 we saw that to calculate the average gradient of a curve we need two points so that we can calculate the gradient of the secant.

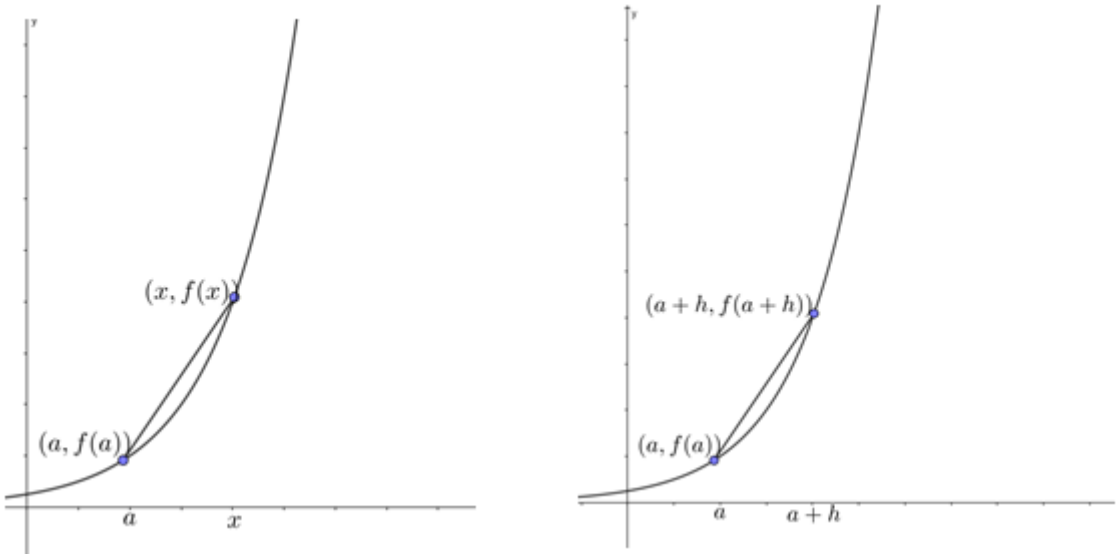


Figure 2: Calculating the gradient of a secant

In Figure 2 we can see that we can calculate the slope of a secant line in two ways. $m_{\text{sec } ant} = \frac{f(x) - f(a)}{x - a}$ or $m_{\text{sec } ant} = \frac{f(a+h) - f(a)}{(a+h) - a}$.

In the second gradient calculation we find the slope of the secant by replacing x with $a + h$, where h is any distance between the points. We can then calculate the slope of the line through the points $(a,\ f(a))$ and $(a\ +\ h,\ f(a\ +\ h))$. We simplify the denominator of the second equation for the gradient of the secant to get:

$$\begin{aligned} m_{\text{sec } ant} &= \frac{f(a+h) - f(a)}{a + h - a} \\ &= \frac{f(a+h) - f(a)}{h} \end{aligned}$$

But, how do we find the gradient at a specific point on a curve?

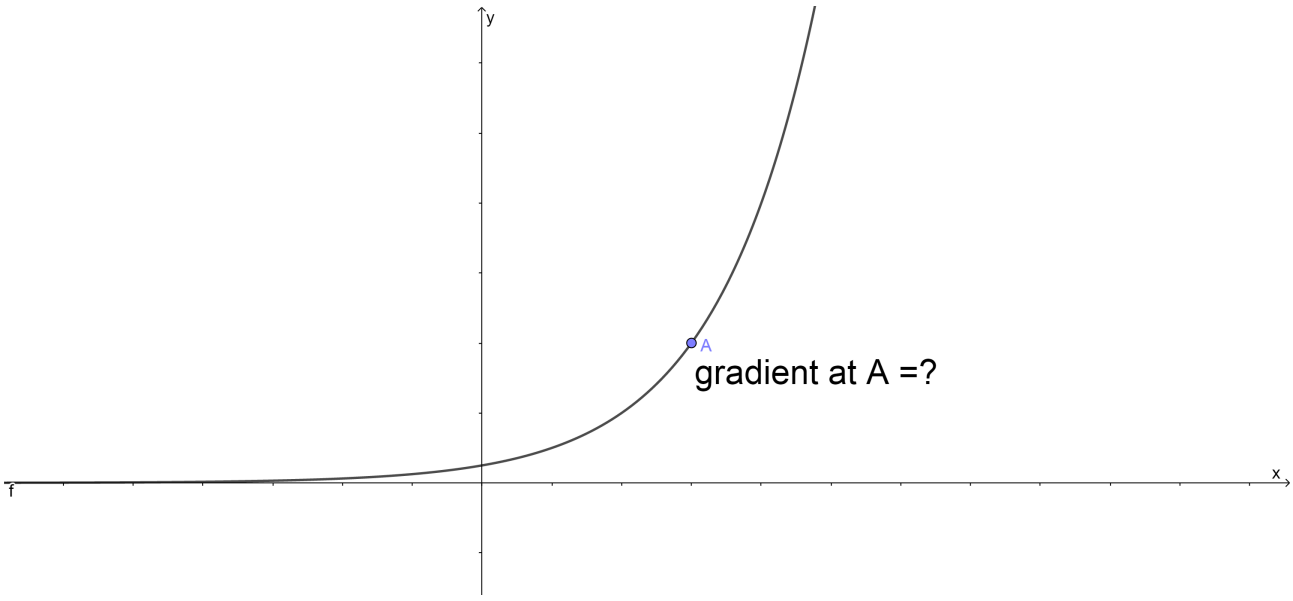


Figure 3: How do we find the gradient at point A?

To find the gradient at a specific point on a curve, we need calculus.

Activity 2.1: Find the gradient at a point on a curve

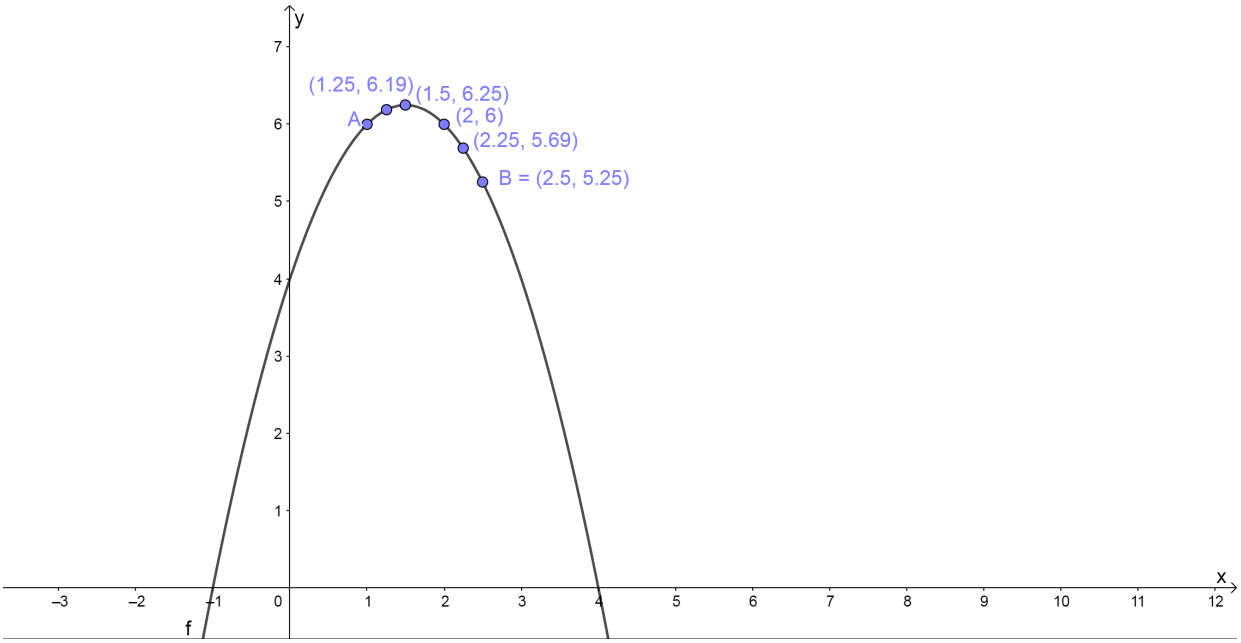
Time required: 20 minutes

What you need:

- pen and paper

What to do:

Imagine that there is a point B that moves closer and closer to a fixed point $\mathrm{A}(1,6)$ as shown.



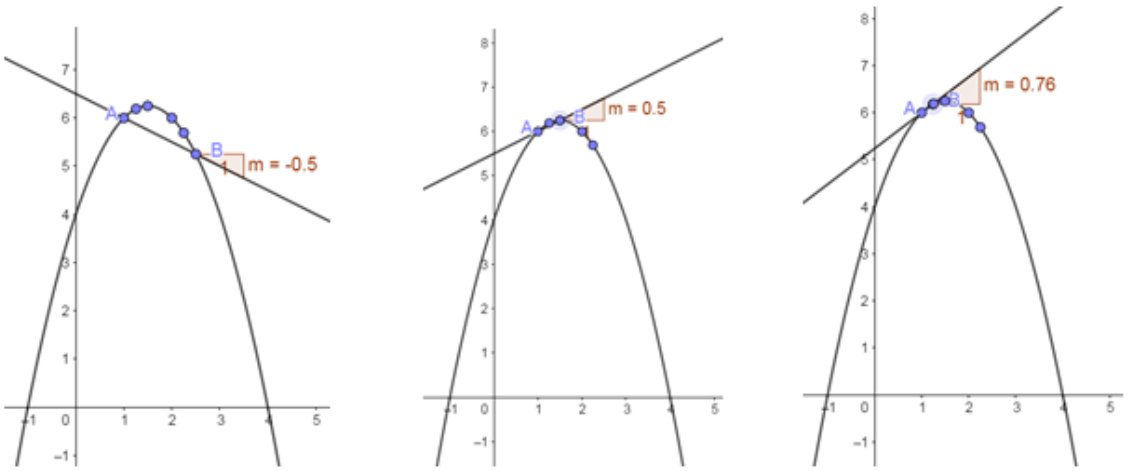
1. Find the gradient of the secant between $\mathrm{A}(1, 6)$ and $\mathrm{B}(2.5,\ 5.25)$.
2. Find the gradient of the secant between $\mathrm{A}(1, 6)$ and $(2.25,\ 5.69)$.
3. Find the gradient of the secant between $\mathrm{A}(1, 6)$ and $(2,\ 6)$.
4. Find the gradient of the secant between $\mathrm{A}(1, 6)$ and $(1.5,\ 6.25)$.
5. Find the gradient of the secant between $\mathrm{A}(1, 6)$ and $(1.25,\ 6.19)$.
6. Let h be the distance between point A and point B. What value does h approach as point B gets closer and closer to point A?
7. What happens to the average gradient as B moves towards point A?
8. What is the average gradient when B overlaps with A?
9. Let A be the point $(a,\ f(a))$ and B be the point $(a + h,\ f(a + h))$, where the distance $h \to 0$ as B moves closer and closer to A. Use your knowledge of limits to write down an expression for the gradient at point A as $\lim_{h \to 0}$.

What did you find?

For parts 1 to 5 of this question the average gradients between the points are shown in the table below. Notice how the sign of the gradient changes as the point B moves towards A. It starts off as a negative slope then reaches a gradient of zero at the turning point and then the slope becomes positive after the graph turns.

Points	Average gradient
$\mathrm{A}(1,6)$ and $(2.5,\ 5.25)$	$m = \dfrac{5.25-6}{2.5-1}$ $= -0.5$
$\mathrm{A}(1,6)$ and $(2.25,\ 5.68)$	$m = \dfrac{5.68-6}{2.25-1}$ $= -0.25$
$\mathrm{A}(1,6)$ and $(2,\ 6)$	$m = \dfrac{6-6}{2-1}$ $= 0$
$\mathrm{A}(1,6)$ and $(1.5,\ 6.25)$	$m = \dfrac{6.25-6}{1.5-1}$ $= 0.5$
$\mathrm{A}(1,6)$ and $(1.25,\ 6.19)$	$m = \dfrac{6.19-6}{1.25-1}$ $= 0.76$

6. The distance h approaches 0 as B moves closer and closer to A.
7. The gradient of the straight line that passes through points A and B changes as B moves closer to A.



At the point where A and B overlap, the straight line only passes through one point on the curve. This line is known as a tangent to the curve. We can now introduce the idea of the gradient at a single point on a curve. The gradient at a point on a curve is the gradient of the tangent to the curve at the given point.

8. We see that as point B approaches point A, h gets closer to 0. If point B lies on point A, then $h = 0$ and the formula for the average gradient is undefined.
9. We use our knowledge of limits to let h tend to 0 and determine the gradient of the curve at point A. The gradient at point A as $\lim\limits_{h \to 0}$ is given by:

$$\begin{aligned} m_A &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{a+h-a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

Activity 2.1 has shown that the gradient at a single point on a curve is the same as the gradient of the tangent to the curve at the given point.

Example 2.2

Given $f(x) = -2x^2$:

1. Determine the average gradient between the points A, where$x = 1$, and B, where $x = 3$.
2. Determine the gradient of the curve at point A.

Solution

1. The average gradient between A and B:
$$\begin{aligned} m_{AB} &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{-2(3)^2 - (-2(1)^2)}{2} \\ &= \frac{-18 + 2}{2} \\ &= -8 \end{aligned}$$
2. The average gradient between any two points on $f(x)$:

   $$\frac{f(a+h) - f(a)}{(a+h) - a}$$

   Substitute $(a + h)$ into the given equation $f(x) = -2x^2$ to get a unique expression for the average gradient of this curve.

$$\begin{aligned} \frac{f(a+h) - f(a)}{(a+h) - a} &= \frac{-2(a+h)^2 - (-2a)^2}{h} \\ &= \frac{-2a^2 - 4ah - 2h^2 + 2a^2}{h} \\ &= \frac{-4ah - 2h^2}{h} \\ &= \frac{h(-4a - 2h)}{h} \\ &= -4a - 2h \end{aligned}$$

   Now, to find the gradient at point A we must add in the limit.

   $$m_A = \lim_{h \to 0} (-4a - 2h)$$

   This is the expression we can use to calculate the gradient at point A.

   At point A: $x = 1$ therefore $a = 1$ as this is the point where we want to find the gradient. At a specific point on the curve $h = 0$ as shown in Activity 2.1, so the gradient is: $-4(1) - 2(0) = -4$.

Exercise 2.1

1. Determine the average gradient of the curve $f(x) = x(x + 3)$ between $x = 5$ and $x = 3$.
2. $A(1, 3)$ is a point on $f(x) = 2x^2 - x + 1$.
   a. Determine the gradient of the curve at point A.
   b. Determine the equation of the tangent line at A.

The full solutions are at the end of the unit.

Differentiation of a function at a point

The type of limit we use to find the slope of the tangent line to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business and growth rates in biology. This limit occurs so frequently that we give it a special name; the **derivative**.

The process of finding a derivative is called differentiation.

To find the gradient of a tangent to a curve at a point we use the formula: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ or $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$. We can use this formula to determine an expression that describes the gradient of the graph (or the gradient of the tangent to the graph) at any point on the graph. This expression (or gradient function) is called the derivative.

The derivative of a function $f(x)$ is written as $f'(x)$ and is $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

Activity 2.2: Estimating a derivative

Time required: 10 minutes

What you need:

- pen and paper

What to do:

1. For $f(x) = x^3$, use $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to write an expression for $f'(2)$.
2. For $f(x) = x^3$, use the table below and the equation you found above to estimate $f'(2)$.

x	$\dfrac{f(x)-f(a)}{x-a}$
1.9	
1.99	
1.999	
2.001	
2.01	
2.1	

3. After examining the table, what is a good estimate of $f'(2)$ as $x \to 2$?
4. Without using a table, estimate $f'(3)$ by using $f'(a) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$.
5. Now, find $f'(3)$ by using $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$.
6. Compare your answers in questions 4 and 5. What do you notice?

What did you find?

1. $f(x) = x^3$

$$f'(2) = \lim_{x \to 2} \frac{x^3 - 2^3}{x-2}$$
$$= \lim_{x \to 2} \frac{x^3 - 8}{x-2}$$

2.

x	$\dfrac{f(x)-f(a)}{x-a}$
1.9	11.41
1.99	11.9401
1.999	11.994001
2.001	12.006001
2.01	12.0601
2.1	12.61

3. After examining the table, we can see that a good estimate for $f'(2)$ is 12.
4. $$f'(3) = \lim_{x \to 3} \frac{x^3 - 3^3}{x-3}$$
$$= \lim_{x \to 3} \frac{x^3 - 27}{x-3}$$

Before we find this limit, here is a quick reminder on how to factorise a difference of cubes.
Step 1: Take the cube root of terms that are perfect cubes

$\sqrt[3]{x^3} = x$
$\sqrt[3]{27} = 3$
These give the terms in the first bracket.
$x^3 - 27 = (x - 3)(\ )$
Step 2: Find the three terms in the second bracket.
Square the first term from the first bracket to get x^2then square the second term from the first bracket to get 9. Lastly, multiply the first term by the second term in the first bracket and change the sign to get $+3x$.
$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

Simplify and factorise the numerator.
$$\lim_{x \to 3} \frac{x^3 - 27}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$$
Cancel the common factor and evaluate the limit.
$$= \lim_{x \to 3} x^2 + 3x + 9$$
$$= 27$$

5.
$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{(3 + h)^3 - 3^3}{h} \\ &= \lim_{h \to 0} \frac{27 + 9h^2 + 27h + h^3 - 27}{h} \\ &= \lim_{h \to 0} \frac{9h^2 + 27h + h^3}{h} \\ &= \lim_{h \to 0} \frac{h(9h + 27 + h^2)}{h} \\ &= \lim_{h \to 0} 9h + 27 + h^2 \\ &= 27 \end{aligned}$$

6.
Without using a table, we saw that the results are the same whether we use
$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ or $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$. $f'(3) = 27$.

The first formula is the definition of the gradient of a tangent line to $f(x)$ passing through the point $(a, f(a))$. In this case $(3, f(3))$, so the slope of the tangent at the point $x = 3$ is 27.

We have also seen that we may define $f'(x)$ passing through the point $(a, f(a))$ as
$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
Alternatively, we may also define $f'(x)$ at a as $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$. So we say the derivative of f at $x = 3$ is 27.

As we have seen, the **derivative** of a function at a given point gives us the slope of the tangent line to the function at that point. Knowing the derivative of a function at every point can give valuable information about the behaviour of various functions.

Note

Take a look at this online tool to explore the development of the derivate.

Tick the 'show tangent' and 'show slope value' boxes and drag the point to explore the gradient.

Differentiation from first principles

We have seen that we use the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of a function. This method is called differentiation from first principles or using the definition.

There are different notations used to refer to derivatives. If we use the common notation $y = f(x)$ where the dependent variable is y and the independent variable is x, then some alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D_x y$$

The symbols D and $\frac{d}{dx}$ are called differential operators because they tell us that we are differentiating whenever we see them.

$\frac{dy}{dx}$ is not a fraction! It means y differentiated with respect to x. Similarly, $\frac{dt}{dx}$ means t differentiated with respect to x.

You must learn to identify the different ways of denoting the derivative and be consistent with how you use them.

Example 2.3

1. Find the derivative of $f(x) = 2x^2 - 1$ from first principles.
2. Calculate $f'(2)$ and interpret the answer.

Solutions

1. **Step 1:** Write down the formula for finding the derivative using first principles.

   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Determine $f(x+h)$

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 1 \\ &= 2(x^2 + 2xh + h^2) - 1 \\ &= 2x^2 + 4xh + 2h^2 - 1 \end{aligned}$$

Step 3: Substitute into the formula and simplify.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 1 - (2x^2 - 1)}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 1 - 2x^2 + 1}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h)}{h} \\ &= \lim_{h \to 0} 4x + 2h \\ &= 4x \end{aligned}$$

Step 4: Write the final answer.

$f'(x) = 4x$

2. $f'(x) = 4x$

$$\begin{aligned} f'(2) &= 4(2) \\ &= 8 \end{aligned}$$

- The derivative of $f(x)$ at $x = 2$ is 8.
- The gradient of the function f at $x = 2$ is equal to 8.
- The gradient of the tangent to $f(x)$ at $x = 2$ is equal to 8.



Exercise 2.2

1. $g(x) = 4x - 1$
   a. Write down the gradient of $g(x)$.
   b. Find the derivative of $g(x)$.
   c. Compare your answers from a) and b). What do you notice?
   d. Write down $g'(-3)$. Interpret your answer.
   e. What is the gradient anywhere along $g(x)$? Why?
2. Calculate $\frac{dk}{dx}$ from first principles if $k(x) = -\frac{2}{x}$.
3. Differentiate $f(x) = \frac{1}{2}$ from first principles and interpret the answer.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to define a tangent and secant to a curve.
- How to find the average gradient between two points on a curve.
- How to calculate the gradient of a tangent to a curve at a specific point.
- The definition of a derivative and its relation to the gradient of a tangent to a curve at a single point.
- To differentiate a function by using the formula for first principles.

Unit 2: Assessment

Suggested time to complete: 25 minutes

1. Determine the gradient of the tangent to $f(x) = \frac{3}{x}\ (x \neq 0)$ at $x = a$.
2. Determine the equation of the tangent to $g(x) = x^2 + 3x$ at $x = -1$.
3. Determine $f'(x)$ from first principles if $f(x) = -2 - 3x$.
4. If $f(x) = 3x^2 - 3$ determine:
   a. $f'(x)$ from first principles
   b. $f'(2)$.
   c. What does your answer to $4\text{b})$ represent?

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1. Start by finding the functions values at the given x-values:
   $f(5) = 5(5 + 3)$ and $f(3) = 3(3 + 3)$
   $= 40$ $= 18$

   $$m = \frac{f(3) - f(5)}{3 - 5}$$
   $$= \frac{18 - 40}{-2}$$
   $$= 11$$
2.

a. Gradient at a point on $f(x) = 2x^2 - x + 1$:

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a} \\ &= \lim_{h \to 0} \frac{2(a+h)^2 - (a+h) + 1 - [2(a)^2 - a + 1]}{h} \\ &= \lim_{h \to 0} \frac{2(a^2 + 2ah + h^2) - a - h + 1 - 2a^2 + a - 1}{h} \\ &= \lim_{h \to 0} \frac{2a^2 + 4ah + 2h^2 - h - 2a^2}{h} \\ &= \lim_{h \to 0} \frac{h(4a + 2h - 1)}{h} \\ &= 4a - 1 \end{aligned}$$

At $\mathrm{A}\,(1, 3)$

$a = 1$

$h = 0$

$4(1) - 1 = 3$

Gradient at $\mathrm{A}\,(1, 3) = 3$

b. $m_{\tan} = 3$

Equation of the tangent:

We know that $m = \dfrac{\text{change in } y}{\text{change in } x}$

By re-arranging the formulae we get:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= 3(x - 1) \\ y &= 3x - 3 + 3 \\ y &= 3x \end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1. $g(x) = 4x - 1$

   a. The gradient of the straight line $g(x)$ is 4.

   b.

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{4(x+h) - 1 - (4x - 1)}{h} \\ &= \lim_{h \to 0} \frac{4x + 4h - 1 - 4x + 1}{h} \\ &= \lim_{h \to 0} \frac{4h}{h} \\ &= 4 \end{aligned}$$

   c. The answers to a and b are the same. The derivate of a function is equivalent to finding the gradient or slope of the function.

   d. $g'(-3) = 4$ tells us that the slope of the graph at $x = -3$ is 4.

e. The gradient anywhere along $g(x) = 4$ because $g(x) = 4x - 1$ is a straight line. The gradient anywhere along a straight line is constant.

2.

$$\begin{aligned}
\frac{dk}{dx} &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} \\
&= \lim_{h \to 0} \frac{-\frac{2}{(x+h)} - \left(\frac{-2}{x}\right)}{h} \\
&= \lim_{h \to 0} \frac{-\frac{2}{(x+h)} + \frac{2}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{-2(x) + 2(x+h)}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{-2x + 2x + 2h}{x(x+h)h} \\
&= \lim_{h \to 0} \frac{2h}{x(x+h)h} \\
&= \lim_{h \to 0} \frac{2}{x(x+h)} \\
&= \frac{2}{x^2}
\end{aligned}$$

Notice: even though h remains in the denominator, we can take the limit since it does not result in division by 0.

3.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{2} - \frac{1}{2}}{h} \\
&= 0
\end{aligned}$$

The gradient of $f(x) = \frac{1}{2}$ is 0 at any point on the graph. The slope of a constant function (graph parallel to the x-axis) is always equal to 0.

Back to Exercise 2.2

Unit 2: Assessment

1. Gradient of tangent at $x = a$:

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{a+h-a} \\
&= \lim_{h\to 0} \frac{\dfrac{3}{(a+h)} - \dfrac{3}{a}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \times \left(\frac{3}{(a+h)} - \frac{3}{a} \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(\frac{3a - 3(a+h)}{a(a+h)} \right) \\
&= \lim_{h\to 0} \left(\frac{3a - 3a - 3h}{ah(a+h)} \right) \\
&= \lim_{h\to 0} \left(\frac{-3h}{ah(a+h)} \right) \\
&= \lim_{h\to 0} \left(\frac{-3}{a(a+h)} \right) \\
&= \frac{-3}{a^2}
\end{aligned}
$$

2. Gradient of $g(x) = x^2 + 3x$ is given by:

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h} \\
&= \lim_{h\to 0} \frac{h(2x + h + 3)}{h} \\
&= 2x + 3
\end{aligned}
$$

Gradient of tangent at $x = -1$:

Substitute the x-value into the general form of the gradient expression:

$$
\begin{aligned}
g'(-1) &= 2(-1) + 3 \\
&= -2 + 3 \\
&= 1
\end{aligned}
$$

3. Therefore, the gradient of tangent at $x = -1$ is 1.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{-2 - 3(x+h) - (-2-3x)}{h} \\
&= \lim_{h\to 0} \frac{-2 - 3x - 3h + 2 + 3x}{h} \\
&= \lim_{h\to 0} \frac{-3h}{h} \\
&= -3
\end{aligned}
$$

4.

a.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 - 3 - (3x^2 - 3)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - 3 - 3x^2 + 3}{h} \\
&= \lim_{h \to 0} \frac{6xh + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(6x + h)}{h} \\
&= \lim_{h \to 0} 6x + h \\
&= 6x
\end{aligned}
$$

b.

$$
\begin{aligned}
f'(2) &= 6(2) \\
&= 12
\end{aligned}
$$

c.

The gradient of the curve $f(x) = 3x^2 - 3$ at $x = 2$ is 12.

OR

The gradient of the tangent to curve $f(x) = 3x^2 - 3$ at $x = 2$ is 12.

Back to Unit 2: Assessment

Media Attributions

- Img 01_Newton and Leibniz © Wikimedia is licensed under a CC BY (Attribution) license
- Img02_Example 2.1 average gradient © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Fig01_both secants in one figure © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Fig 02_how do you find gradient at a point © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Fig 03Activity_gradient at a point © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Fig 5_Activity_gradient as A gets closer to B © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 3: Find the derivative using rules for differentiation

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Apply the constant, sum and difference rules for differentiation.
- Use notation appropriately.

What you should know

Before you start this unit, make sure you can:

- Find the derivative by using first principles as shown in unit 2 of this subject outcome.

Introduction

It can be quite a long and challenging process to find derivatives of functions by using the definition of the derivative or first principles. For example, when we previously found the derivative by first principles of $k(x) = -\frac{2}{x}$ (unit 2 Exercise 2.2) we saw that we had to work with complicated fractions before evaluating the limit.

In this unit, we will learn how to find the derivative of a function with rules instead of with first principles. Unless a question specifically requires the use of the definition to find the derivative, we can always use the rules to derive.

The basics

The functions $f(x) = c$ and $g(x) = x^n$, where n is a positive integer, are the building blocks from which all polynomials and rational functions are created. To find derivatives of polynomials and rational functions without using the definition of the derivative, we must first develop formulae for differentiating these two basic functions.

Activity 3.1: Discover the basic rules for differentiation

Time required: 35 minutes

What you need:

- pen and paper

What to do

1. Differentiate the following by first principles:
   a. $f(x) = 2$
   b. $f(x) = 6$
   c. $f(x) = x$
   d. $f(x) = -2x$
   e. $f(x) = x^2$
   f. $f(x) = 4x^2$
2. Complete the table below:

$f(x)$	$f'(x)$	Power of x in $f(x)$	Co-efficient of $f'(x)$	Power of x in $f'(x)$
2				
6				
x				
$-2x$				
x^2				
$4x^2$				

3. Can you see a pattern for finding the derivative of a constant function c?
4. Can you see a pattern for finding the derivative of a power function x^n?

What did you find?

1.
   a. $f(x) = 2$
   For this function, both $f(x) = 2$ and $f(x+h) = 2$, so we get:
   $$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2-2}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$
   b. $f(x) = 6$
   For this function, both $f(x) = 6$ and $f(x+h) = 6$, so we get:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{6-6}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

c. $f(x) = x$

For this function, $f(x) = x$ and $f(x+h) = x+h$, so we get:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)-(x)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= 1 \end{aligned}$$

d. $f(x) = -2x$

For this function, $f(x) = -2x$ and

$$\begin{aligned} f(x+h) &= -2(x+h) \\ &= -2x-2h \end{aligned}$$

so we get:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{-2(x+h)-(-2x)}{h} \\ &= \lim_{h\to 0} \frac{-2x-2h+2x}{h} \\ &= \lim_{h\to 0} \frac{-2h}{h} \\ &= -2 \end{aligned}$$

e. $f(x) = x^2$

For this function, $f(x) = x^2$ and

$$\begin{aligned} f(x+h) &= (x+h)^2 \\ &= x^2+2xh+h^2 \end{aligned}$$

so we get:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x \end{aligned}$$

f. $f(x) = 4x^2$

For this function, $f(x) = 4x^2$ and

$$f(x+h) = 4(x+h)^2$$
$$= 4x^2 + 8xh + 4h^2$$

so we get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h}$$
$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(8x + 4h)}{h}$$
$$= \lim_{h \to 0} (8x + 4h)$$
$$= 8x$$

2.

$f(x)$	$f'(x)$	Power of x in $f(x)$	Co-efficient of $f'(x)$	Power of x in $f'(x)$
2	0	0	2×0	None
6	0	0	6×0	None
x	1	1	1×1	0
$-2x$	-2	1	-2×1	0
x^2	$2x$	2	2×1	1
$4x^2$	$8x$	2	2×4	1

3. From a) and b) we see that the derivative of a number (constant) is equal to 0.
4. From c) to e) we can see that when we derive a power function, the exponent is multiplied by the co-efficient of the original function and the derivative's power is one less than the original function's power.
   For example, when we found the derivative of $f(x) = 4x^2$ we got $f'(x) = 8x$. The power of x in $f(x)$ is 2 and the coefficient is 4. The power of x in $f'(x)$ is 1, which is one less than the power of $f(x)$. The coefficient of x in $f'(x)$ is 8, which is the power of 2 from $f(x)$ multiplied by the coefficient of 4.

Although it is often unwise to draw general conclusions from specific examples, Activity 3.1 is a useful way to get an idea of the patterns that are forming. We note from the activity that when we differentiate $f(x) = x$, $f(x) = -2x$, $f(x) = x^2$ and $f(x) = 4x^2$, the power of x is multiplied by the coefficient of $f(x)$ in the derivative. And, the power of x in the derivative decreases by 1.

We also saw that when we derive a constant we end up with an answer of zero.

The following two rules are the basic rules for differentiation that you will start off with.

The constant rule:

If $f(x) = c$ where c is any constant then $f'(x) = 0$.

The power rule:

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{R},\ \mathrm{n} \neq 0$

Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract or multiply by a constant.

The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative.

If $f(x) = kx^n$ then $f'(x) = n \cdot kx^{n-1}$ where $n \in \mathbb{R},\ \mathrm{n} \neq 0$

The derivative of a sum is equal to the sum of the derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The derivative of a difference is equal to the difference of the derivatives.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Applying the power rule

Let's find the derivative of $f(x) = 2x^5$ by applying the power rule.

We see that $n = 5$ and $k = 2$ by the power rule and the constant multiplied by a function rule $f'(x) = n \cdot kx^{n-1}$ so $f'(x) = 2 \times 5x^{5-1}$. We multiply the coefficient 5 of $f(x)$ by the power of 2 and subtract 1 from the exponent to get $f'(x) = 10x^4$.



Example 3.1

Use the rules of differentiation to find the derivative of:

1. $y = 3x^{-2} + 2$
2. $f(x) = x^4 + 2x^3 - x$

Solutions

1. Remember from unit 2 there are different ways to write the derivative:
   $f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D_x y$
   $y = 3x^{-2} + 2$
   $\therefore y' = 3(-2)x^{-2-1} + 0$
   $= -6x^{-3}$
   You will often be required to write final answers with positive exponents.

So $y' = \dfrac{-6}{x^3}$

2. Here we apply the sum and difference rules.

$$f(x) = x^4 + 2x^3 - x$$
$$f'(x) = 4x^{4-1} + 2(3)x^{3-1} - x^{1-1}$$
$$= 4x^3 + 6x^2 - 1$$

Exercise 3.1

Use the rules of differentiation to find the derivative of:

1. $y = 2x^3$
2. $p = \dfrac{1}{2}q^4$
3. $g(x) = 25$
4. $f(x) = 2x^5 + 8$
5. $h(x) = 2x^3 - 6x^2 + 3x$

The full solutions are at the end of the unit.

When to differentiate using first principles:

- If the question says 'use first principles.'
- If we are required to differentiate using the definition of a derivative, then we use first principles.

When to use the rules for differentiation:

- If the question does not specify how to find the derivative, then we use the rules for differentiation.

Take note!

A good knowledge of exponents is essential when deriving.

Before deriving, make sure:

- There are no variables in the denominator. Any variables in the denominator should be moved to the numerator and the sign of the exponents changed.
- There are no surds. Surds should be written in the equivalent power form. For example, write $\sqrt[3]{x^4}$ as $x^{\frac{4}{3}}$.
- There are only separate terms (separated by plus or minus signs) in the simplified function.
- All brackets have been removed by expanding or simplifying.

Summary

In this unit you have learnt the following:

- How to apply the constant, sum and difference rules for differentiation.
- When to use the rules to differentiate.
- When to use first principles to differentiate.

Unit 3: Assessment

Suggested time to complete: 20 minutes

1. Differentiate the following:
   a. $y = 12x^{-2}$
   b. $y = 10(7 - 3)$
   c. $g(x) = 3x^{\frac{2}{3}} - 4x + 10$
   d. $g(x) = x(x + 3) + 5x$
   e. $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$
   f. $y = \sqrt{x^3} + \dfrac{1}{3x^3}$
2. Find $D_x \left[x^{\frac{2}{3}} - \dfrac{3}{x^{\frac{1}{2}}} \right]$

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1. $y' = 6x^2$
2. $p' = \dfrac{1}{2} \times 4q^{4-1}$
   $= 2q^3$
3. $g'(x) = 0$
4.

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(2x^5 + 8) \\
&= \frac{d}{dx}(2x^5) + \frac{d}{dx}(8) && \text{Apply the sum rule} \\
&= 2\frac{d}{dx}(x^5) + \frac{d}{dx}(8) && \text{Apply the constant multiple rule} \\
&= 2(5x^4) + 0 && \text{Apply the power rule and the constant rule} \\
&= 10x^4
\end{aligned}
$$

5.

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}(2x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(3x) \\
&= 6x^2 - 12x + 3
\end{aligned}
$$

Back to Exercise 3.1

Unit 3: Assessment

1.

a.

$$
\begin{aligned}
y' &= -24x^{-3} \\
&= \frac{-24}{x^3}
\end{aligned}
$$

b.

$$y' = 0$$

c.

$$
\begin{aligned}
g(x) &= 3x^{\frac{2}{3}} - 4x + 10 \\
g'(x) &= 3 \times \frac{2}{3}x^{\frac{2}{3}-1} - 4 \\
&= \frac{2}{x^{\frac{1}{3}}} - 4
\end{aligned}
$$

d.

$$
\begin{aligned}
g(x) &= x(x+3) + 5x \\
&= x^2 + 3x + 5x && \text{Expand brackets and simplify} \\
&= x^2 + 8x \\
\therefore g'(x) &= 2x + 8
\end{aligned}
$$

e.

$$
\begin{aligned}
f(x) &= \frac{x^2 - 5x + 6}{x - 3} && \text{Factorise and cancel} \\
&= \frac{(x-3)(x-2)}{(x-3)} \\
&= x - 2 \\
\therefore f'(x) &= 1
\end{aligned}
$$

f.

$$
\begin{aligned}
y &= \sqrt{x^3} + \frac{1}{3x^3} \\
&= x^{\frac{3}{2}} + \frac{1}{3}x^{-3} && \text{Get rid off roots and variables in denominator}
\end{aligned}
$$

$$y' = \frac{3}{2}x^{\frac{1}{2}} - x^{-4}$$
$$= \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{x^4}$$

2. $$D_x\left[x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{2}}}\right] = D_x\left[x^{\frac{2}{3}} - 3x^{\frac{-1}{2}}\right]$$ Simplify inside brackets before deriving

3. $$D_x\left[x^{\frac{2}{3}} - 3x^{\frac{-1}{2}}\right] = \frac{2}{3}x^{\frac{-1}{3}} + \frac{3}{2}x^{\frac{-3}{2}}$$
$$= \frac{2}{3x^{\frac{1}{3}}} + \frac{3}{2x^{\frac{3}{2}}}$$

Back to Unit 3: Assessment

Unit 4: Use derivatives to solve maxima and minima problems

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Use differentiation to find optimal solutions.

What you should know

Before you start this unit, make sure you can:

- Use the rules of differentiation to find the derivative. The rules of differentiation are covered in unit 3 of this subject outcome.

Introduction

One common application of calculus is to calculate the smallest or largest value of a function. Finding the maximum and minimum values of a function has many real-life applications because we can use this method to solve optimisation problems – problems that require some variable to be maximised or minimised.

Examples of optimisation problems include maximising profit, minimising the amount of material used in manufacturing or finding the maximum height a rocket can reach. In this unit, we use derivatives to find the maximum and minimum values of a function.

Activity 4.1: Functions and optimisation

Time required: 20 minutes

What you need:

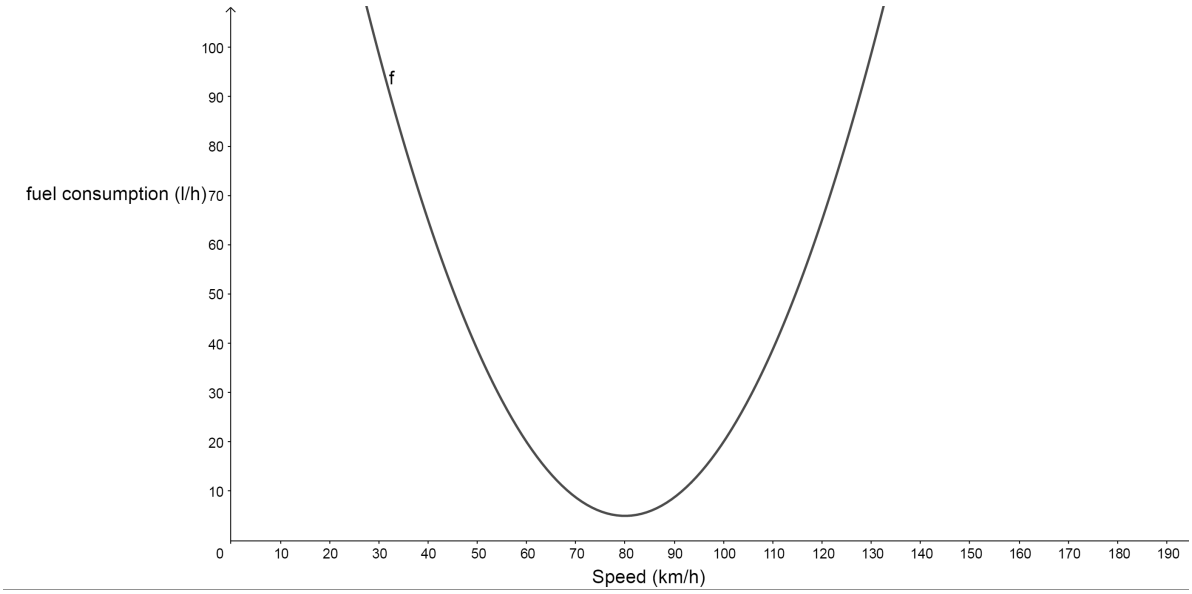
- pen and paper

What to do:

The fuel used by a car is defined by $f(v) = \frac{3}{80}v^2 - 6v + 245$, where v is the travelling speed in km/h.

1. Plot speed on the x-axis and fuel consumption on the y-axis to draw the graph of f. Use a scale of $1:10$ on the axes. Notice that the function is a quadratic function.
2. At what speed is the least fuel consumed?
3. Find $f'(v)$.
4. Let $f'(v) = 0$ and solve for v. What have you found?
5. Comment on the answers you got in questions 2 and 5.

What did you find?

1. 

2. The least fuel usage will be found at the lowest part of the graph or the minimum value of the graph. The speed at the minimum would then give the most economical speed. Therefore, the least fuel is consumed at 80km/h.

3. $$f(v) = \frac{3}{80}v^2 - 6v + 245$$
$$f'(v) = \frac{3}{80}(2)v^{2-1} - 6v^{1-1} + 0$$
$$= \frac{3}{40}v - 6$$

4. $$f'(v) = 0$$
$$\frac{3}{40}v - 6 = 0$$
$$\therefore v = \frac{6 \times 40}{3}$$
$$= 80$$
We have found the speed that corresponds to the minimum fuel consumption.

5. We see that we can find the x-coordinate of the turning point of a function by differentiating (speed in this example) and letting the derivative equal zero. If we set $f'(v) = 0$ we can calculate the speed that corresponds to the minimum turning point, which is also known as a stationary point.

This means that the most economical speed of 80 km/h can also be found by equating the derivative to zero.

Take note!

To find the optimum point using calculus:

Let $f'(x) = 0$ and solve for x.

To check whether the optimum point is a local minimum or a local maximum without drawing the graph, we must find the second derivative. You will learn more about the second derivative in level 4.

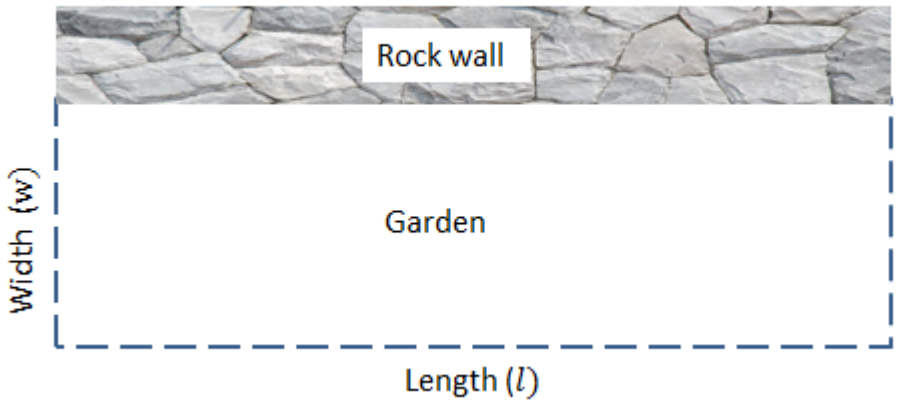
- If $f''(x) < 0$, then the point is a local maximum.
- If $f''(x) > 0$, then the point is a local minimum.

Example 4.1

Jacky wants to put a fence around her vegetable garden. She decides to fence it off in the shape of a rectangle and only has 160 m of fencing so she uses a rock wall as the one border of the garden. Calculate the width and length of the garden that corresponds to the largest possible area that Jacky can fence off.

Solution

Step 1: Read the question carefully, draw a diagram if necessary and decide what variables need to be used.



Let the width be w and the length be l.

Step 2: Determine which quantity needs to be maximised or minimised. Write a formula for the quantity to be maximised or minimised in terms of the variables.

We are given information about the area and the modified perimeter of the garden. We know that the area of the garden is given by the formula:

$\text{Area} = l \times w$

Write any other equations relating the independent variables in the formula found above. Use these equations to write the quantity to be maximised or minimised (area in this case) as a function of one variable.

The fencing is only required for three sides and the three sides must add up to $160\ \text{m}$. So we can say the perimeter is:

$w + w + l = 160$

Rearrange the formula and make l the subject of the formula.

$l = 160 - 2w$

Substitute the expression for l into the formula for the area of the garden. Notice that this formula now contains only one unknown variable.

$$\begin{aligned}\text{Area} &= w(160 - 2w) \\ &= 160w - 2w^2\end{aligned}$$

Step 3: Differentiate with respect of the variable to find the maximum or minimum value.

We are interested in maximising the area of the garden, so we differentiate to get the following:

$$\frac{dA}{dw} = A'(w) = 160 - 4w$$

Step 4: Calculate the stationary point (point on a graph where a minimum or maximum value is found).

To find the stationary point, we set $A' = 0$ and solve to find the value(s) of w that maximises the area.

$$\begin{aligned}A'(w) &= 160 - 4w \\ 0 &= 160 - 4w \\ 4w &= 160 \\ w &= 40\end{aligned}$$

Therefore, the width of the garden is $40\ \text{m}$.

Substitute to solve for the length.

$$\begin{aligned}l &= 160 - 2w \\ &= 160 - 2(40) \\ &= 80\end{aligned}$$

Therefore, the length of the garden is $80\ \text{m}$.

We can check that these dimensions give a maximum area by finding the second derivative.

For a maximum value $A''(w) < 0$:

$$\begin{aligned}A'(w) &= 160 - 4w \\ \therefore A''(w) &= -4\end{aligned}$$

$\because A'' < 0$ The maximum area of the garden is reached when $w = 40$.

Therefore, a width of $40\ \text{m}$ and a length of $80\ \text{m}$ gives the maximum area of the garden.

Take note!

Here is a problem-solving strategy for maxima and minima questions:

1. Read the question carefully, draw a diagram if necessary and decide what variables need to be used.
2. Determine which quantity needs to be maximised or minimised. If necessary, write a formula for the quantity to be maximised or minimised in terms of the variables.
3. The quantity that needs to be minimised or maximised must be expressed in terms of only one variable. So write any other information relating to the variables in another equation and use this equation to substitute for one variable in terms of the other.
4. Find the derivative of your equation with only one variable.
5. Set the derivative equal to zero and solve to find the value that will minimise or maximise the problem.
6. Find the second derivative to confirm that the value you found is the minimum or maximum value.

The next example looks at maximising profit.

Example 4.2

A factory has x employees and makes a profit of P rand per week. The relationship between the profit and number of employees is given by $P(x) = -2x^3 + 600x + 1\ 000$.

a. Calculate the number of employees needed for the factory to make a maximum profit.

b. Confirm that the value you found in a) will maximise profit.

c. Calculate the maximum profit.

Solution

a. In this example we are given the equation and variables to work with, which will make things a little easier.
$P(x) = -2x^3 + 600x + 1\ 000$
We can move straight onto the step where we find the derivative and equate it to zero to solve for the x-value that will maximise profit.

$$\begin{aligned} P'(x) &= -6x^2 + 600 \\ P'(x) &= 0 \\ -6x^2 + 600 &= 0 \\ x^2 &= 100 \\ x &= \pm 10 \\ \therefore x &= 10 \end{aligned}$$

Since x represents employees, it cannot be negative and we only accept the positive solution for

the number of employees that will maximise profit.

b. To check that $x = 10$ maximises profit we must find $P''(10)$.

$P'(x) = -6x^2 + 600$

$P''(x) = -12x$

$P''(10) = -120$

Since the second derivative is negative this means that 10 employees will maximise profit.

c. To find the maximum profit, substitute the value we found for x back into the original equation, which shows the relationship between profit and employees.

$P(x) = -2x^3 + 600x + 1\ 000$

$P(10) = -2(10)^3 + 600(10) + 1000$

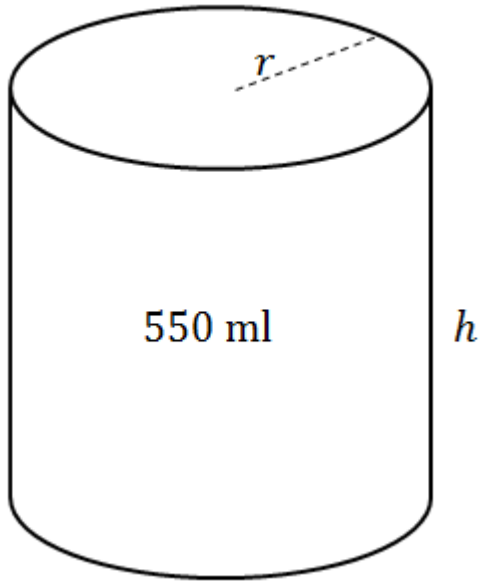
$= 5000$

Maximum profit is $\mathrm{R}\ 5000$ when there are 10 employees.

In the last example we minimise material used.

Example 4.3

The diagram below shows a closed metal cylinder with a capacity of $550\ \mathrm{ml}$. The volume of the cylinder is given by the formula $V = \pi r^2 h$ where r is the radius and h is the height.



a. Express the height h of the cylinder in terms of r.

b. Show that the surface area of cylinder can be written as $S = 2\pi r^2 + \frac{1100}{r}$.

c. Calculate the radius of the cylinder with a minimum surface area.

Solutions

a. We are given the capacity of the cylinder. When the thickness of the walls of a cylinder are insignificant, volume and capacity is basically the same thing.
$V = \pi r^2 h = 550$
We must use the equation for volume to make the height the subject of the formula.
$\pi r^2 h = 550$
$$h = \frac{550}{\pi r^2}$$

b. For any closed container the easiest way to find the surface area is to use the formula:
$\text{TSA} = 2 \times \text{area of the base} + \text{perimeter of the base} \times h$
Since the base of a cylinder is a circle, its area is πr^2 and the circumference (perimeter) is $2\pi r$. Substitute these formulae into the equation for total surface area and replace TSA with S.
$S = 2 \times \pi r^2 + 2\pi r \times h$
$= 2\pi r^2 + 2\pi r h$
Next, we need to rewrite the equation in terms of one variable only. So replace h with the value you found in part a) and simplify to get the required expression.
$S = 2 \times \pi r^2 + 2\pi r \times h$
$= 2\pi r^2 + 2\pi r h$
$= 2\pi r^2 + 2\pi r \left(\frac{550}{\pi r^2}\right)$
$= 2\pi r^2 + \frac{1\ 100}{r}$

c. Calculate the radius of the cylinder with a minimum surface area.
To find a minimum value we need to find the derivative. In this case, we need to minimise the surface area of material needed.
$S = 2\pi r^2 + 1\ 100 r^{-1}$
$S' = 4\pi r - 1\ 100 r^{-2}$
Note: we have changed to a negative exponent to get rid of the variable in the denominator before deriving, as you were shown in unit 3.

For a minimum value $S'(r) = 0$
$4\pi r - 1\ 100 r^{-2} = 0$
$$4\pi r = \frac{1100}{r^2}$$
$$4\pi r^3 = 1\ 100$$
$$r^3 = \frac{1\ 100}{4\pi}$$
$$r = \sqrt[3]{\frac{275}{\pi}}$$
$$= 4.44$$
A radius of 4.44 units will produce a cylinder with a minimum surface area, in other words, using the least material.

Note

When you have an internet connection, watch the video "Optimisation: box volume (Part 1)" about optimising the volume of a box.

Optimisation: box volume (Part 1) (Duration: 09.49)

Exercise 4.1

1. The volume of a rectangular box is given by the equation $f(x) = x^3 - 8x^2 + 5x + 50$. Calculate the value of x for which the volume is a maximum.
2. The dimensions of a cylinder are such that $r + h = 12$. Determine the value of r if the volume of the cylinder is at its maximum.
   HINT: $V = \pi r^2 h$
3. The radius of the base of a solid metal cylinder is x mm while its height is $(300 - x)$ mm.
   Find the value of x which results in a maximum volume if the volume is given by $V = 300\pi x^2 - \pi x^3$.

The full solutions are at the end of the unit.

Summary

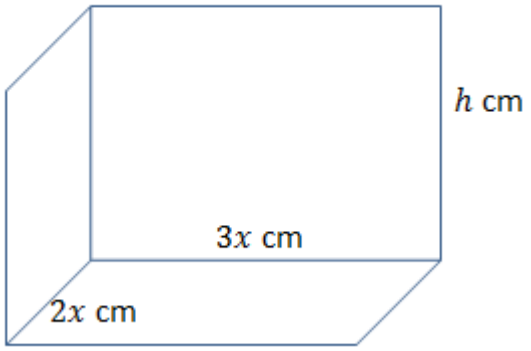
In this unit you have learnt the following:

- How to use the largest and smallest values of a function to maximise and minimise real life problems.
- How to derive to find the optimal solutions.
- How to check if your solutions are maximum or minimum values.

Unit 4: Assessment

Suggested time to complete: 30 minutes

1. The sum of two positive numbers is 10. One of the numbers is multiplied by the square of the other. If each number is greater than zero, find the numbers that make this product a maximum.
2. A cat climbs into a rectangular box without a lid. The box is made of cardboard of insignificant thickness. The sides of the base are $2x$ cm by $3x$ cm, and the height is h cm.



a. If the total external surface area is $200\ \text{cm}^2$, show that $h = \frac{20}{x} - \frac{3x}{5}$.

b. Find the dimensions of the box that will maximise the volume.

3. You need to build a fence around an area of $200\ \text{m}$. What are the dimensions of a rectangular fence that will minimise the amount of material needed?

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1.

$f'(x) = 3x^2 - 16x + 5$

To find the maximum/minimum let $f'(x) = 0$.

$$f'(x) = 3x^2 - 16x + 5 = 0$$

$$\therefore (3x - 1)(x - 5) = 0$$

$$\therefore x = \frac{1}{3} \text{ or } x = 5$$

For maximum volume $f''(x) < 0$

$$f''(x) = 6x - 16$$

$$f''(\frac{1}{3}) = 6(\frac{1}{3}) - 16$$

$$= -14$$

$$\therefore f''(\frac{1}{3}) < 0$$

$$f''(5) = 6(5) - 16$$

$$= 14$$

$$\therefore f''(5) > 0$$

Maximum volume when $x = \frac{1}{3}$.

2.

$r + h = 12$

$\therefore h = 12 - r$

$$V = \pi r^2 h$$
$$= \pi r^2 (12 - r)$$
$$= 12\pi r^2 - \pi r^3$$

For the maximum volume find the derivative with respect to r and equate to zero.

$$V' = 24\pi r - 3\pi r^2 = 0$$
$$\therefore 3\pi r(8 - r) = 0$$
$$\therefore r = 0 \text{ n/a or } r = 8$$
$$\therefore r = 8$$

3.

$$V' = 600\pi x - 3\pi x^2 = 0$$
$$\therefore 3\pi x(200 - x) = 0$$
$$\therefore x = 0 \text{ n/a or } x = 200$$
$$\therefore x = 200$$

Back to Exercise 4.1

Unit 4: Assessment

1. Let the two numbers be x and y and the product be P.
   $x + y = 10 ... (1)$
   $P = x \times y^2 ... (2)$

   Make y the subject of equation (1) and substitute into equation (2).
   $y = 10 - x ... (1)$
   $P = x \times y^2$
   $= x(10 - x)^2$
   $= x(100 - 20x + x^2)$
   $= 100x - 20x^2 + x^3$
   Differentiate with respect to x.
   $P'(x) = 100 - 40x + 3x^2$
   Find the value of x which makes P a maximum by setting the derivative equal to zero.
   $P'(x) = 3x^2 - 40x + 100$
   $0 = (3x - 10)(x - 10)$
   $\therefore x = \frac{10}{3} \text{ or } x = 10$
   Substitute values into equation (1) to solve for y:
   If $x = 10$
   $y = 10 - (10)$
   $= 0$
   This is not possible since each number is greater than zero.

   If $x = \frac{10}{3}$
   $y = 10 - (\frac{10}{3})$
   $= \frac{20}{3}$
   Therefore, the product is maximised when the two numbers are $\frac{10}{3}$ and $\frac{20}{3}$.

2.

   a. Since the box has no lid the formula for surface area becomes:
      $\text{TSA} = \text{area of the base} + \text{perimeter of the base} \times h$

$$
\begin{aligned}
200 &= 2x \cdot 3x + 2(2x + 3x) \cdot h \\
&= 6x^2 + 10xh \\
200 - 6x^2 &= 10xh \\
\frac{200 - 6x^2}{10x} &= h \\
\frac{20}{x} - \frac{3}{5}x &= h
\end{aligned}
$$

b. $\text{Volume} = \text{length} \times \text{breadth} \times \text{height}$

$$
\begin{aligned}
V &= 2x \cdot 3x \cdot h \\
&= 6x^2 h
\end{aligned}
$$

Substitute the value for h found in part a).

$$
\begin{aligned}
V &= 6x^2\left(\frac{20}{x} - \frac{3}{5}x\right) \\
&= 120x - \frac{18}{5}x^3
\end{aligned}
$$

For maximum:

$$
\begin{aligned}
V' &= 0 \\
V' &= 120 - \frac{54}{5}x^2 \\
120 - \frac{54}{5}x^2 &= 0 \\
\frac{54}{5}x^2 &= 120 \\
x^2 &= \frac{100}{9} \\
\therefore x &= \frac{10}{3}
\end{aligned}
$$

Substitute back into part a) to find h.

$$
\begin{aligned}
h &= \frac{20}{\left(\frac{10}{3}\right)} - \frac{3\left(\frac{10}{3}\right)}{5} \\
&= 4 \text{ cm}
\end{aligned}
$$

Sides are:

$2\left(\frac{10}{3}\right) = \frac{20}{3}$ cm and $3\left(\frac{10}{3}\right) = 10$ cm

Dimensions for the box to have a maximum volume are $\frac{20}{3}$ cm, 10 cm and 4 cm.

3. Let the length be x and the width be y.

$$
\begin{aligned}
\text{Area} &= x \times y \\
xy &= 200 \\
\therefore y &= \frac{200}{x}
\end{aligned}
$$

To find the amount of material for the fence find the perimeter.

$$
\begin{aligned}
\text{P} &= 2(x + y) \\
&= 2x + 2y
\end{aligned}
$$

Substitute the y-value from the Area equation.

$$
\begin{aligned}
P &= 2x + 2\left(\frac{200}{x}\right) \\
&= 2x + \frac{400}{x}
\end{aligned}
$$

Derive and solve for the minimum x-value.

$$P = 2x + 400x^{-1}$$
$$P' = 2 - 400x^{-2}$$
$$2 - \frac{400}{x^2} = 0$$
$$2x^2 - 400 = 0$$
$$x^2 = 200$$
$$x = \pm 10\sqrt{2}$$

Since dimension is positive

$$x = 10\sqrt{2}$$
$$= 14.14 \text{ m}$$
$$\therefore y = \frac{200}{10\sqrt{2}}$$
$$= 14.14 \text{ m}$$

Dimensions of a rectangular fence that will minimise the amount of material needed are 14.14 m by 14.14 m.

Back to Unit 4: Assessment

Media Attributions

- Unit 4_Activity_Functions and optimisation © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- Unit 4 Example 4.1 © DHET is licensed under a CC BY (Attribution) license
- Unit 4 Example 4.3 © DHET is licensed under a CC BY (Attribution) license
- Unit 4 Assessment Q2 © DHET is licensed under a CC BY (Attribution) license

SUBJECT OUTCOME VIII

SPACE, SHAPE AND MEASUREMENT: CALCULATE THE SURFACE AREA AND VOLUME OF TWO- AND THREE-DIMENSIONAL SHAPES

Subject outcome

Subject outcome 3.1: Calculate the surface area and volume of two- and three-dimensional shapes

Learning outcomes

- Calculate the surface area and volume of the following geometrical objects:
  - right pyramids (with square, equilateral triangle or regular hexagonal bases)
  - right cones
  - spheres.
- Calculate the surface area and volume of a combination of the above mentioned geometrical objects.

Unit 1 outcomes

By the end of this unit you will be able to:

- Apply the appropriate formulae to find the volume of:
  - right pyramids (with square, equilateral triangle or regular hexagonal bases)
  - right cones
  - spheres.
- Determine and calculate the volume of combinations of these geometric objects.

Unit 2 outcomes

By the end of this unit you will be able to:

- Apply the appropriate formulae to find the surface area of:
  - right pyramids (with square, equilateral triangle or regular hexagonal bases)
  - right cones
  - spheres.
- Determine and calculate the surface area of combinations of these geometric objects.

Unit 1: Finding the volume of 3-D shapes

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Apply the appropriate formulae to find the volume of:
  - right pyramids (with square, equilateral triangle or regular hexagonal bases)
  - right cones
  - spheres.
- Determine and calculate the volume of combinations of these geometric objects.

What you should know

Before you start this unit, make sure you can:

- calculate perimeter and area of two-dimensional shapes, and surface area and volume of prisms. You can revise this in level 2 subject outcome 3.2.

Introduction

Volume is the amount of three-dimensional (3-D) space taken up by a solid, liquid or gas. Volume is measured in cubes or cubic units. The terms volume and capacity are often used to mean the same thing but there is a slight difference.

Capacity is the amount of liquid or other substance a container can hold. Capacity is only used in relation to containers. A can of cooldrink has volume, as it takes up a certain amount of space and has capacity to be filled with liquid. So, when you are filling a container with liquid or other substances you are working with capacity.

We saw in level 2 that we find the volume of right prisms and cylinders by multiplying the area of the base by the height of the 3-D shape. Now, we will find the volume of right pyramids, cones and spheres.

Pyramids

Pyramids have only one polygon base and the side faces are triangles that meet at a vertex called the apex. There are many types of pyramids, and they are named after the shape of their base. We call a pyramid a "right pyramid" if the line between the apex and the centre of the base is perpendicular to the base.

For example, in the **square based pyramid** the base is a square and the sides are isosceles triangles, which meet at the apex.

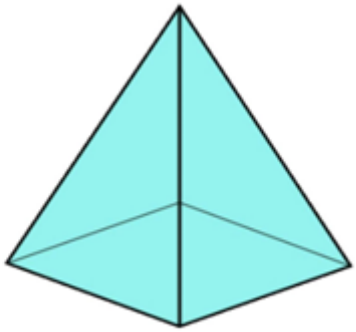


Figure 1: Square based pyramid

A **tetrahedron or triangular pyramid** has a triangular base and sides. It is made up of four equilateral triangles, has six edges and four vertices.

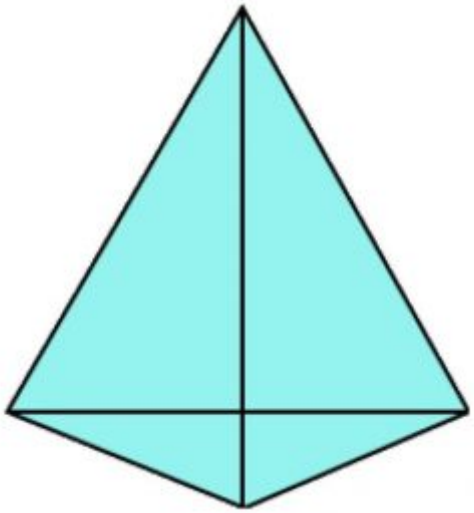


Figure 2: Tetrahedron

A **hexagonal pyramid** has a hexagonal base and six isosceles triangle faces as its sides.

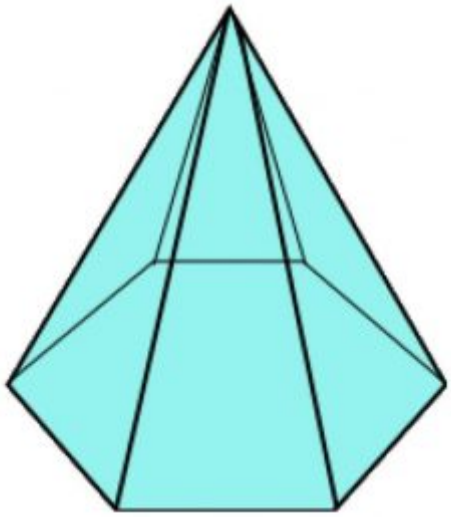


Figure 3: Hexagonal pyramid

To find the volume of a pyramid we divide the area of its base by three and multiply by its height.

$$\text{Volume of pyramid} = \frac{1}{3}\text{area of base} \times \text{height}$$

Example 1.1

Find the volume of a regular square pyramid with base sides $5\ \text{cm}$ and altitude $10\ \text{cm}$.

Solution

First, we find the area of the base, which is a square.

$$\begin{aligned}\text{Area of square base} &= 5\ \text{cm} \times 5\ \text{cm} \\ &= 25\ \text{cm}^2\end{aligned}$$

Altitude is the same as the height. Now we multiply the area of the base by the height of the pyramid and divide by three to find its volume.

$$\begin{aligned}\text{Volume} &= \frac{1}{3}(25\ \text{cm}^2) \times 10\ \text{cm} \\ &= 83.3\ \text{cm}^3\end{aligned}$$

Example 1.2

A paperweight is in the shape of a hexagonal pyramid with base sides $12\ \text{cm}$ and a vertical height of $15\ \text{cm}$. Find the volume of the paperweight.

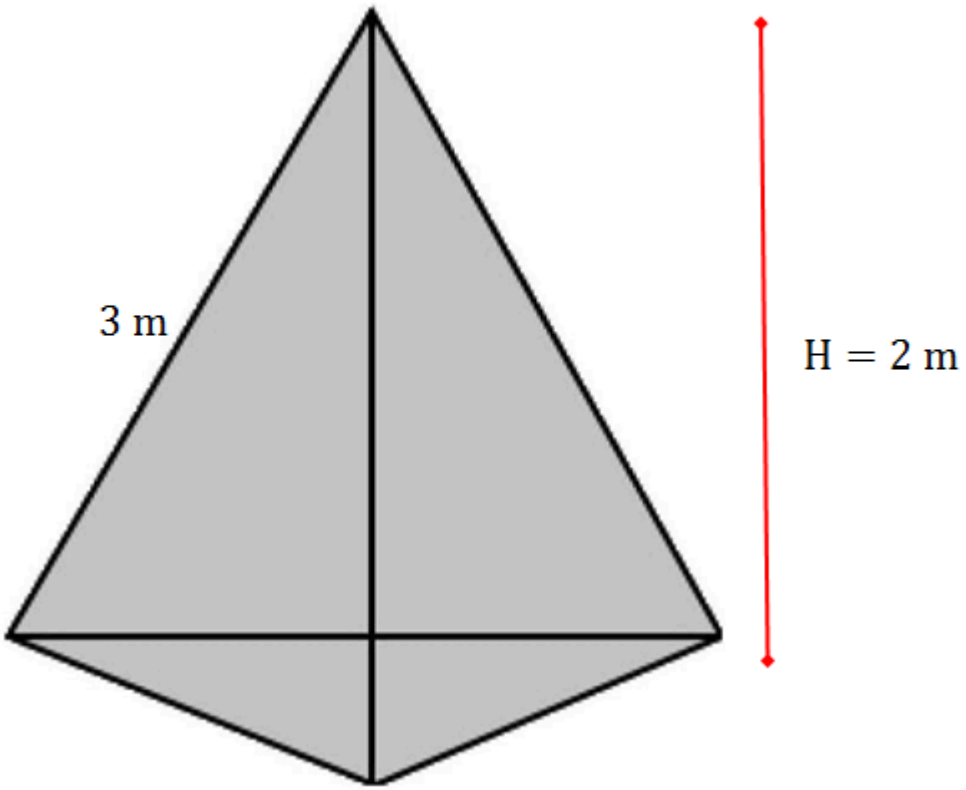
Solution

The area of a hexagon is given by: $\frac{3\sqrt{3}}{2}s^2$.

$$\begin{aligned}\text{Volume} &= \frac{1}{3}\left[\frac{3\sqrt{3}}{2}(12\ \text{cm})^2\right] \times 15\ \text{cm} \\ &= 1\ 870.61\ \text{cm}^3\end{aligned}$$

Exercise 1.1

1. The figure below shows a triangular pyramid shaped container with altitude of $2\ \text{m}$. The side length is $3\ \text{m}$ and the height of the triangle is $2.6\ \text{m}$.

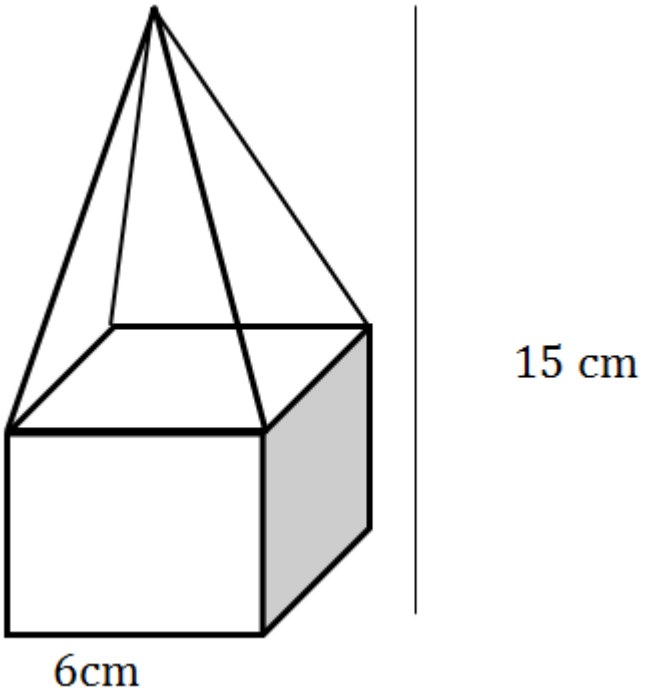


a. Find the volume of the container.

b. The container must be packed into a box. If the box has a volume of $2.5\ \text{m}^3$ will the container fit into the box?

2. We need to fit square pyramid ornaments with base side of $10\ \text{mm}$ and height of $5\ \text{mm}$ into a rectangular box to be transported from Cape Town to Johannesburg. If the box has a length of $25\ \text{cm}$, a breadth of $15\ \text{cm}$and a height of $5\ \text{cm}$, how many ornaments will fit into the box?

3. The solid below is made of a cube and a square pyramid. Determine the volume of the shape.



The full solutions are at the end of the unit.

Cones

Cones are similar to pyramids except that their bases are circles instead of polygons.

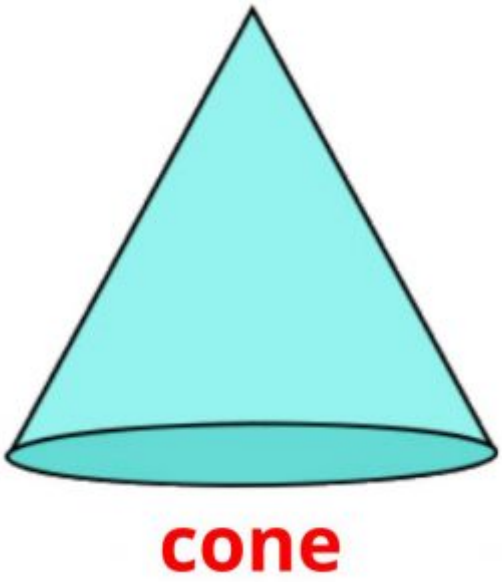


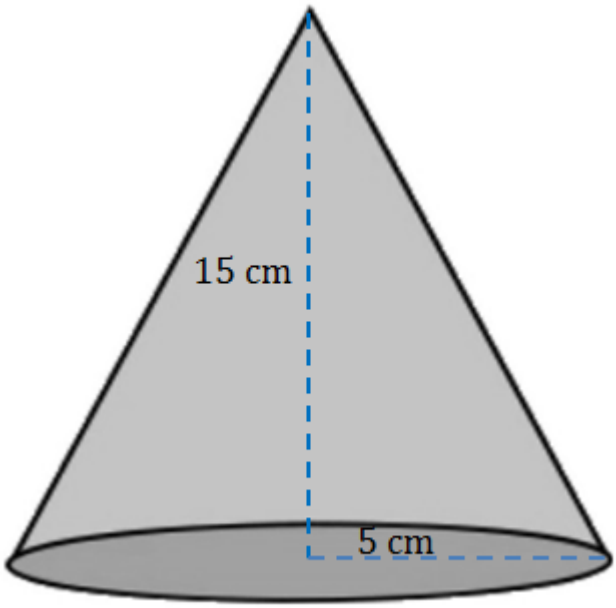
Figure 4: Cone

We find the volume of a cone using a similar formula to that for right pyramids.

$$\text{Volume of cone} = \frac{1}{3}\text{area of base} \times \text{height of cone}$$
$$= \frac{1}{3}\pi r^2 H$$

Example 1.3

Find the volume of the following cone (correct to 1 decimal place).



Solution

First, find the area of the base.

$$A = \pi r^2$$
$$= \pi(5\text{ cm})^2$$
$$= 25\pi\text{ cm}^2$$

Next, calculate the volume of the cone.

$$V = \frac{1}{3}\text{area of base} \times \text{height of cone}$$
$$= \frac{1}{3}(25\pi) \times 15$$
$$= 125\pi\text{ cm}^3$$
$$= 392.7\text{ cm}^3$$

Spheres

Spheres are solids that are perfectly round and look the same from any direction.

Note

Watch this video to see an example of calculating the volume of a sphere.

Calculating the volume of a sphere (Duration: 2.36)

Example 1.4

A world globe is in the shape of a sphere with radius of $12\ \text{cm}$. Find its volume correct to one decimal place.

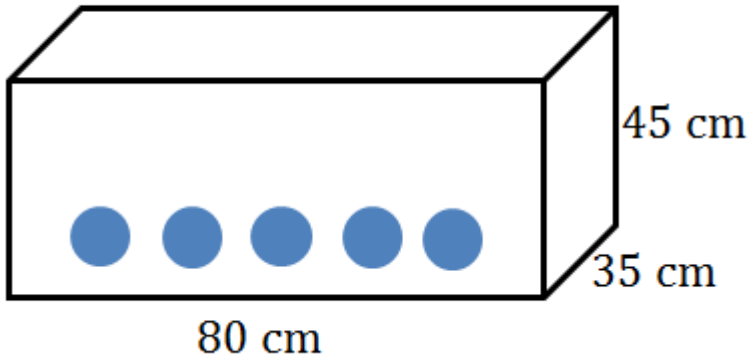


Solution

$$\begin{aligned} V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(12)^3 \\ &= 7238.2\ \text{cm}^3 \end{aligned}$$

Exercise 1.2

1. Tom has 20 new marbles. If the marbles are spherical in shape and each one has a diameter of $4\ \text{mm}$ what is the total volume of Tom's marbles?
2. The vertical height of a cone is 7 units and the slant height is 7.28 units; the radius of the cone is 2 units. Calculate the volume of the cone. Round your answer off to two decimal places.
3. An open rectangular fish tank, $80\ \text{cm}$ long, $35\ \text{cm}$ wide and $45\ \text{cm}$ high, is full to the top with water. John puts 30 marbles, each with a diameter of $1\ \text{cm}$, into the tank resulting in some of the water spilling out of the fish tank.



   a. Determine the initial volume of the water in the tank.
   b. Calculate the volume of one marble.
   c. Calculate the volume of the water that spilled out of the tank after all the marbles have been added in.

The full solutions are at the end of the unit.

Summary

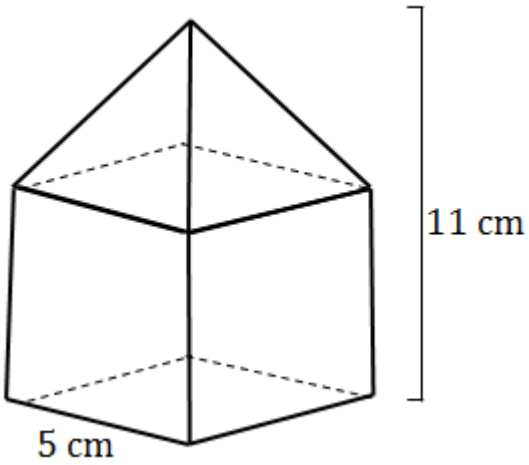
In this unit you have learnt the following:

- How to find the volume of right pyramids.
- How to find the volume of spheres.
- How to find the volume of cones.

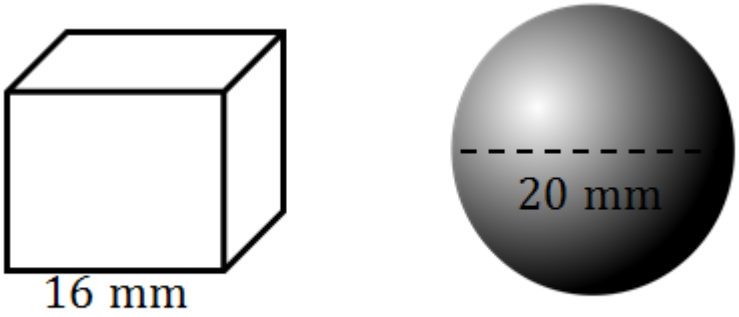
Unit 1: Assessment

Suggested time to complete: 30 minutes

1. A solid is made up of a cube and a square pyramid. Find its volume if the height of the solid is $11\ \text{cm}$ and the sides of the base are $5\ \text{cm}$.



2. An insecticide company has mistakenly manufactured cube-shaped moth balls instead of spherical-shaped moth balls. The cube-shaped moth balls have a side length of $16\ \text{mm}$. The spherical-shaped moth balls must have a diameter of $20\ \text{mm}$.



   Calculate:

   1. The volume of each cube-shaped moth ball.
   2. The volume of one spherical-shaped moth ball.
   3. The extra amount of material required if the original order was for 2 million $(2\ 000\ 000)$spherical moth balls.

3. A solid golden sphere with a diameter of $14\ \text{mm}$ is to be melted and cast into a piece of jewellery. The piece of jewellery is in the shape of a right pyramid with a square base. The square pyramid has a vertical height of $10\ \text{mm}$ and a side length of $20\ \text{mm}$. Calculate the excess gold (gold left over) after the piece of jewellery has been made.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

   a.

$$V = \frac{1}{3}\text{area of base} \times \text{height}$$
$$\therefore V = \frac{1}{3}\left(\frac{1}{2}(3) \times 2.6\right) \times 2$$
$$= 2.6 \text{ m}^3$$

b. The container will not fit into the box since its volume of 2.7 m^3 is greater than the volume of the box.

2. Remember to convert lengths from millimetres to centimetres.

$$V_{\text{pyramid}} = \frac{1}{3}\text{area of base} \times \text{height}$$
$$= \frac{1}{3}(1 \text{ cm} \times 1 \text{ cm}) \times 0.5 \text{ cm}$$
$$= \frac{1}{6} \text{ cm}^3$$
$$V_{\text{box}} = 25 \text{ cm} \times 15 \text{ cm} \times 5 \text{ cm}$$
$$= 1875 \text{ cm}^3$$

Number of ornaments:

$$\frac{V_{\text{box}}}{V_{\text{ornaments}}} = 1875 \div \frac{1}{6}$$
$$= 11\ 250$$

3.

$$V_{\text{cube}} = s^3$$
$$= 6^3$$
$$= 216 \text{ cm}^3$$
$$V_{\text{pyramid}} = \frac{1}{3}(6)^2 \times (15 - 6)$$
$$= 108 \text{ cm}^3$$

Total volume:

$$216 \text{ cm}^3 + 108 \text{ cm}^3 = 324 \text{ cm}^3$$

Back to Exercise 1.1

Exercise 1.2

1. Remember to divide 4 mm by two to get the length of the radius.

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(2)^3$$
$$= \frac{32}{3}\pi \text{ mm}^3 \text{ volume of each marble}$$

Total volume:

$$\frac{32}{3}\pi \text{ mm}^3 \times 20 = 670.2 \text{ mm}^3$$

2.

$$\text{Volume of cone} = \frac{1}{3}\text{area of base} \times \text{height of cone}$$
$$= \frac{1}{3}\pi(2)^2 \times 7$$
$$= 29.32 \text{ cubic units}$$

3.

a. Volume of the water in the tank:

$$
\begin{aligned}
V &= l \times b \times h \\
&= 80 \text{ cm} \times 35 \text{ cm} \times 45 \text{ cm} \\
&= 126\ 000 \text{ cm}^3
\end{aligned}
$$

b. Volume of one marble:
Radius is 0.5 cm

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi(0.5 \text{ cm})^3 \\
&= \frac{1}{6}\pi \text{ cm}^3 \\
&= 0.52 \text{ cm}^3
\end{aligned}
$$

c. Total volume of marbles will give the amount of water that spilled over:
$\frac{1}{6}\pi \text{ cm}^3 \times 30 = 15.71 \text{ cm}^3$

Back to Exercise 1.2

Unit 1: Assessment

1.

$$
\begin{aligned}
V_{\text{cube}} &= s^3 \\
&= 5^3 \\
&= 125 \text{ cm}^3 \\
V_{\text{pyramid}} &= \frac{1}{3}(5)^2 \times (11 - 5) \\
&= 50 \text{ cm}^3
\end{aligned}
$$

Total volume:
$125 \text{ cm}^3 + 50 \text{ cm}^3 = 175 \text{ cm}^3$

2.

a.

$$
\begin{aligned}
V_{\text{cube}} &= s^3 \\
&= 16^3 \\
&= 4\ 096 \text{ mm}^3
\end{aligned}
$$

b.

Diameter of 20 mm. ∴ radius of 10 mm.

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi(10)^3 \\
&= \frac{4\ 000}{3}\pi \text{ mm}^3 \\
&= 4188.79 \text{ mm}^3 \text{ correct to 2 decimal places}
\end{aligned}
$$

c.

$2\ 000\ 000 \times (\frac{4000}{3}\pi \text{ mm}^3 - 4\ 096 \text{ mm}^3) = 185\ 580\ 409 \text{ mm}^3$

3.

$$
\begin{aligned}
V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi(7)^3 \\
&= \frac{1\ 372}{3}\pi\ \text{mm}^3
\end{aligned}
$$

$$
\begin{aligned}
V_{\text{pyramid}} &= \frac{1}{3}(20)^2 \times 10 \\
&= \frac{4\ 000}{3}\ \text{mm}^3
\end{aligned}
$$

$$\text{Excess gold:}$$

$$\frac{1\ 372}{3}\pi\ \text{mm}^3 - \frac{4\ 000}{3}\ \text{mm}^3 = 103.42\ \text{mm}^3$$

Back to Unit 1: Assessment

Media Attributions

- Square pyramid © kids-flashcards.com is licensed under a All Rights Reserved license
- triangular pyramid © kids-flashcards.com is licensed under a All Rights Reserved license
- hexagonal pyramid © kids-flashcards.com is licensed under a All Rights Reserved license
- Ex 1.1 Q1 © kids-flashcards.com is licensed under a All Rights Reserved license
- Ex 1.1 Q3 © kids-flashcards.com is licensed under a All Rights Reserved license
- cone © kids-flashcards.com is licensed under a All Rights Reserved license
- Example 1.3 © kids-flashcards.com is licensed under a All Rights Reserved license
- Example 1.4 © kids-flashcards.com is licensed under a All Rights Reserved license
- Ex 1.2 Q3 © DHET is licensed under a CC BY (Attribution) license
- Assess Q1 © DHET is licensed under a CC BY (Attribution) license
- Assess Q2 © DHET is licensed under a CC BY (Attribution) license

Unit 2: Finding the surface area of 3-D shapes

NATASHIA BEARAM-EDMUNDS



Unit outcomes

By the end of this unit you will be able to:

- Apply the appropriate formulae to find the surface area of:
  - right pyramids (with square, equilateral triangle or regular hexagonal bases)
  - right cones
  - spheres.
- Determine and calculate the surface area of combinations of these geometric objects.

What you should know

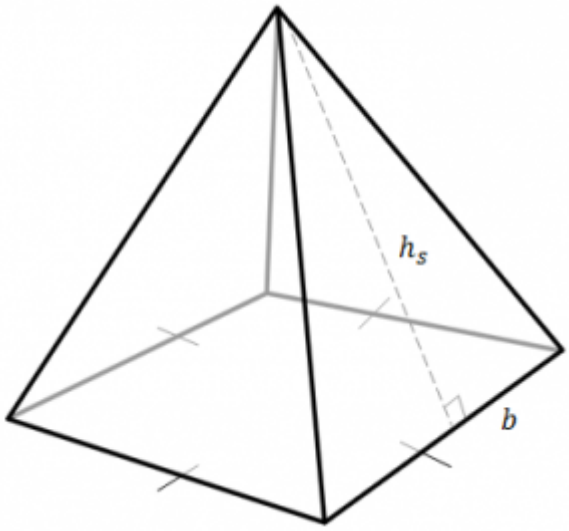
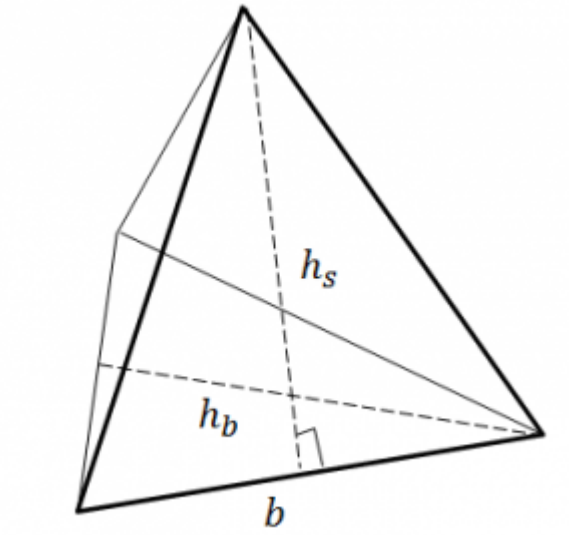
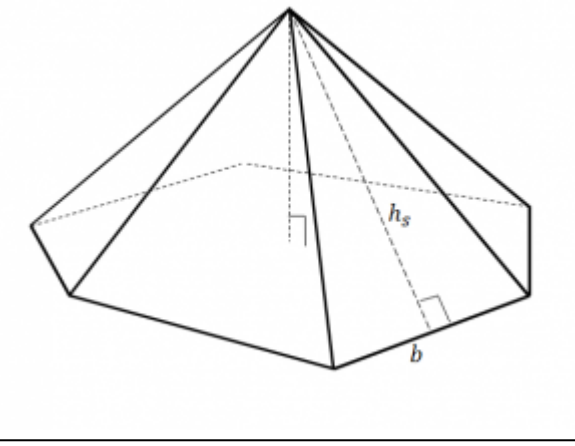
Before you start this unit, make sure you can:

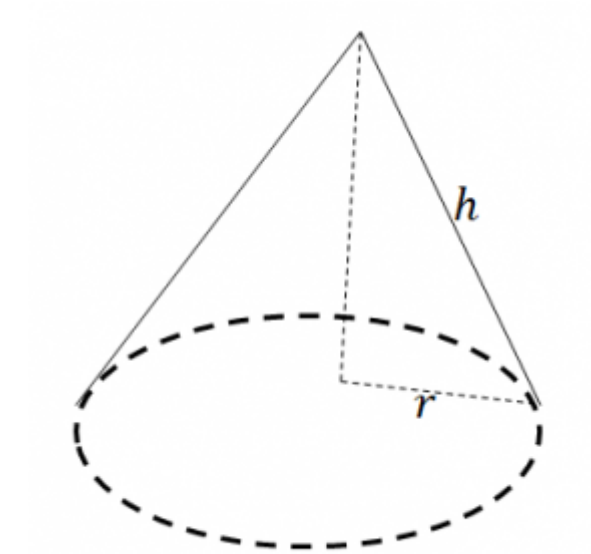
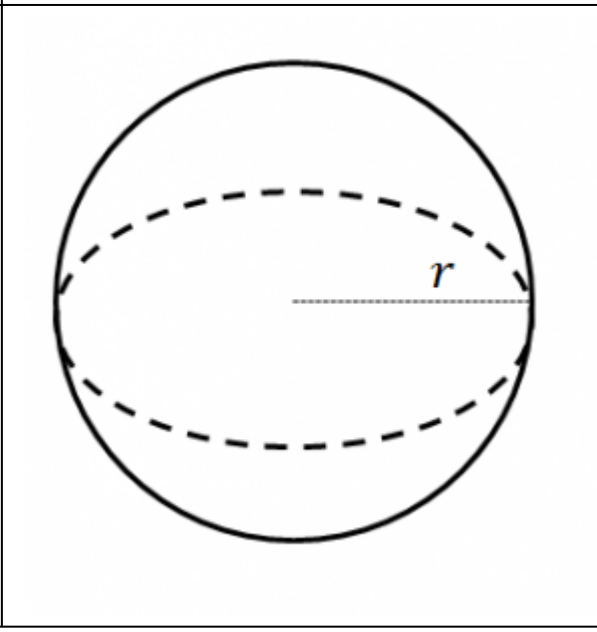
- find the surface area of right prisms. You can revise surface area of prisms in level 2 subject outcome 3.2 unit 3.

Introduction

The total surface area (TSA) of a 3-D shape is the sum of the areas of all its exposed faces (or outer surfaces). In unit 1 of this subject outcome we calculated the volume of right pyramids, cones and spheres. Now, we will find the surface area of these shapes.

Below is a useful table of formulae we can use to find the surface area of right pyramids, cones and spheres.

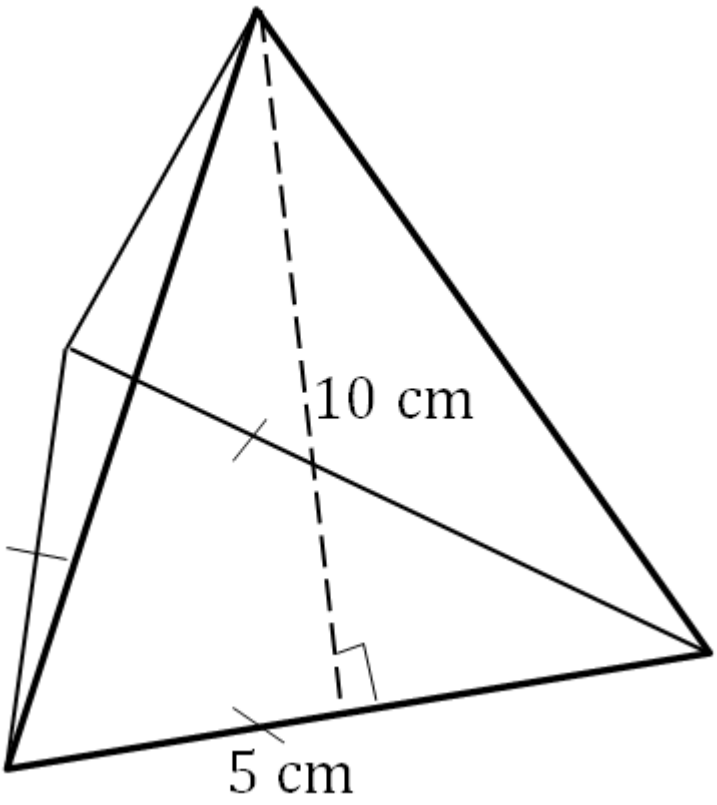
Square pyramid		$\text{TSA} = \text{area of base} + \text{area of triangular sides}$ $= b^2 + \left(4\frac{1}{2}bh_s\right)$ h_s is the slant height $= b(b + 2h_s)$
Triangular pyramid/ tetrahedron		$\text{TSA} = \text{area of base} + \text{area of triangular sides}$ $= \left(\frac{1}{2}b \times h_b\right) + 3\left(\frac{1}{2}b \times h_s\right)$
Hexagonal pyramid		$\text{TSA} = \text{area of base} + \text{area of triangular sides}$ $= \frac{3\sqrt{3}}{2}b^2 + 6\left(\frac{1}{2}b \cdot h_s\right)$ $= \frac{3\sqrt{3}}{2}b^2 + 3b \cdot h_s$ $= 3b\left(\frac{\sqrt{3}}{2}b + h_s\right)$

Cone		$\begin{aligned}\text{TSA} &= \text{area of base} + \text{area of walls} \\ &= \pi r^2 + \frac{1}{2} \times 2\pi r h \\ &= \pi r(r + h)\end{aligned}$
Sphere		$\text{TSA} = 4\pi r^2$



Example 2.1

Find the surface area of the following pyramid (correct to two decimal places):



Solution

Here we are given a triangular pyramid with the slant height and length of a base side. If we use the formula:

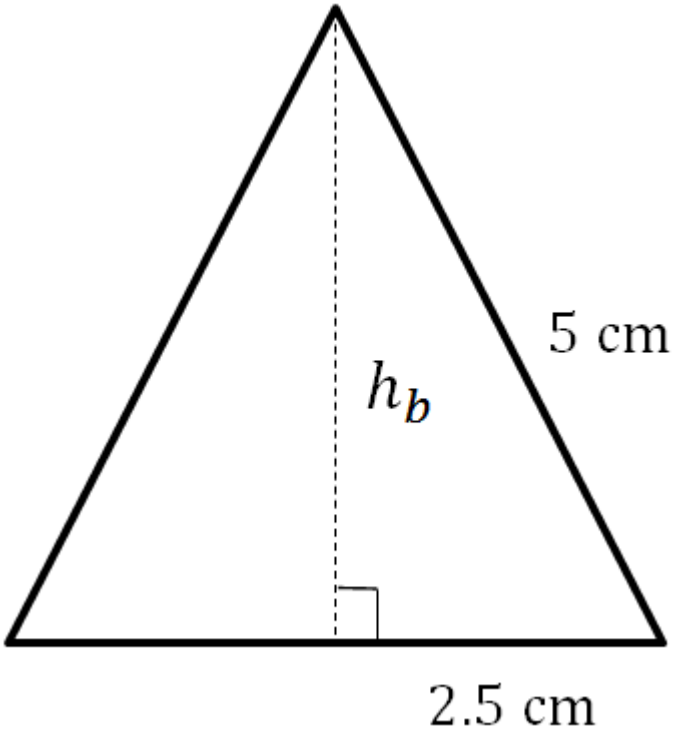
$$\begin{aligned}\text{TSA} &= \text{area of base} + \text{area of triangular sides} \\ &= (\frac{1}{2}b \times h_b) + 3(\frac{1}{2}b \times h_s)\end{aligned}$$

we see that we need to calculate the length of the height of the base triangle to find the area of the base triangle.

Step 1: Find the area of the base

$$\text{Area of base triangle} = (\frac{1}{2}b \times h_b)$$

To find the height of the base triangle h_b we use the theorem of Pythagoras.



Note that: the altitude of an equilateral triangle bisects its base.

$$5^2 = {h_b}^2 + (2.5)^2$$
$$h_b = \sqrt{5^2 - (2.5)^2}$$
$$= \sqrt{\frac{75}{4}}$$
$$= \frac{5\sqrt{3}}{2}$$
$$\text{Area of base triangle} = \left(\frac{1}{2}(5) \times \frac{5\sqrt{3}}{2}\right)$$
$$= \frac{25\sqrt{3}}{4} \text{ cm}^2$$

Step 2: Find the area of the sides

$$\text{Area of sides} = 3\left(\frac{1}{2}b \times h_s\right)$$
$$= 3\left(\frac{1}{2}(5) \times 10\right)$$
$$= 75 \text{ cm}^2$$

Step 3: Find the sum of the areas to find the total surface area

$$\frac{25\sqrt{3}}{4} \text{ cm}^2 + 75 \text{ cm}^2 = 85.83 \text{ cm}^2$$

The surface area of the triangular pyramid is 85.83 cm^2.

Example 2.2

Find the surface area of a square based pyramid with base length $6\ \text{cm}$ and slant height of $11\ \text{cm}$.

Solution

We use the formula:

$$\begin{aligned}\text{TSA} &= \text{area of base} + \text{area of triangular sides} \\ &= b^2 + 4\left(\frac{1}{2}bh_s\right) \quad h_s \text{ is the slant height} \\ &= b(b + 2h_s)\end{aligned}$$

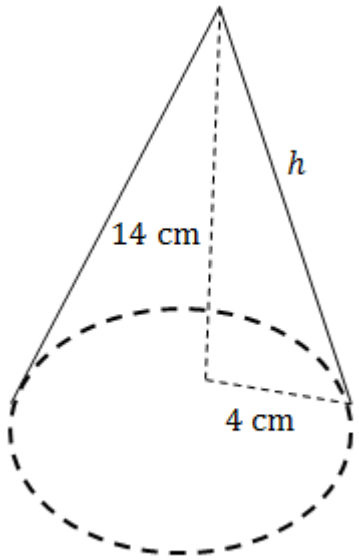
$b = 6\ \text{cm}\ h_s = 11\ \text{cm}$

$$\begin{aligned}\text{TSA} &= b(b + 2h_s) \\ &= 6(6 + 2 \times 11) \\ &= 168\ \text{cm}^2\end{aligned}$$

The surface area of the square based pyramid is $168\ \text{cm}^2$.

Example 2.3

Find the surface area of the following cone (correct to 1 decimal place):



Solution

Write down the formula.

$$\begin{aligned}\text{TSA} &= \text{area of base} + \text{area of walls} \\ &= \pi r^2 + \frac{1}{2} \times 2\pi rh \\ &= \pi r(r + h)\end{aligned}$$

To find the slant height we use the theorem of Pythagoras.

$$\begin{aligned}h^2 &= 14^2 + 4^2 \\ &= 212 \\ h &= 2\sqrt{53}\end{aligned}$$

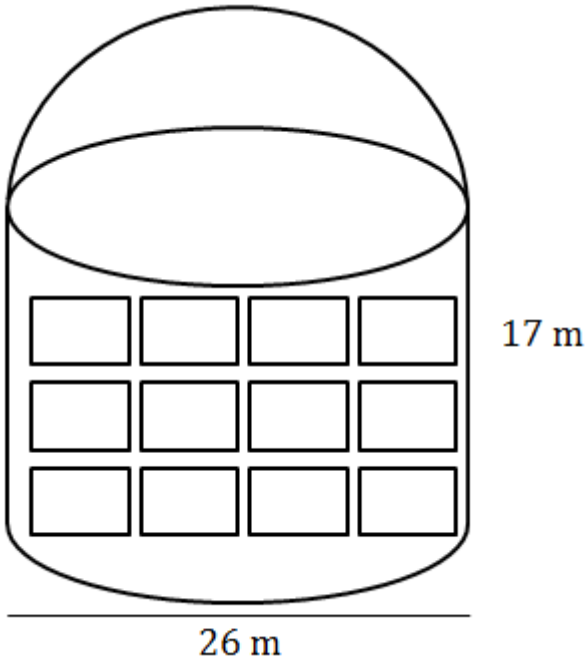
Substitute the values for r and h into the formula.

$$\begin{aligned}\text{TSA} &= \pi r(r + h) \\ &= \pi(4)(4 + 2\sqrt{53}) \\ &= 233.2 \text{ cm}^2\end{aligned}$$

The surface area of the cone is 233.2 cm^2.

Exercise 2.1

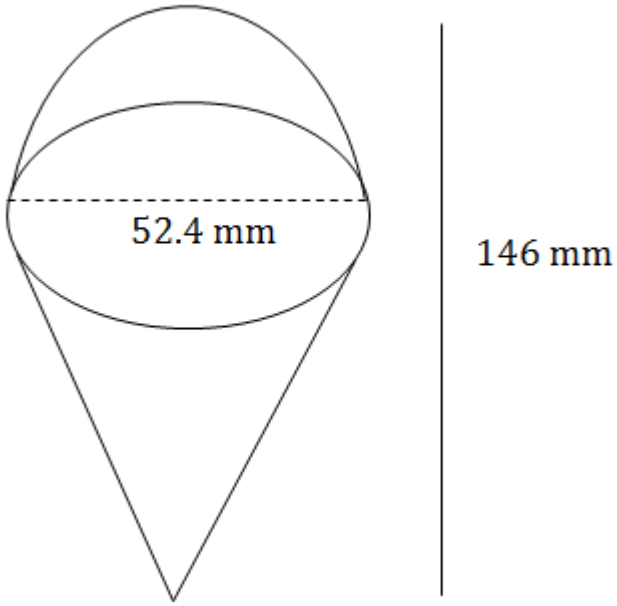
1. The Southern African Large Telescope (SALT) is housed in a cylindrical building with a domed roof in the shape of a hemisphere. The height of the building's wall is 17 m and the diameter is 26 m.



   a. Calculate the total surface area of the building. (HINT: The surface area of a hemisphere is half the surface area of a sphere).

   b. Calculate the total volume of the building.

2. An ice-cream cone has a diameter of 52.4 mm and a total height of 146 mm. Calculate the surface area of the ice-cream and the cone.



3. A pyramid with a square base has base sides each of 4 units long. The vertical height of the pyramid is 8.77 units, and the slant height of the pyramid is 9 units. Determine the surface area of the pyramid.

The full solutions are at the end of the unit.

Summary

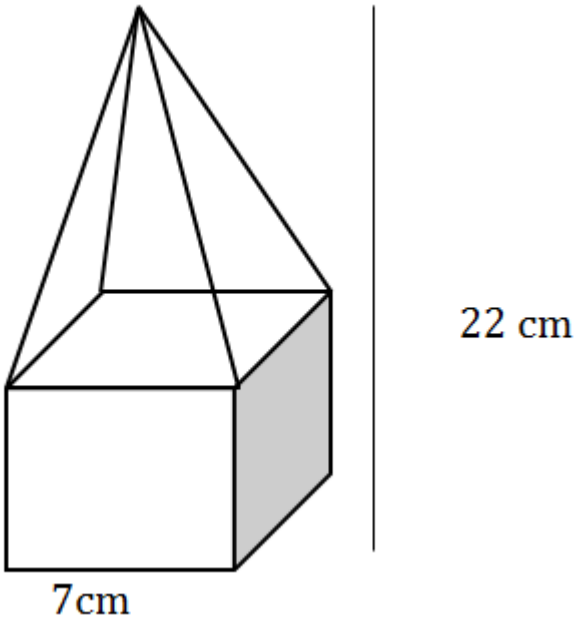
In this unit you have learnt the following:

- How to find the surface area of right pyramids.
- How to find the surface area of cones.
- How to find the surface area of spheres.

Unit 2: Assessment

Suggested time to complete: 20 minutes

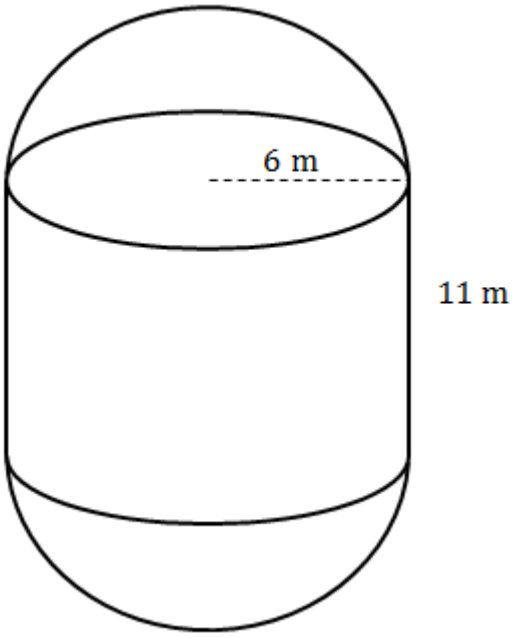
1. The solid below is made of a cube and a square pyramid:



a. Find the surface area of the solid. Give your answers to two decimal places.

b. Now determine the volume of the shape. Give your answer to the nearest integer value.

2. Find the surface area of the following capsule:



3. Sammy needs to cover a pyramid with a hexagonal base and sides of length of $550\ \text{mm}$. The slant height of the pyramid is $200\ \text{mm}$. If the cost of the material needed to cover the pyramid is $\text{R}100$ per square metre. What is the total cost to cover the pyramid?

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

a. $\text{Total surface area} = \text{surface area of dome} + \text{surface area of cylinder}$

$$\begin{aligned}\text{TSA} &= \frac{1}{2}(4\pi r^2) + 2\pi rh \\ &= \frac{1}{2}[4\pi(13)^2] + 2\pi(13)(17) \\ &= 780\pi \text{ m}^2 \\ &= 2\ 450 \text{ m}^2\end{aligned}$$

b. $\text{Total volume} = \text{volume of dome} + \text{volume of cylinder}$

$$\begin{aligned}V &= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \pi r^2 h \\ &= \frac{2}{3}\pi(13)^3 + \pi(13)^2(17) \\ &= 13\ 627 \text{ m}^3\end{aligned}$$

2.

The vertical height of the cone is $146 \text{ mm} - 52.4 \text{ mm} = 93.6 \text{ mm}$.
Work out the slant height using the theorem of Pythagoras.

$$\begin{aligned}{h_s}^2 &= (26.2)^2 + (93.6)^2 \\ h_s &= \sqrt{9447.4} \\ &= 97.19... \text{ mm}\end{aligned}$$

For the surface area of the cone we consider only the surface area of the walls that make up the cone.
$\text{Total surface area} = \text{surface area of hemisphere} + \text{surface area of cone}$

$$\begin{aligned}\text{TSA} &= \frac{1}{2}(4\pi r^2) + \pi rh \\ &= \frac{1}{2}(4\pi(26.2)^2) + \pi(26.2)(97.19...) \\ &= 12\ 313.35 \text{ mm}^2\end{aligned}$$

3.

$$\begin{aligned}\text{TSA} &= \text{area of base} + \text{area of triangular sides} \\ &= b^2 + 4(\frac{1}{2}bh_s) \ h_s \text{ is the slant height} \\ &= b(b + 2h_s) \\ &= 4(4 + 2 \cdot 9) \\ &= 88 \text{ units squared}\end{aligned}$$

Back to Exercise 2.1

Unit 2: Assessment

1.

a. Note: we must include only the exposed surfaces in the calculation of surface area.
First, find h_s by using the theorem of Pythagoras.
$22 \text{ cm} - 7 \text{ cm} = 15 \text{ cm}$ is the vertical height of the pyramid

$$h_s{}^2 = 15^2 + 3.5^2$$
$$h_s = \sqrt{\frac{949}{4}}$$
$$= \frac{\sqrt{949}}{2} \text{ cm}$$

Next, find the total surface area.

$$\text{TSA} = \text{Surface area of square base pyramid} + \text{surface area of cube}$$
$$\text{TSA} = 4(\frac{1}{2}bh_s) + [b^2 + 4bh]$$
$$= 2(7 \cdot \frac{\sqrt{949}}{2}) + [(7^2) + 4(7)7]$$
$$= 460.64 \text{ cm}^2$$

b. $\text{Total volume} = \text{volume of pyramid} + \text{volume of cube}$

$$V = \frac{b^2h}{3} + b^3$$
$$= \frac{(7)^2(15)}{3} + (7)^3$$
$$= 588 \text{ cm}^3$$

2. $\text{Total surface area} = \text{surface area of sphere} + \text{surface area of cylinder}$

$$\text{TSA} = 4\pi r^2 + 2\pi rh$$
$$= 4\pi(6)^2 + 2\pi(6)(11)$$
$$= 276\pi$$
$$= 867.08 \text{ m}^2$$

3.

$$\text{TSA} = \text{area of base} + \text{area of triangular sides}$$
$$= \frac{3\sqrt{3}}{2}b^2 + 6\left(\frac{1}{2}b \cdot h_s\right)$$
$$= \frac{3\sqrt{3}}{2}b^2 + 3b \cdot h_s$$
$$= 3b\left(\frac{\sqrt{3}}{2}b + h_s\right)$$
$$= 3(550)(\frac{\sqrt{3}}{2}(550) + 200)$$
$$= 1\ 115\ 918.05 \text{ mm}^2$$

Next, covert from mm^2 to m^2

$$1 \text{ m}^2 = 1\ 000\ 000 \text{ mm}^2$$
$$1\ 115\ 918.05 \text{ mm}^2 = 1.115... \text{ m}^2$$

$$\text{Total cost} = 100 \times 1.\ 115...$$
$$= \text{R}111.59$$

Back to Unit 2: Assessment

Media Attributions

- Fig 1 Square pyramid © "svg-WikimediaCommons " is licensed under a Public Domain license
- Fig 2 Triangular pyramid © DHET is licensed under a CC BY (Attribution) license
- Fig 3 Hex pyramid © DHET is licensed under a CC BY (Attribution) license
- Fig 4 cone © DHET is licensed under a CC BY (Attribution) license
- Fig 5 sphere © DHET is licensed under a CC BY (Attribution) license

- Fig 6 Example 2.1 © DHET is licensed under a CC BY (Attribution) license
- Fig 7 Example 2.1 © DHET is licensed under a CC BY (Attribution) license
- Fig 8 Example 2.3 © DHET is licensed under a CC BY (Attribution) license
- Fig 9 Ex 2.1 Q1 © DHET is licensed under a CC BY (Attribution) license
- Fig 10 Ex 2.1 Q2 © DHET is licensed under a CC BY (Attribution) license
- Fig 11 Assess Q1 © DHET is licensed under a CC BY (Attribution) license
- Fig 12 Assess Q2 © DHET is licensed under a CC BY (Attribution) license

SUBJECT OUTCOME IX

SPACE, SHAPE AND MEASUREMENT: USE THE CARTESIAN CO-ORDINATE SYSTEM TO DERIVE AND APPLY EQUATIONS

Subject outcome

Subject outcome 3.2: Use the Cartesian co-ordinate system to derive and apply equations

Learning outcomes

- Use the Cartesian co-ordinate system to derive the equation of a line through two given points.
- Use the Cartesian co-ordinate system to derive the equation of a line parallel or perpendicular to another line.
- Use the Cartesian co-ordinate system to derive and use the angle of inclination of a line.

Unit 1 outcomes

By the end of this unit you will be able to:

- Use the Cartesian co-ordinate system to derive the equation of a line through two given points using:
  - the two-point form
  - the gradient-point form
  - the gradient-intercept form.

Unit 2 outcomes

By the end of this unit you will be able to:

- Determine the equation of a straight line parallel to another straight line on the Cartesian plane.
- Determine the equation of a straight line perpendicular to another straight line on the Cartesian plane.

Unit 3 outcomes

By the end of this unit you will be able to:

- Use the gradient of a straight line to determine the angle of inclination of the line with the x-axis.
- Use the angle of inclination to determine the gradient of a straight line.

Unit 1: Finding the equation of a line

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use the Cartesian co-ordinate system to derive the equation of a line through two given points using:
  - the two-point form
  - the gradient-point form
  - the gradient-intercept form.

What you should know

Before you start this unit, make sure you can:

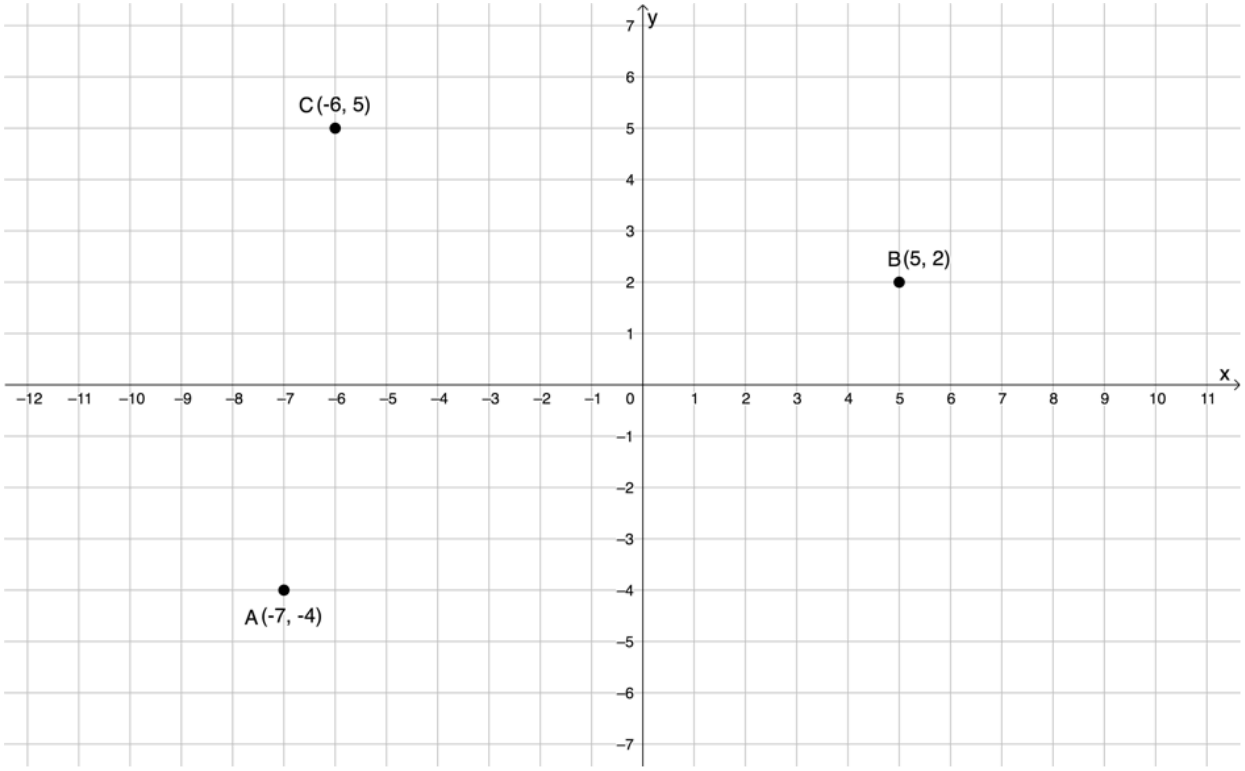
- Plot points on the Cartesian plane. Refer to level 2 subject outcome 3.3 unit 1 if you need help with this.
- Calculate the distance, gradient and midpoint between two points on the Cartesian plane. Refer to level 2 subject outcome 3.3 unit 2 if you need help with this.
- Solve a system of two linear equations simultaneously. Refer to level 2 subject outcome 2.3 unit 4 if you need help with this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

1. Plot the following points on a Cartesian plane:
   1. $A(-7,-4)$
   2. $B(5,2)$
   3. $C(-6,5)$
2. Calculate the distance AB.
3. Calculate the midpoint of BC.
4. Calculate the gradient of AC.

Solutions

1.



2.

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5 - (-7))^2 + (2 - (-4))^2} \\ &= \sqrt{12^2 + 6^2} \\ &= \sqrt{144 + 36} \\ &= \sqrt{180} \\ &= \sqrt{36 \times 5} \\ &= 6\sqrt{5} \end{aligned}$$

3.

$$\begin{aligned} M_{BC} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= \left(\frac{5 - 6}{2}, \frac{2 + 5}{2} \right) \\ &= \left(-\frac{1}{2}, 3\frac{1}{2} \right) \end{aligned}$$

4.

$$\begin{aligned} m_{AC} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - (-4)}{-6 - (-7)} \\ &= \frac{9}{1} \\ &= 9 \end{aligned}$$

Introduction

In level 2 subject outcome 3.3 we started to learn about an area of mathematics called analytical geometry. Analytical geometry is the study of geometric objects and shapes (points, lines quadrilaterals, etc.) on the

Cartesian plane. In that subject outcome we learnt some of the basics involved, such as how to plot points in the Cartesian plane, how to find the distance between any two points, how to find the midpoint between any two points and how to find the gradient of the line between any two points.

Take note!

The distance formula (to calculate the distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$) is:
$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Take note!

The midpoint formula (to calculate the midpoint between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$) is:
$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Take note!

The gradient formula (to calculate the gradient between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$) is:
$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

A great deal of analytical geometry relies on us being able to find the equations of straight lines and line segments between different points. Therefore, in this unit, we are going to learn about three different methods to find the equation of any straight line or line segment on the Cartesian plane.

Take note!

A **straight line** is infinite and has no beginning or end. A **line segment** is a finite portion of a straight line between two specific points.

The first of these methods is useful when we know any two points that lie on the straight line.

Find the equation of a straight line using the two-point method

We already know that the gradient between any two points A and B on the Cartesian plane is given by $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$.

Figure 1 shows a situation where we know the coordinates of two points on a straight line. These are the points $A(2, 5)$ and $B(-2, -3)$.

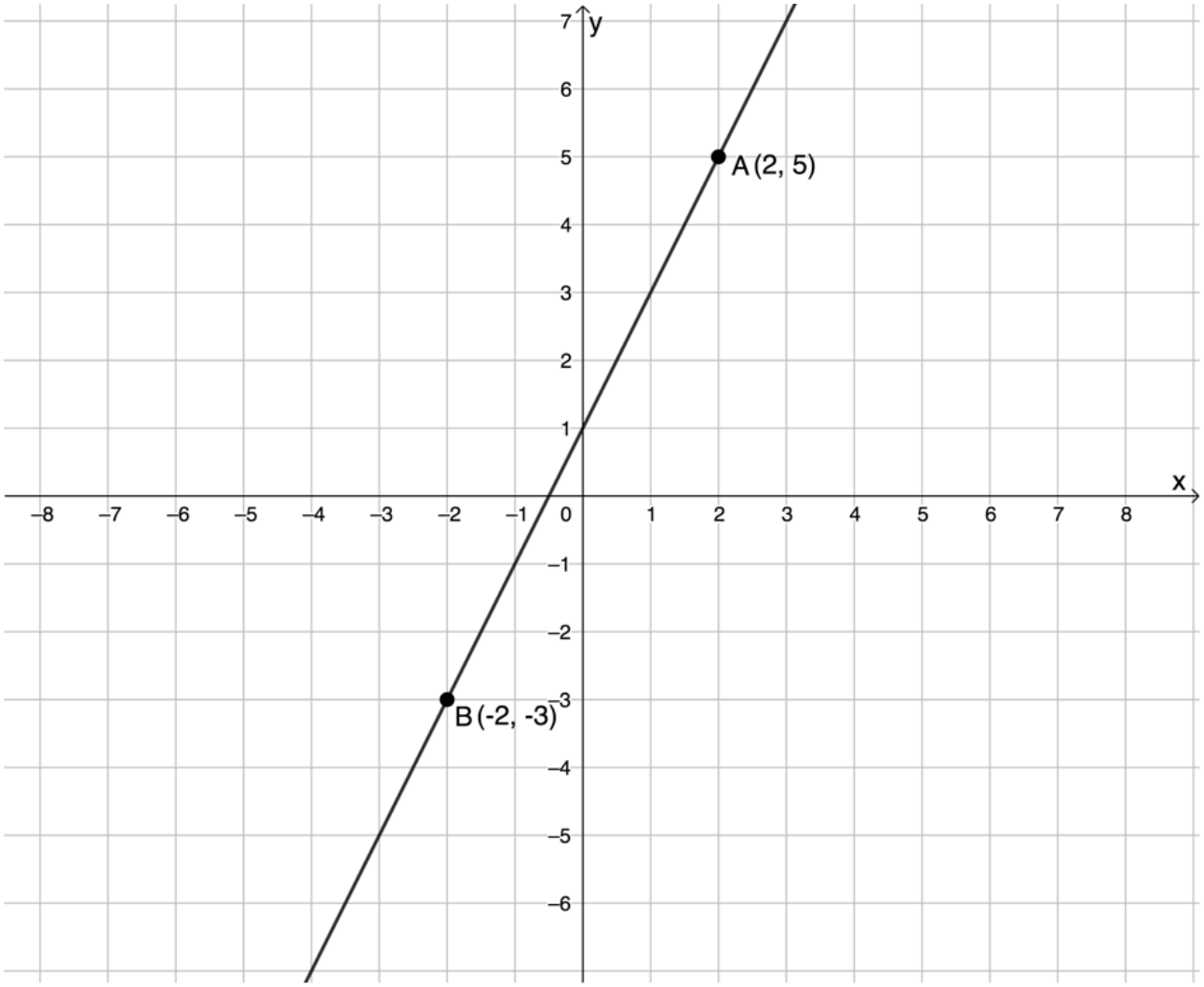


Figure 1: Straight line with two known points $A(2, 5)$ and $B(-2, -3)$

However, we need to find the equation that describes the relationship between the x- and y-coordinates of every point on that line.

Suppose we choose any random point on this straight line, say $C(x, y)$. We know that the gradient between this point and either of the other two points will be the same as the gradient between points A and B. All the points lie on the same straight line and the gradient is constant (see Figure 2).

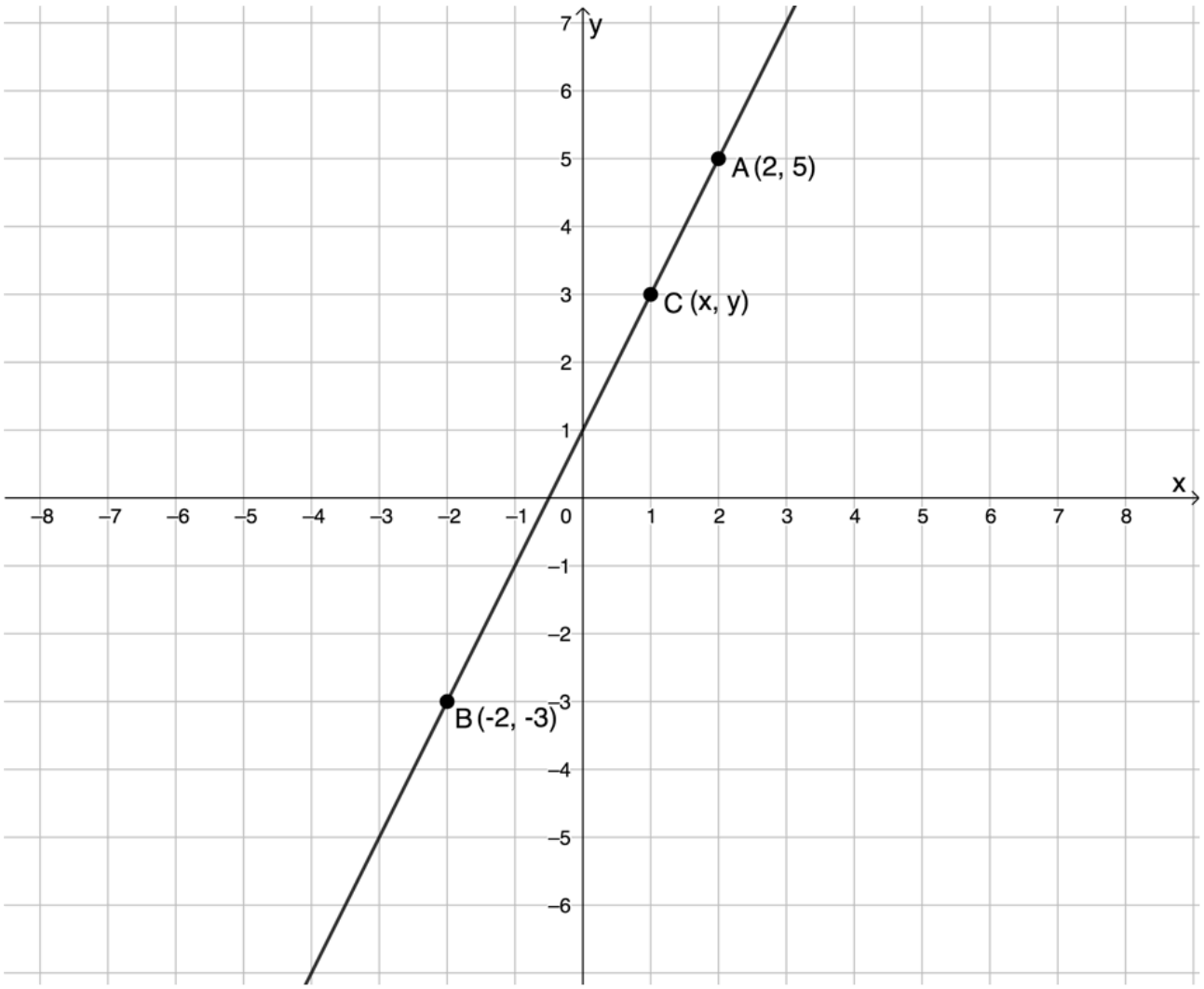


Figure 2: Straight line with unknown point C

Therefore, we can say that $m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$ is equal to $m_{AC} = \dfrac{y - y_1}{x - x_1}$.

In other words, $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

If we substitute in the values we have, and rearrange the equation into the familiar $y = mx + c$ form of the straight line we get:

$$\frac{y - 5}{x - 2} = \frac{-3 - 5}{-2 - 2}$$

$$\therefore \frac{y - 5}{x - 2} = \frac{8}{4}$$

$$\therefore y - 5 = 2(x - 2)$$

$$\therefore y - 5 = 2x - 4$$

$$\therefore y = 2x + 1$$

This is the equation of the straight line that passes through points A and B. Have a closer look at figure 2 to satisfy yourself that the gradient of this line is 2 and that the y-intercept is 1.

So, when we are given any two points on a straight line we can find the equation of that line by using the two-point form of the straight line equation.

The two-point form of the straight-line equation:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

Example 1.1

Find the equation of the straight line passing through $R(3,-4)$ and $S(-5,4)$.

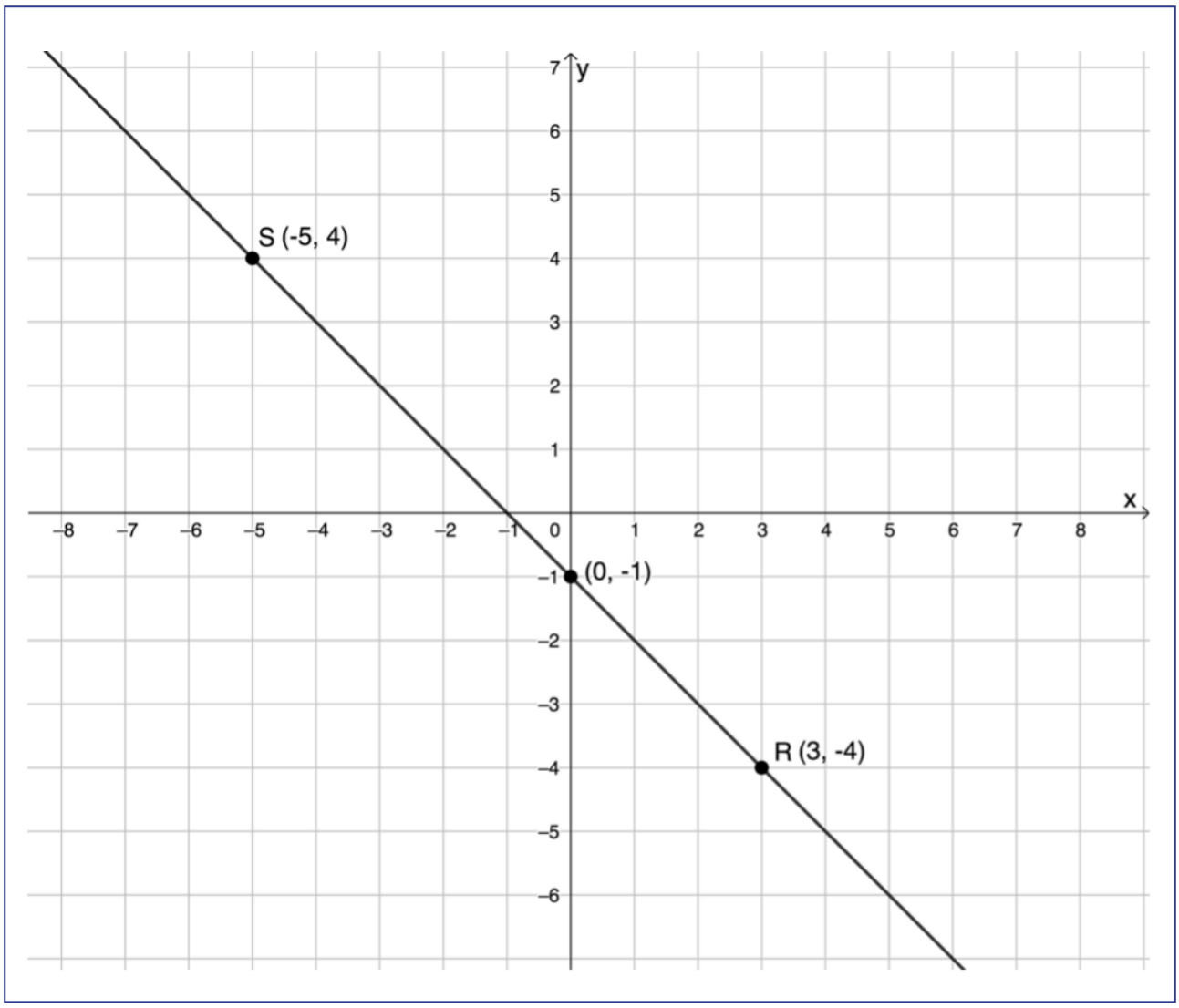
Solution

Because we have been given two points on the straight line, we can use the two-point form of the straight line equation.

Let $R(3,-4)$ be (x_1,y_1). Let $S(-5,4)$ be (x_2,y_2).

$$\begin{aligned}
&\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}\\
\therefore\ &\frac{y-(-4)}{x-3}=\frac{4-(-4)}{-5-3}\\
\therefore\ &\frac{y+4}{x-3}=-\frac{8}{8}=-1\\
\therefore\ &y+4=-(x-3)\\
\therefore\ &y+4=-x+3\\
\therefore\ &y=-x-1
\end{aligned}$$

It is always a good idea to make a quick sketch of the points and join them with a straight line to make sure that you are correct. Here we can see that the gradient is indeed -1 and the y-intercept is at -1.



Exercise 1.1

Determine the equations of the straight lines passing through the following points:

1. $(3, 7)$ and $(-6, 1)$
2. $(3, 5)$ and $(2, 3)$
3. $(-7, -5)$ and $(1, -5)$
4. (a, b) and (b, a)
5. $(0, r)$ and $(2s, -r)$

The full solutions are at the end of the unit.

Find the equation of a straight line using the gradient-point method

From the previous section, we know that we can find the equation of a straight line between any two points by using the two-point form of the equation of a straight line.

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

But what if, we are given the gradient of the line and any one point on the line? If we know the gradient, we can replace $\frac{y_2-y_1}{x_2-x_1}$ in the two-point form with the value of m. The equation then becomes $\frac{y-y_1}{x-x_1}=m$ which we can simplify to $y-y_1=m(x-x_1)$. This is the gradient-point form of the straight-line equation.

The gradient-point form of the straight-line equation:
$y-y_1=m(x-x_1)$



Example 1.2

Determine the equation of the straight line with a gradient of $\frac{3}{2}$ passing through the point $(2,-7)$.

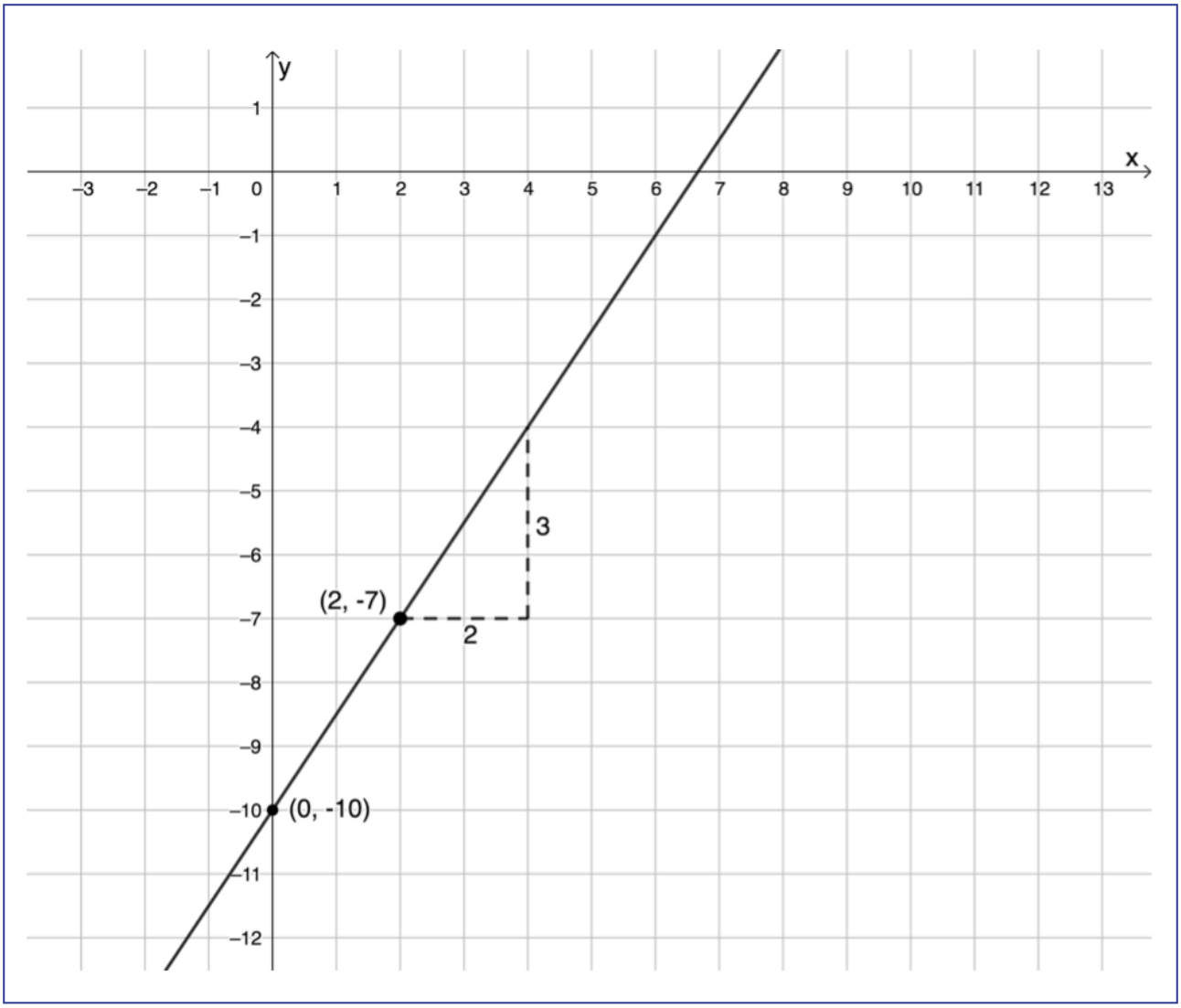
Solution

Because we have been given the gradient and one point on the straight line, we can use the gradient-point form of the straight line equation.

$m=\frac{3}{2}$ and (x_1,y_1) is $(2,-7)$

$$\begin{aligned} y-y_1&=m(x-x_1)\\ \therefore y-(-7)&=\frac{3}{2}(x-2)\\ \therefore y+7&=\frac{3}{2}x-3\\ \therefore y&=\frac{3}{2}x-10 \end{aligned}$$

It is a good idea to make a sketch of the straight line to confirm that your equation is correct.



Exercise 1.2

Determine the equations of the following straight lines:

1. passing through the point $(-3, 4)$ with a gradient of -2
2. with a gradient of 0.75 and passing through $(2, -3)$
3. where $m = -\frac{4}{5}$ passing through $\left(\frac{3}{4}, \frac{1}{4}\right)$
4. with a zero gradient and passing through (a, b)
5. with an undefined gradient and passing through $(-4, y)$
6. passing through the point $(-2, -6a + b)$ with a gradient of $m = -2a$

The full solutions are at the end of the unit.

Find the equation of a straight line using the gradient-intercept form of the straight line equation

Obviously, we can always use the **standard form** of the straight-line equation – $y = mx + c$ (or $y = ax + q$). This is the one that is probably most familiar to you. If you are ever given the gradient and the y-intercept, it is simply a matter to substitute these values into the equation.

However, questions will seldom be this straightforward!

Example 1.3

Determine the equation of the straight line passing through $(-1, 2)$ with a gradient $m = \frac{2}{3}$ using the gradient-intercept form of the straight line equation.

Solution

The gradient intercept form is $y = mx + c$. We know that $m = \frac{2}{3}$. Therefore, $y = \frac{2}{3}x + c$.

We have also been given the point $(-1, 2)$ that lies on the line, so we can substitute these values in for x and y and solve the equation for c.

$$y = \frac{2}{3}x + c$$
$$\therefore 2 = \frac{2}{3}(-1) + c$$
$$\therefore 2 + \frac{2}{3} = c$$
$$\therefore c = \frac{8}{3}$$

Therefore, the equation of the straight line is $y = \frac{2}{3}x + \frac{8}{3}$.

Example 1.4

Determine the equation of the straight line that passes through $(-2, 1)$ and $(7, -6)$ by using the standard form of the straight line equation.

Solution

We have been asked to find the equation of the straight line using the gradient-intercept (or standard) form of the equation. To do so, we use each point to set up an equation in standard form and then solve for m and c simultaneously.

Using $(-2, 1)$:

$$1 = m(-2) + c$$
$$\therefore 1 = -2m + c \quad (1)$$

Using $(7, -6)$:

$$-6 = m(7) + c$$

$$\therefore -6 = 7m + c \quad (2)$$

From (1): $c = 1 + 2m$ (3)

Substitute (3) into (2):

$$-6 = 7m + 1 + 2m$$

$$\therefore 9m = -7$$

$$\therefore m = -\frac{7}{9}$$

Substitute $m = -\frac{7}{9}$ into (3):

$$c = 1 + 2\left(-\frac{7}{9}\right)$$

$$= 1 - \frac{14}{9}$$

$$= \frac{9 - 14}{9}$$

$$= -\frac{5}{9}$$

Therefore, the equation of the straight line is $y = -\frac{7}{9}x - \frac{5}{9}$



Exercise 1.3

Use the gradient-intercept form of the straight line equation to find the equation of the following straight lines:

1. passing through $(1, 8)$ and $(\frac{1}{2}, 4)$
2. passing through $(4, -2)$ with $m = -2$
3. $m = \frac{3}{5}$ and cuts the y-axis at 3
4. $m = -\frac{1}{2}$ and cuts the x-axis at 7
5. passing through $(3, -4)$ and the origin

The full solutions are at the end of the unit.

Note

Generally speaking, the choice of which form of the straight line equation to use is yours. If you are given any two points, using the two-point form is usually easiest. If you are given the gradient and a point,

using the gradient-point form is usually easiest. You may, also use the gradient-intercept method as well.

Summary

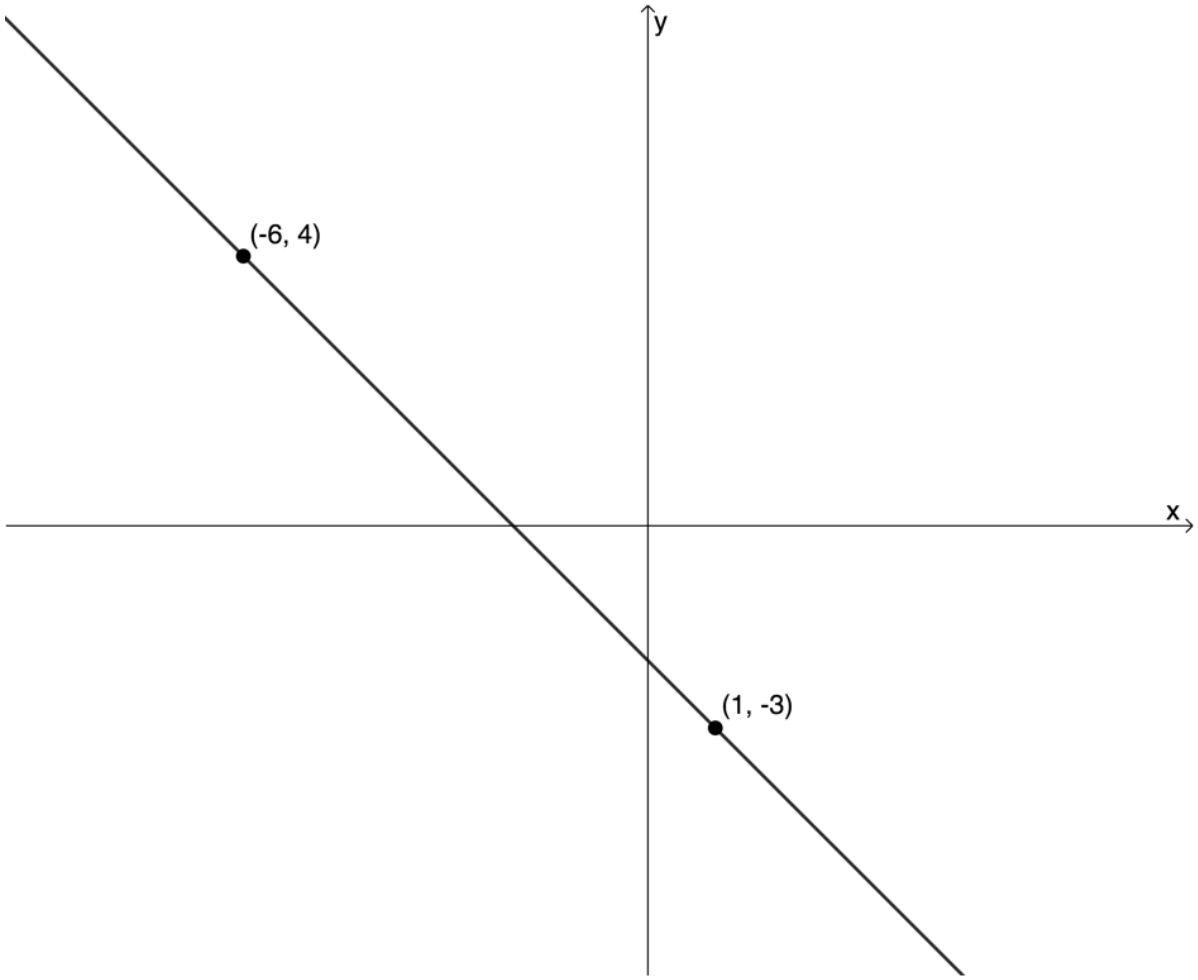
In this unit you have learnt the following:

- How to find the equation of a straight line using the two-point form of the equation – $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$.
- How to find the equation of a straight line using the gradient-point form of the equation – $y-y_1=m(x-x_1)$.
- How to find the equation of a straight line using the gradient-intercept form of the equation – $y=mx+c$.

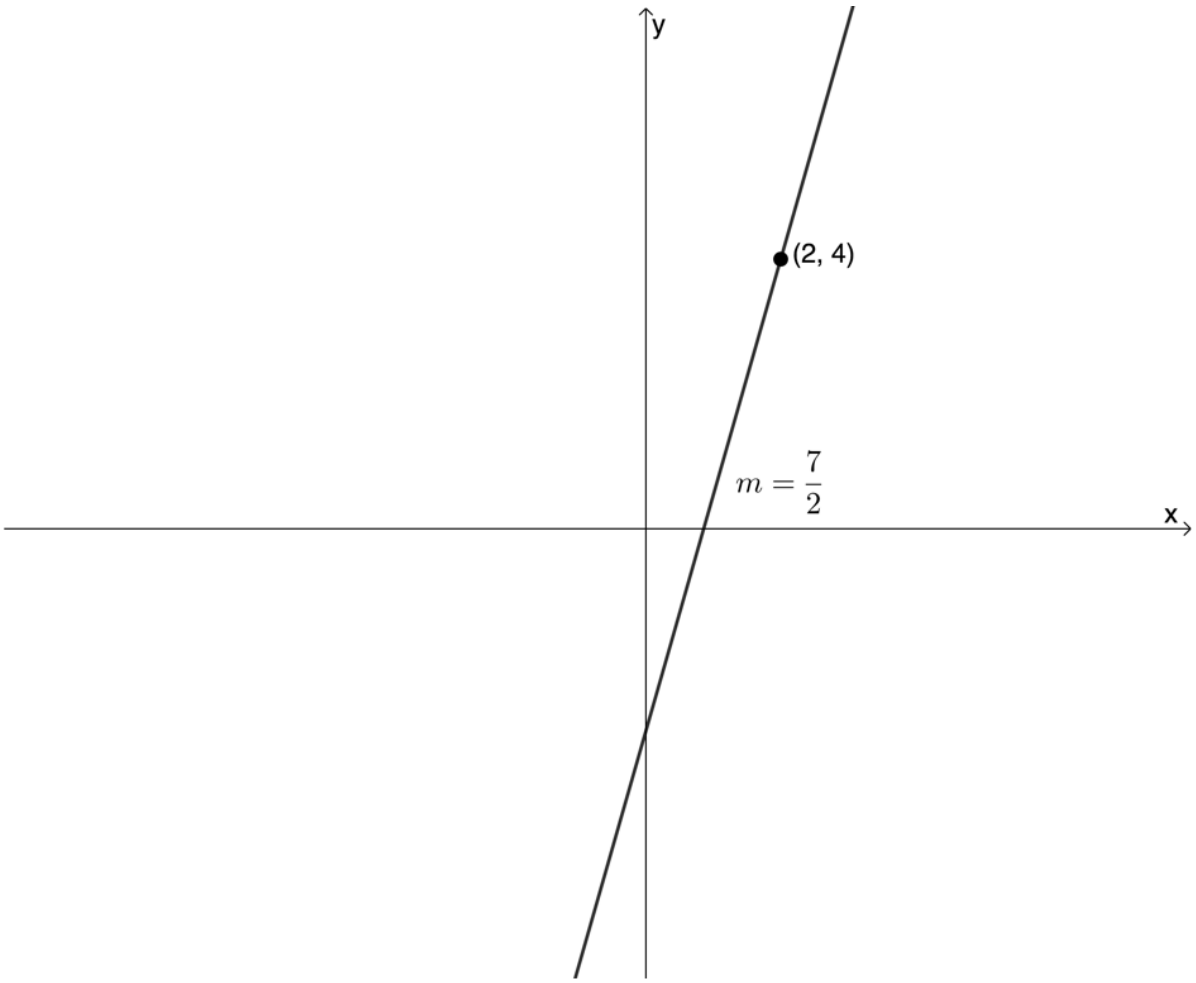
Unit 1: Assessment

Suggested time to complete: 30 minutes

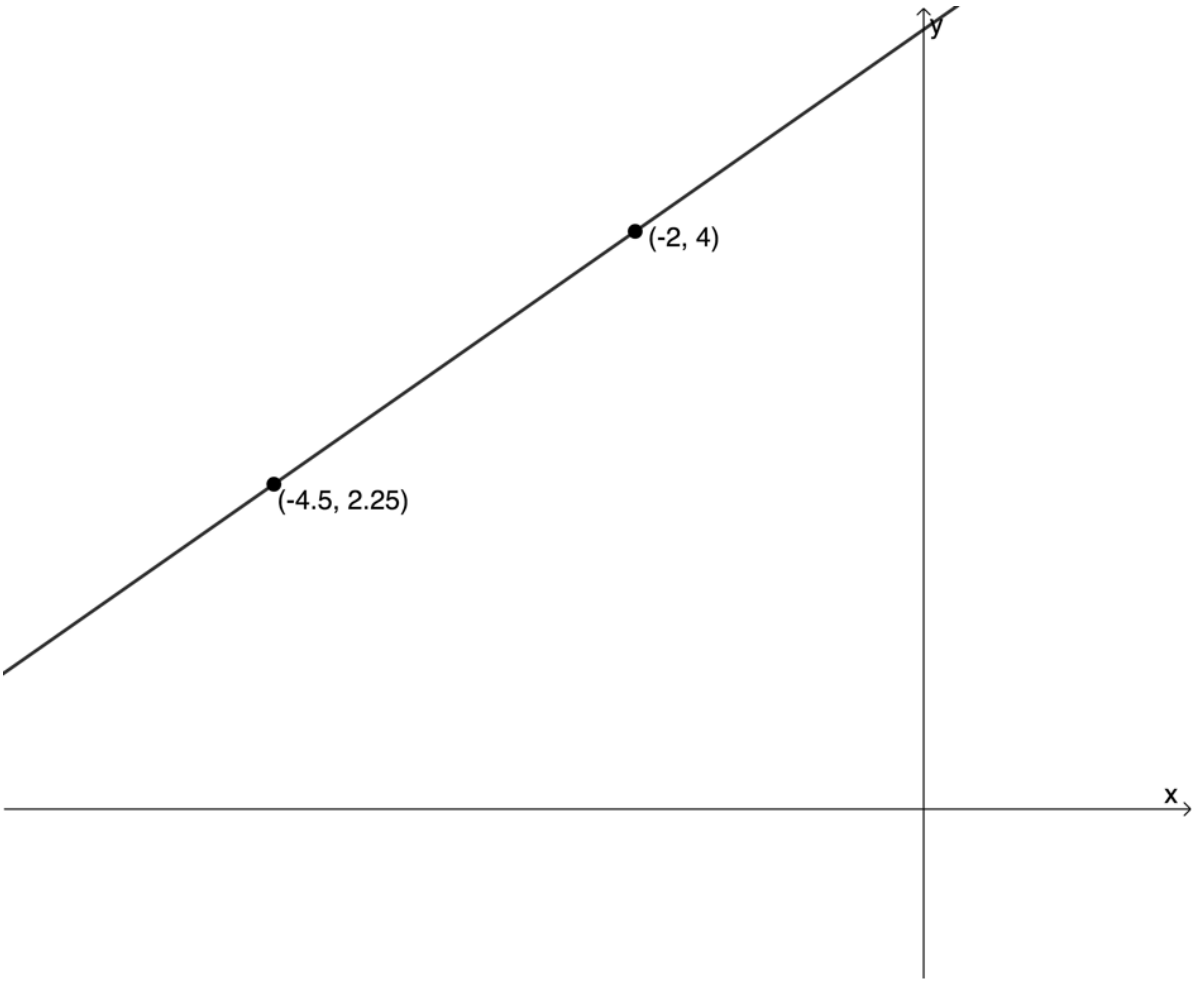
1. Determine the equations of the following straight lines:
   a. passing through $(0,6)$ and $(5,0)$
   b. parallel to the x-axis and passing through $(-6,-2)$
   c. a gradient of $m=\frac{5}{2}$ and passing through $(5,2)$
   d. passing through the points $(2a,b)$ and $(a-b,2b)$
   e. parallel to the y-axis and passing through the origin
2. Determine the equations of the following straight lines:
   a.



b.



c.



The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $(3, 7)$ and $(-6, 1)$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\therefore \frac{y - 7}{x - 3} = \frac{1 - 7}{-6 - 3}$$
$$\therefore \frac{y - 7}{x - 3} = \frac{6}{9} = \frac{2}{3}$$
$$\therefore y - 7 = \frac{2}{3}(x - 3)$$
$$\therefore y - 7 = \frac{2}{3}x - 2$$
$$\therefore y = \frac{2}{3}x + 5$$

2. $(3, 5)$ and $(2, 3)$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\therefore \frac{y-5}{x-3}=\frac{3-5}{2-3}$$
$$\therefore \frac{y-5}{x-3}=\frac{-2}{-1}=2$$
$$\therefore y-5=2(x-3)$$
$$\therefore y-5=2x-6$$
$$\therefore y=2x-1$$

3. $(-7,-5)$ and $(1,-5)$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\therefore \frac{y-(-5)}{x-(-7)}=\frac{-5-(-5)}{1-(-7)}$$
$$\therefore \frac{y+5}{x+7}=\frac{0}{8}=0$$
$$\therefore y+5=0(x+7)$$
$$\therefore y=-5$$

4. (a,b) and (b,a)

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\therefore \frac{y-b}{x-a}=\frac{a-b}{b-a}$$
$$\therefore \frac{y-b}{x-a}=\frac{a-b}{-(-b+a)}=\frac{a-b}{-(a-b)}=-1$$
$$\therefore y-b=-1(x-a)$$
$$\therefore y-b=-x+a$$
$$\therefore y=-x+a+b$$

5. $(0,r)$ and $(2s,-r)$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\therefore \frac{y-r}{x-0}=\frac{-r-r}{2s-0}$$
$$\therefore \frac{y-r}{x}=\frac{-2r}{2s}=-\frac{r}{s}$$
$$\therefore y-r=-\frac{r}{s}x$$
$$\therefore y=-\frac{r}{s}x+r$$

Back to Exercise 1.1

Exercise 1.2

1.

$$y-y_1=m(x-x_1)$$
$$\therefore y-4=-2\left(x-(-3)\right)$$
$$\therefore y-4=-2(x+3)$$
$$\therefore y-4=-2x-6$$
$$\therefore y=-2x-2$$

2.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
\therefore y - (-3) &= 0.75(x - 2) \\
\therefore y + 3 &= 0.75x - 1.5 \\
\therefore y &= 0.75x - 4.5 \\
\therefore y &= \frac{3}{4}x - \frac{9}{2}
\end{aligned}
$$

3.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
\therefore y - \frac{1}{4} &= -\frac{4}{5}\left(x - \frac{3}{4}\right) \\
\therefore y - \frac{1}{4} &= -\frac{4}{5}x + \frac{12}{20} \\
\therefore y &= -\frac{4}{5}x + \frac{12}{20} + \frac{1}{4} \\
\therefore y &= -\frac{4}{5}x + \frac{12}{20} + \frac{5}{20} \\
\therefore y &= -\frac{4}{5}x + \frac{17}{20}
\end{aligned}
$$

4.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
\therefore y - b &= 0(x - a) \\
\therefore y &= b
\end{aligned}
$$

5.

m is undefined. Therefore, the straight line is vertical passing through $(-4, y)$. Therefore $x = -4$.

6.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
\therefore y - (-6a + b) &= -2a\,(x - (-2)) \\
\therefore y + 6a - b &= -2a(x + 2) \\
\therefore y + 6a - b &= -2ax - 4a \\
\therefore y &= -2ax - 4a - 6a + b \\
\therefore y &= -2ax - 10a + b
\end{aligned}
$$

Back to Exercise 1.2

Exercise 1.3

1.

$(1, 8)$: $8 = m + c$ (1)
$(\frac{1}{2}, 4)$: $4 = \frac{1}{2}m + c$ (2)
From (1): $c = 8 - m$ (3)
Substitute (3) into (2):

$$
\begin{aligned}
4 &= \frac{1}{2}m + 8 - m \\
\therefore \frac{1}{2}m &= 4 \\
\therefore m &= 8
\end{aligned}
$$

Substitute $m = 8$ into (3):
$c = 8 - 8 = 0$
Therefore, the equation of the straight line is $y = 8x$.

2.

$y = -2x + c$
Substitute in $(4, -2)$:

$$-2 = -2(4) + c$$
$$\therefore -2 = -8 + c$$
$$\therefore c = 6$$

Therefore, the equation of the straight line is $y = -2x + 6$.

3. $m = \frac{3}{5}$ and $c = 3$. Therefore, the equation of the straight line is $y = \frac{3}{5}x + 3$.

4. $y = -\frac{1}{2}x + c$

Substitute in $(7, 0)$:

$$0 = -\frac{1}{2}(7) + c$$
$$\therefore c = \frac{7}{2}$$

Therefore, the equation of the straight line is $y = -\frac{1}{2}x + \frac{7}{2}$.

5. Passing through $(3, -4)$ and the origin.

$(3, -4)$: $-4 = 3m + c$ (1)

$(0, 0)$: $0 = c$ (2)

Substitute (2) into (1):

$$-4 = 3m + 0$$
$$\therefore m = -\frac{4}{3}$$

Therefore, the equation of the straight line is $y = -\frac{4}{3}x$.

Back to Exercise 1.3

Unit 1: Assessment

1.

a.
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\therefore \frac{y - 6}{x - 0} = \frac{0 - 6}{5 - 0}$$
$$\therefore \frac{y - 6}{x} = -\frac{6}{5}$$
$$\therefore y - 6 = -\frac{6}{5}x$$
$$\therefore y = -\frac{6}{5}x + 6$$

b. Parallel to the x-axis means that $m = 0$. Therefore, the equation of the line is $y = -2$.

c.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 2 = \frac{5}{2}(x - 5)$$
$$\therefore y - 2 = \frac{5}{2}x - \frac{25}{2}$$
$$\therefore y = \frac{5}{2}x - \frac{25}{2} + 2$$
$$\therefore y = \frac{5}{2}x - \frac{21}{2}$$

d.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\therefore \frac{y - b}{x - 2a} = \frac{2b - b}{(a - b) - 2a}$$
$$\therefore \frac{y - b}{x - 2a} = \frac{b}{a - b - 2a} = \frac{b}{-a - b} = -\frac{b}{(a + b)}$$
$$\therefore y - b = -\frac{b}{(a + b)}(x - 2a)$$
$$\therefore y - b = -\frac{b}{(a + b)}x + \frac{2ab}{(a + b)}$$
$$\therefore y = -\frac{b}{(a + b)}x + \frac{2ab}{(a + b)} + b$$
$$\therefore y = -\frac{b}{(a + b)}x + \frac{b(2ab) + b(a + b)}{(a + b)} = -\frac{b}{(a + b)}x + \frac{2ab^2 + ab + b^2}{(a + b)}$$

e. Parallel to the y-axis means that the gradient is undefined. Therefore, the equation is $x = 0$.

2.

a.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\therefore \frac{y - 4}{x - (-6)} = \frac{-3 - 4}{1 - (-6)}$$
$$\therefore \frac{y - 4}{x + 6} = -\frac{7}{7} = -1$$
$$\therefore y - 4 = -(x + 6)$$
$$\therefore y = -x - 2$$

b.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 4 = \frac{7}{2}(x - 2)$$
$$\therefore y - 4 = \frac{7}{2}x - 7$$
$$\therefore y = \frac{7}{2}x - 3$$

c.

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$
$$\therefore \frac{y-4}{x-(-2)} = \frac{2.25-4}{-4.5-(-2)}$$
$$\therefore \frac{y-4}{x+2} = \frac{1.75}{2.5} = 0.7$$
$$\therefore y-4 = 0.7(x+2)$$
$$\therefore y = 0.7x + 1.4 + 4$$
$$\therefore y = 0.7x + 5.4$$

Back to Unit 1: Assessment

Media Attributions

- pre-assessmentA1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example1.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example1.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2c © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 2: Equations of parallel and perpendicular lines

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Determine the equation of a straight line parallel to another straight line on the Cartesian plane.
- Determine the equation of a straight line perpendicular to another straight line on the Cartesian plane.

What you should know

Before you start this unit, make sure you can:

- Determine the equation of a straight line by using any of the following forms of the straight line equation:
  - the standard (or gradient-intercept) form
  - the gradient-point form
  - the two-point form.

  Refer to unit 1 in this subject outcome if you need help with this.

Introduction

Sometimes in analytical geometry, we don't know the gradient of a line and we do not have enough information to find it directly. However, if the line whose equation we are trying to find is parallel or perpendicular to another line, then we can use this information to **infer** its gradient.

In this unit we will look more closely at what we mean by parallel and perpendicular and how we can infer the gradient of a line based on this.

Parallel lines

Parallel lines in flat 2-D geometry are straight lines that run next to each other but never meet. Parallel lines are like the tracks of a railway line. The distance between them is always the same. They never get nearer or further apart.



Figure 1: Two sets of parallel railway tracks

Figure 2 shows two parallel lines on the Cartesian plane. Measure the gradient of each line using the two points given. Remember that $m = \frac{y_2 - y_1}{x_2 - x_1}$. What can we say about the gradient of each line?

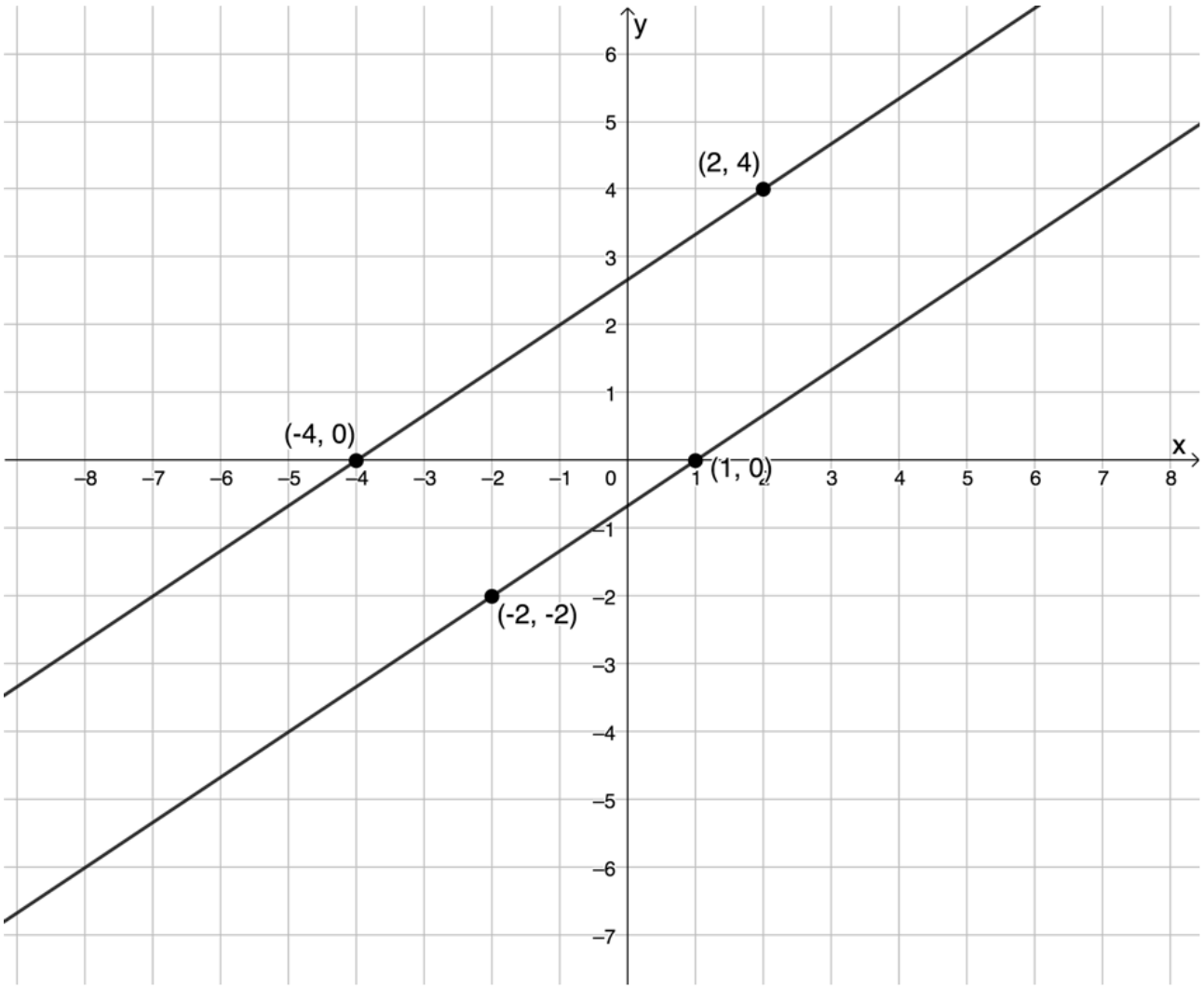


Figure 2: Two parallel lines on the Cartesian plane

You should have found that the gradient of both lines is $m = \frac{2}{3}$ (see Figure 3). Therefore, we can see that if two straight lines on the Cartesian plane are parallel, then they have the same gradient.

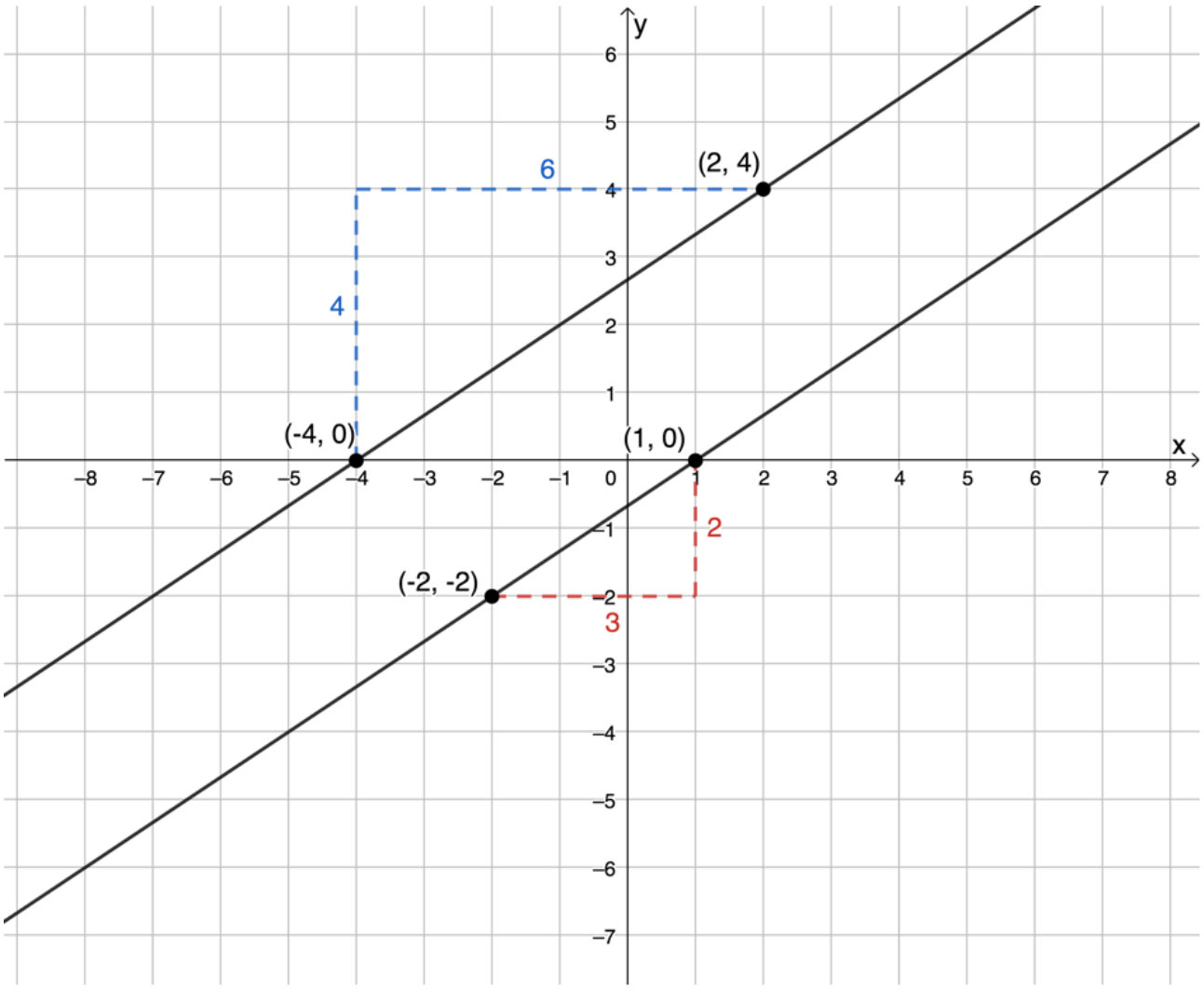


Figure 3: Parallel lines have the same gradient

We can use the fact that one line is parallel to another line to help us find its equation because, if the lines are parallel, we can infer the gradient of the line.

Take note!

If two straight lines are parallel, then $m_1 = m_2$.

Example 2.1

Determine the equation of the line passing through $(-2, 5)$ parallel to $2y - x + 4 = 0$.

Solution

We are told that the line whose equation we want to find is parallel to the line with the equation

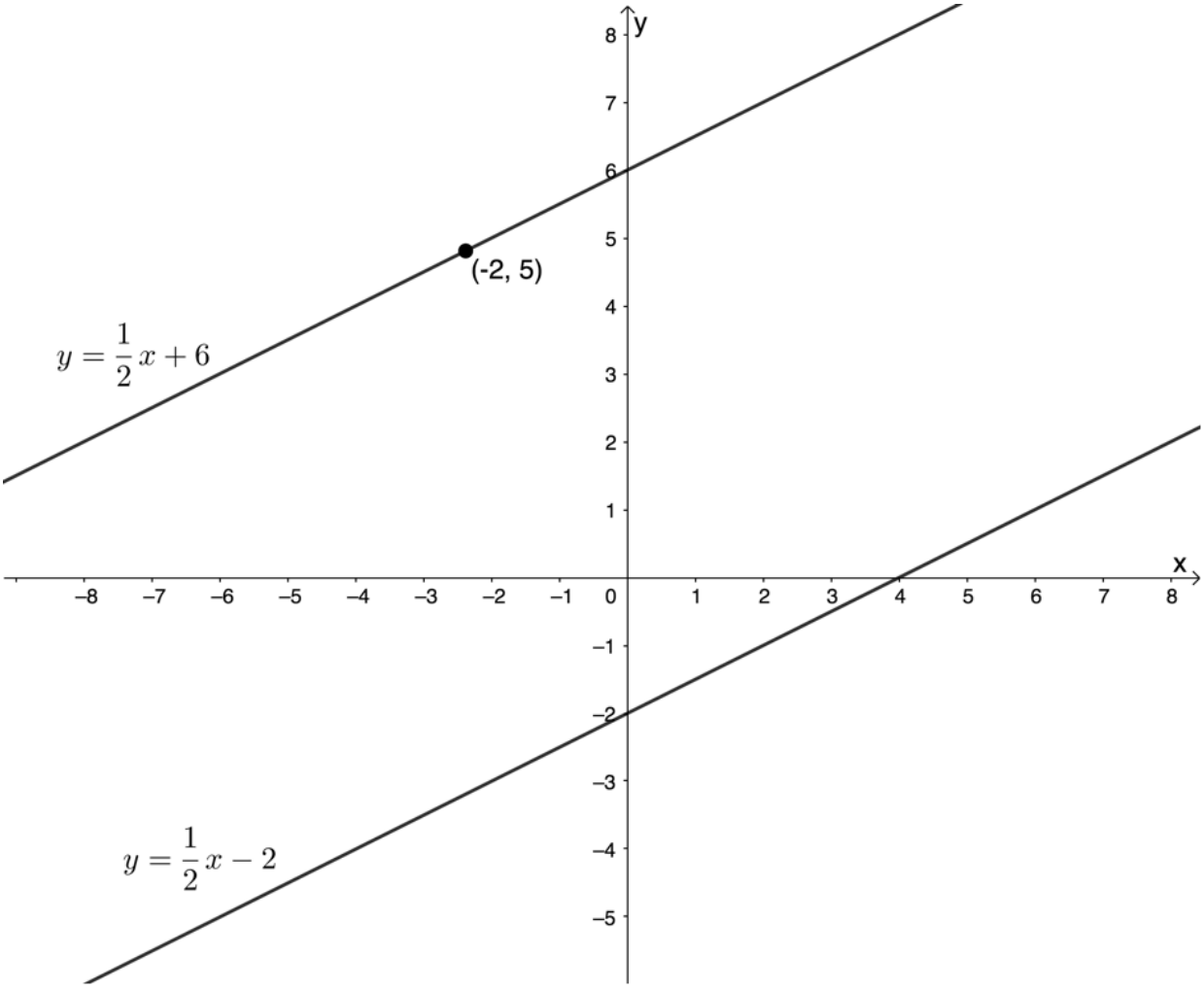
$2y - x + 4 = 0$. We need to rewrite this equation in standard form in order to determine the value of m and hence its gradient.

$$\begin{aligned} 2y - x + 4 &= 0 \\ \therefore 2y &= x - 4 \\ \therefore y &= \frac{1}{2}x - 2 \end{aligned}$$

Therefore, $m = \frac{1}{2}$. So, the line whose equation we want to find also has a gradient of $m = \frac{1}{2}$. The lines are parallel. We also know a point that this line passes through. Therefore, we can use the gradient-point form of the straight line equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - 5 &= \frac{1}{2}(x - (-2)) \\ \therefore y - 5 &= \frac{1}{2}(x + 2) \\ \therefore y - 5 &= \frac{1}{2}x + 1 \\ \therefore y &= \frac{1}{2}x + 6 \end{aligned}$$

Here is a sketch of the two lines.



Exercise 2.1

1. Determine the equations of the following straight lines:
   a. passing through $(2, -3)$ and parallel to the line $y + 3x - 4 = 0$
   b. passing through $(-4, 1)$ and parallel to the line $3(y + 2x) = 7$
   c. passing through $(-3, -3)$ and parallel to the line passing through $(4, 2)$ and $(-2, 6)$
2. Determine whether the following straight lines are parallel or not:
   a. $y + 3x = 8$ and $6x = 3y - 2$
   b. $\frac{y}{3} + x + 5 = 0$ and $2y + 6x = 1$
   c. $y - 1 = \frac{1}{3}x$ and the line passing through $(-2, 4)$ and $(1, 5)$.

The full solutions are at the end of the unit.

Perpendicular lines

In flat 2-D geometry, we say that two lines that meet or intersect at $90°$ or at right angles are **perpendicular** to each other (see Figure 4).

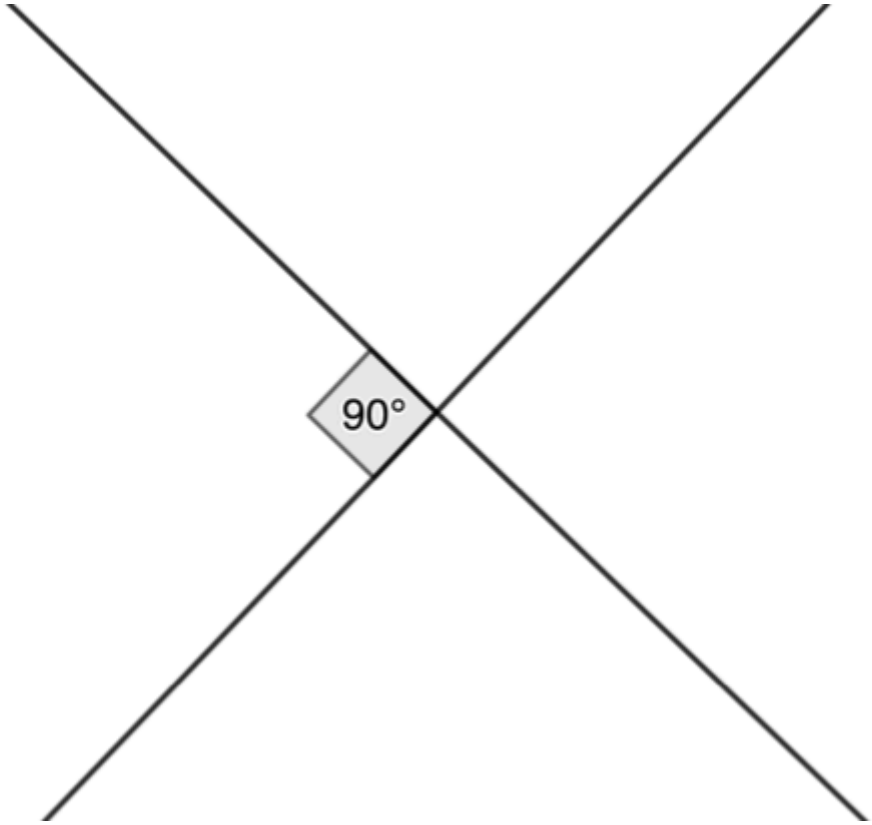


Figure 4: Perpendicular line meet or intersect at $90°$

But what can we say about the gradients of these lines?

Activity 2.1: Gradients of perpendicular lines

Time required: 10 minutes

What you need:

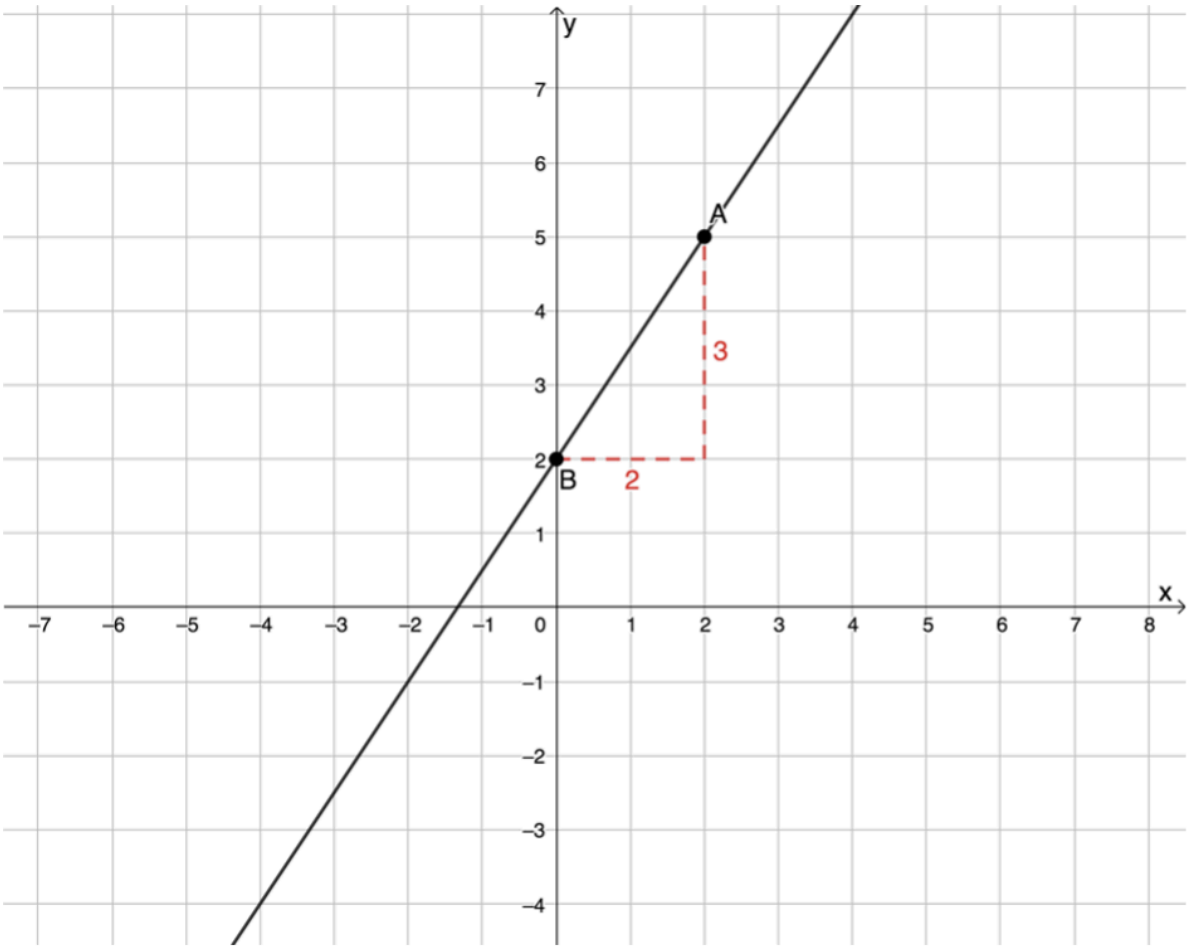
- a blank piece of paper or graph paper
- a pen or pencil
- a ruler
- a protractor

What to do:

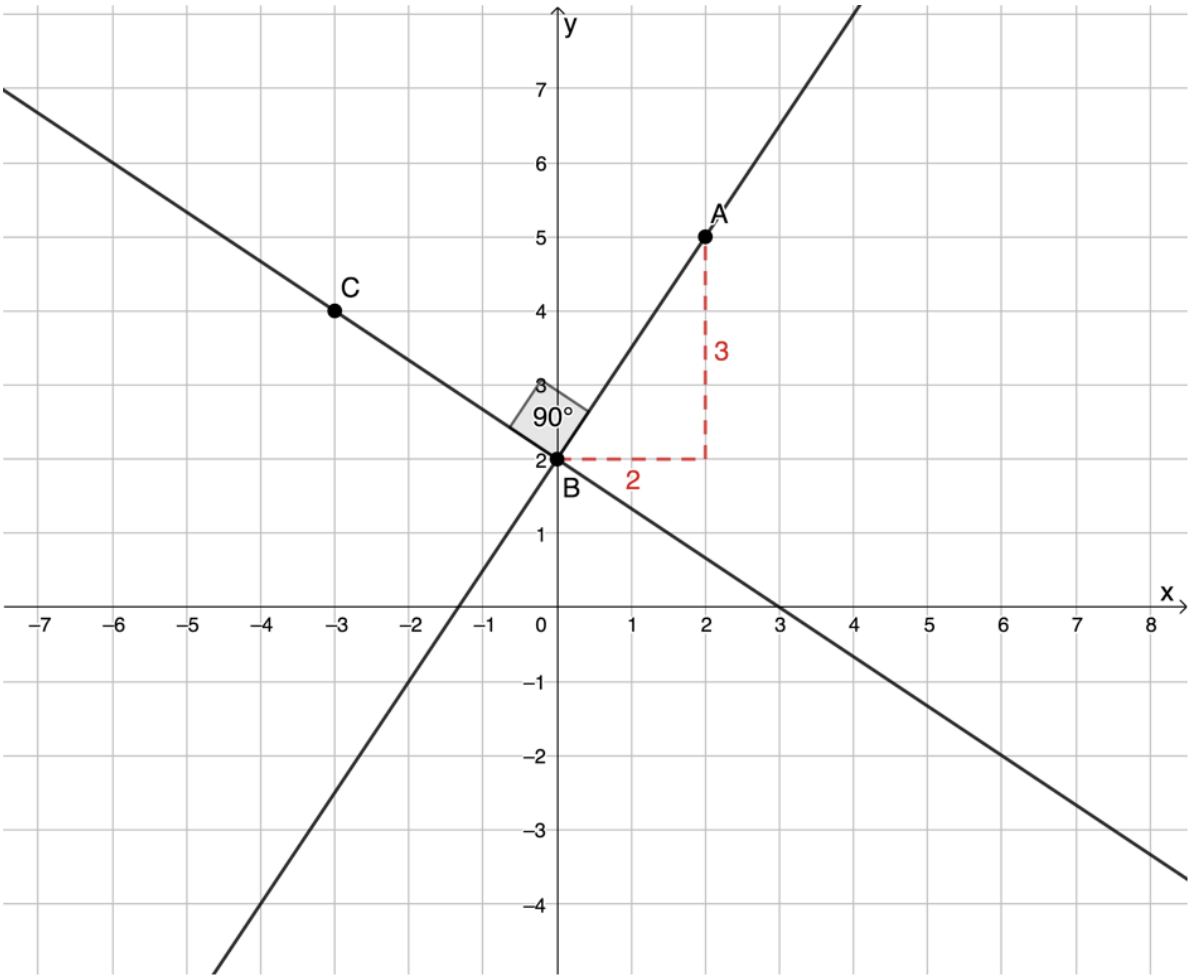
1. On your blank piece of paper or graph paper, create a Cartesian plane. Now draw any straight line with a gradient of $\frac{3}{2}$ on the Cartesian plane.
2. Use your protractor to measure an angle of 90° to this straight line. Draw another straight line at 90° (or perpendicular) to the first line.
3. Measure the gradient of this second straight line.
4. How are these two gradients related?
5. Write an expression for this relationship.

What did you find?

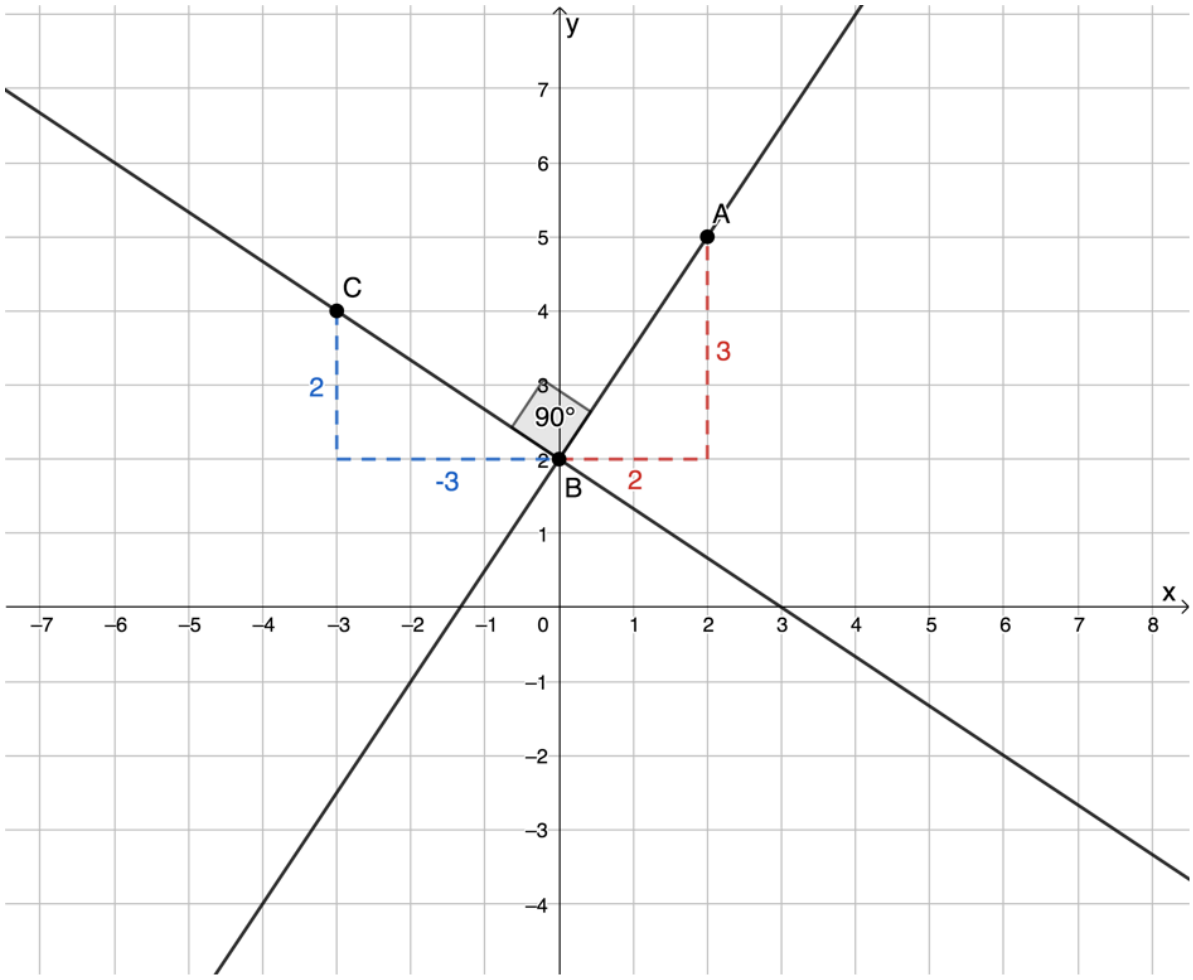
1. Here is a line with a gradient of $m = \frac{3}{2}$. The gradient has been measured between points A and B



2. Another line (through points B and C) is drawn perpendicular to the first line.



3. The gradient of the second line is measured between points B and C. Notice that the run is -3. Therefore, the gradient is $m = -\frac{2}{3}$.



4. The gradients are $m_1 = \frac{3}{2}$ and $m_2 = -\frac{2}{3}$. The one gradient is the **negative reciprocal** or **negative multiplicative** inverse of the other.
5. $m_1 \times m_2 = \frac{3}{2} \times -\frac{2}{3} = -1$

In Activity 2.1, we discovered that the gradients of perpendicular lines are negative reciprocals. Another way of saying this is that, if the lines are perpendicular, then $m_1 \times m_2 = -1$.

Take note!

If two straight lines are perpendicular, then $m_1 \times m_2 = -1$.

Example 2.2

Determine the equation of the straight line passing through $(7, 4)$ and perpendicular to $\frac{y}{3} - \frac{x}{5} = 1$.

Solution

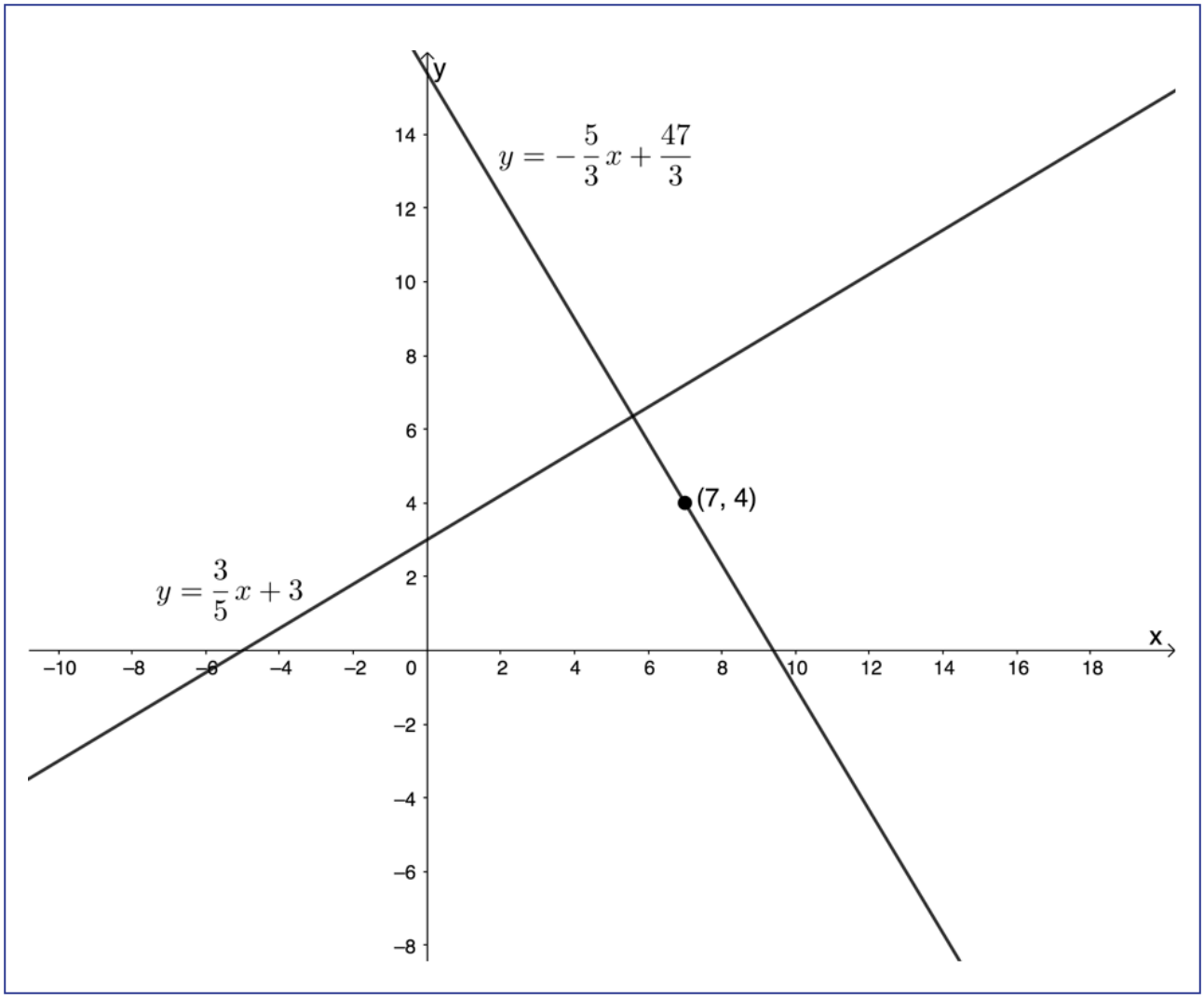
We are told that the line whose equation we want to find is perpendicular to the line with the equation $\frac{y}{3} - \frac{x}{5} = 1$. We need to rewrite this equation in standard form in order to determine the value of m and hence its gradient.

$$\frac{y}{3} - \frac{x}{5} = 1$$
$$\therefore \frac{y}{3} = \frac{x}{5} + 1$$
$$\therefore y = \frac{3x}{5} + 3$$

Therefore, $m_1 = \frac{3}{5}$. For perpendicular lines, we know that $m_1 \times m_2 = -1$. Therefore, $m_2 = -\frac{5}{3}$.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 4 = -\frac{5}{3}(x - 7)$$
$$\therefore y - 4 = -\frac{5}{3}x + \frac{35}{3}$$
$$\therefore y = -\frac{5}{3}x + \frac{35 + 12}{3}$$
$$= -\frac{5}{3}x + \frac{47}{3}$$

Here is a sketch of the two lines.



Exercise 2.2

1. Determine the equation of the following straight lines:
   a. passing through $(2, -3)$ and perpendicular to the line $y + 3x - 4 = 0$
   b. passing through $(-4, 1)$ and perpendicular to the line $3(y + 2x) = 7$
   c. passing through $(-3, -3)$ and perpendicular to the line passing through $(4, 2)$ and $(-2, 6)$
2. Determine if the following straight lines are perpendicular or not:
   a. $4y + 2 = -x$ and $4x = y - 1$
   b. $\frac{1}{3}y = x + 3$ and $3y + x = -9$
   c. $y = 2$ and $x = -3$
3. For what value of a will $y + \frac{x}{a} = c$ and $2y - \frac{3x}{a} + k = 0$ be perpendicular?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- If two straight lines on the Cartesian plane are parallel, then $m_1 = m_2$.
- If two straight lines on the Cartesian plane are perpendicular, then $m_1 \times m_2 = -1$.

Unit 2: Assessment

Suggested time to complete: 30 minutes

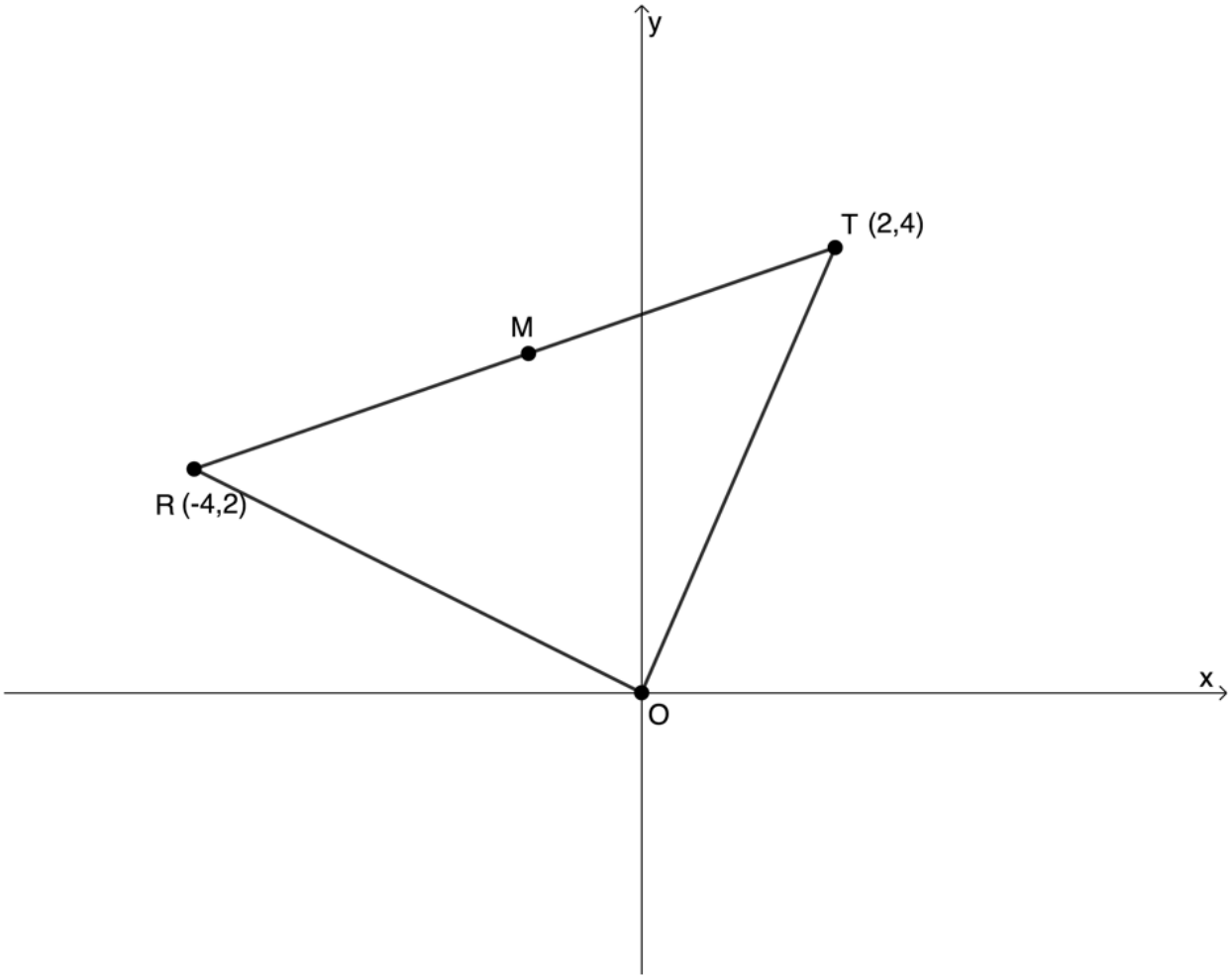
1. Determine the equation of the line that passes through the point $(1, -1)$ and is parallel to the line $y - 2x + 1 = 0$.
2. Determine the equation of the straight line that passes through the point $(2, 7)$ and is perpendicular to the line $5y - x = 0$.

Question 3 is taken from NC(V) Mathematics Level 3 Paper 2 March 2013 question 1.3

3. $A(2, 3)$ and $B(-5, -2)$ are points on the Cartesian plane. CD is perpendicular to AB and passes through the point $(5, 0)$.
   a. Determine the gradient of line segment AB.
   b. Determine the equation of line segment CD.

Question 4 is taken from NC(V) Mathematics Level 3 Paper 2 March 2014 question 1.3

4. $R(-4, 2)$ and $T(2, 4)$ are points in the Cartesian plane. M is the midpoint of RT.



a. Calculate the length of RT.
b. Calculate the co-ordinates of M, the midpoint of RT.
c. Write down the equation of OM.
d. Show by calculation that ΔROT is a right-angle triangle.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

a.

$$y + 3x - 4 = 0$$
$$\therefore y = -3x + 4$$

Therefore $m = -3$

$$y - y_1 = m(x - x_1)$$
$$\therefore y - (-3) = -3\,(x - 2)$$
$$\therefore y + 3 = -3x + 6$$
$$\therefore y = -3x + 3$$

b.

$$3(y+2x)=7$$
$$\therefore 3y+6x=7$$
$$\therefore 3y=-6x+7$$
$$\therefore y=-2x+\frac{7}{3}$$

Therefore $m=-2$

$$y-y_1=m(x-x_1)$$
$$\therefore y-1=-2\left(x-(-4)\right)$$
$$\therefore y-1=-2(x+4)$$
$$\therefore y-1=-2x-8$$
$$\therefore y=-2x-7$$

c.

Gradient of the line passing through $(4,2)$ and $(-2,6)$:

$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{6-2}{-2-4}$$
$$=-\frac{4}{6}$$
$$=-\frac{2}{3}$$
$$y-y_1=m(x-x_1)$$
$$\therefore y-(-3)=-\frac{2}{3}(x-(-3))$$
$$\therefore y+3=-\frac{2}{3}(x+3)$$
$$\therefore y+3=-\frac{2}{3}x-2$$
$$\therefore y=-\frac{2}{3}x-5$$

2.

a.

$$y+3x=8$$
$$\therefore y=-3x+8$$

Therefore, $m_1=-3$

$$6x=3y-2$$
$$\therefore 3y=6x+2$$
$$\therefore y=2x+\frac{2}{3}$$

Therefore, $m_2=2$

$m_1\neq m_2$. Therefore, the lines are not parallel.

b.

$$\frac{y}{3}+x+5=0$$
$$\therefore \frac{y}{3}=-x-5$$
$$\therefore y=-3x-15$$

Therefore, $m_1=-3$

$$2y+6x=1$$
$$\therefore 2y=-6x+1$$
$$\therefore y=-3x+\frac{1}{2}$$

Therefore, $m_2=-3$

$m_1=m_2$. Therefore, the lines are parallel.

c.

$$y - 1 = \frac{1}{3}x$$
$$\therefore y = \frac{1}{3}x + 1$$

Therefore, $m_1 = \frac{1}{3}$

Line passing through $(-2, 4)$ and $(1, 5)$.

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{5 - 4}{1 - (-2)}$$
$$= \frac{1}{3}$$

$m_1 = m_2$. Therefore, the lines are parallel.

Back to Exercise 2.1

Exercise 2.2

1.

a. Passing through $(2, -3)$ and perpendicular to the line $y + 3x - 4 = 0$.

$$y + 3x - 4 = 0$$
$$\therefore y = -3x + 4$$

$m_1 = -3 = -\frac{3}{1}$. Therefore, $m_2 = \frac{1}{3}$.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - (-3) = \frac{1}{3}(x - 2)$$
$$\therefore y + 3 = \frac{1}{3}x - \frac{2}{3}$$
$$\therefore y = \frac{1}{3}x - \frac{2}{3} - \frac{9}{3}$$
$$= \frac{1}{3}x - \frac{11}{3}$$

b. Passing through $(-4, 1)$ and perpendicular to the line $3(y + 2x) = 7$.

$$3(y + 2x) = 7$$
$$\therefore 3y + 6x = 7$$
$$\therefore 3y = -6x + 7$$
$$\therefore y = -2x + \frac{7}{3}$$

$m_1 = -2 = -\frac{2}{1}$. Therefore, $m_2 = \frac{1}{2}$.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 1 = \frac{1}{2}(x - (-4))$$
$$\therefore y - 1 = \frac{1}{2}(x + 4)$$
$$\therefore y - 1 = \frac{1}{2}x + 2$$
$$\therefore y = \frac{1}{2}x + 3$$

c. Passing through $(-3, -3)$ and perpendicular to the line passing through $(4, 2)$ and $(-2, 6)$.

Gradient of the line passing through $(4, 2)$ and $(-2, 6)$:

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{6-2}{-2-4} \\
&= -\frac{4}{6} \\
&= -\frac{2}{3}
\end{aligned}
$$

$m_1 = -\frac{2}{3}$. Therefore, $m_2 = \frac{3}{2}$

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
\therefore y - (-3) &= \frac{3}{2}(x - (-3)) \\
\therefore y + 3 &= \frac{3}{2}(x + 3) \\
\therefore y + 3 &= \frac{3}{2}x + \frac{9}{2} \\
\therefore y &= \frac{3}{2}x + \frac{9-6}{2} \\
&= \frac{3}{2}x + \frac{3}{2}
\end{aligned}
$$

2.

a.

$$
\begin{aligned}
4y + 2 &= -x \\
\therefore 4y &= -x - 2 \\
\therefore y &= -\frac{1}{4}x - \frac{1}{2}
\end{aligned}
$$

$m_1 = -\frac{1}{4}$

$$
\begin{aligned}
4x &= y - 1 \\
\therefore y &= 4x + 1
\end{aligned}
$$

$m_2 = \frac{4}{1}$

$m_1 \times m_2 = -1$. Therefore, the lines are perpendicular.

b.

$$
\begin{aligned}
\frac{1}{3}y &= x + 3 \\
\therefore y &= 3x + 9
\end{aligned}
$$

$m_1 = \frac{3}{1}$

$$
\begin{aligned}
3y + x &= -9 \\
\therefore 3y &= -x - 9 \\
\therefore y &= -\frac{1}{3}x - 3
\end{aligned}
$$

$m_2 = -\frac{1}{3}$

$m_1 \times m_2 = -1$. Therefore, the lines are perpendicular.

c. $y = 2$. The line is horizontal. Therefore, $m_1 = 0$.
$x = -3$. The line is vertical. Therefore, m_2 is undefined.
However, even though $m_1 \times m_2$ cannot be calculated, we know the lines are perpendicular.

3.

$$
\begin{aligned}
y + \frac{x}{a} &= c \\
\therefore y &= -\frac{1}{a}x + c
\end{aligned}
$$

$m_1 = -\frac{1}{a}$

$2y - \frac{3x}{a} + k = 0$

$\therefore 2y = \frac{3}{a}x - k$

$\therefore y = \frac{3}{2a}x - \frac{k}{2}$

$m_2 = \frac{3}{2a}$

For perpendicular lines $m_1 \times m_2 = -1$:

$-\frac{1}{a} \times \frac{3}{2a} = -1, a \neq 0$

$\therefore \frac{-3}{2a^2} = -1$

$\therefore -3 = -2a^2$

$\therefore 2a^2 = 3$

$\therefore a^2 = \frac{3}{2}$

$\therefore a = \pm\sqrt{\frac{3}{2}}$

Back to Exercise 2.2

Unit 2: Assessment

1. $y - 2x + 1 = 0$

   $\therefore y = 2x - 1$

   $m_1 = 2$. Therefore $m_2 = 2$.

   $y - y_1 = m(x - x_1)$

   $\therefore y - (-1) = 2(x - 1)$

   $\therefore y + 1 = 2x - 2$

   $\therefore y = 2x - 3$

2. $5y - x = 0$

   $\therefore 5y = x$

   $\therefore y = \frac{1}{5}x$

   $m_1 = \frac{1}{5}$. Therefore, $m_2 = -5$.

   $y - y_1 = m(x - x_1)$

   $\therefore y - 7 = -5(x - 2)$

   $\therefore y - 7 = -5x + 10$

   $\therefore y = -5x + 17$

3. $A(2, 3)$ and $B(-5, -2)$ are points on the Cartesian plane. CD is perpendicular to AB and passes through the point $(5, 0)$.

   a. $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 3}{-5 - 2} = \frac{5}{7}$

   b. $m_{CD} = -\frac{7}{5}$

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 0 = -\frac{7}{5}(x - 5)$$
$$\therefore y = -\frac{7}{5}x + 7$$

4.

a.

$$\begin{aligned} RT &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - (-4))^2 + (4 - 2)^2} \\ &= \sqrt{6^2 + 2^2} \\ &= \sqrt{36 + 4} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

b.

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-4 + 2}{2}, \frac{2 + 4}{2}\right) \\ &= \left(\frac{-2}{2}, \frac{6}{2}\right) \\ &= (-1, 3) \end{aligned}$$

c. $O(0, 0)$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\therefore \frac{y - 0}{x - 0} = \frac{3 - 0}{-1 - 0}$$
$$\therefore \frac{y}{x} = -3$$
$$\therefore y = -3x$$

d. $m_{RO} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{-4 - 0} = -\frac{1}{2}$

$m_{TO} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{2 - 0} = \frac{2}{1}$

$m_{RO} \times m_{TO} = -1$. Therefore $R\hat{O}T = 90^\circ$.

Back to Unit 2: Assessment

Media Attributions

- figure1 © libreshot is licensed under a CC0 (Creative Commons Zero) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example2.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- activity2.1.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- activity2.1.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- activity2.1.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example2.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 3: Determine the angle of inclination of a line

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use the gradient of a straight line to determine the angle of inclination of the line with the x-axis.
- Use the angle of inclination to determine the gradient of a straight line.

What you should know

Before you start this unit, make sure you can:

- Define the tangent ratio.
- Use a calculator to determine the tangent of an angle.
- Use a calculator to determine the angle from a given tangent ratio.
- Use the CAST diagram to identify where tangent is positive and negative.

Refer to level 2 subject outcome 3.6 unit 1 if you need help with any of this.

Introduction

So far in this subject outcome we have been calculating the gradients of straight lines using the idea of a gradient being $\frac{\text{rise}}{\text{run}}$ – the vertical distance between two points on the straight line divided by the horizontal distance between those points. This idea leads us to an equation for gradient that we know well: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

However, there is another way in which we can measure the gradient of a straight line on the Cartesian plane and this is to measure the angle that the line makes with the x-axis. This is called the **angle of inclination**.

The angle of inclination

Figure 1 shows what we mean by the angle of inclination. It is the angle that a straight line makes with the positive x-axis.

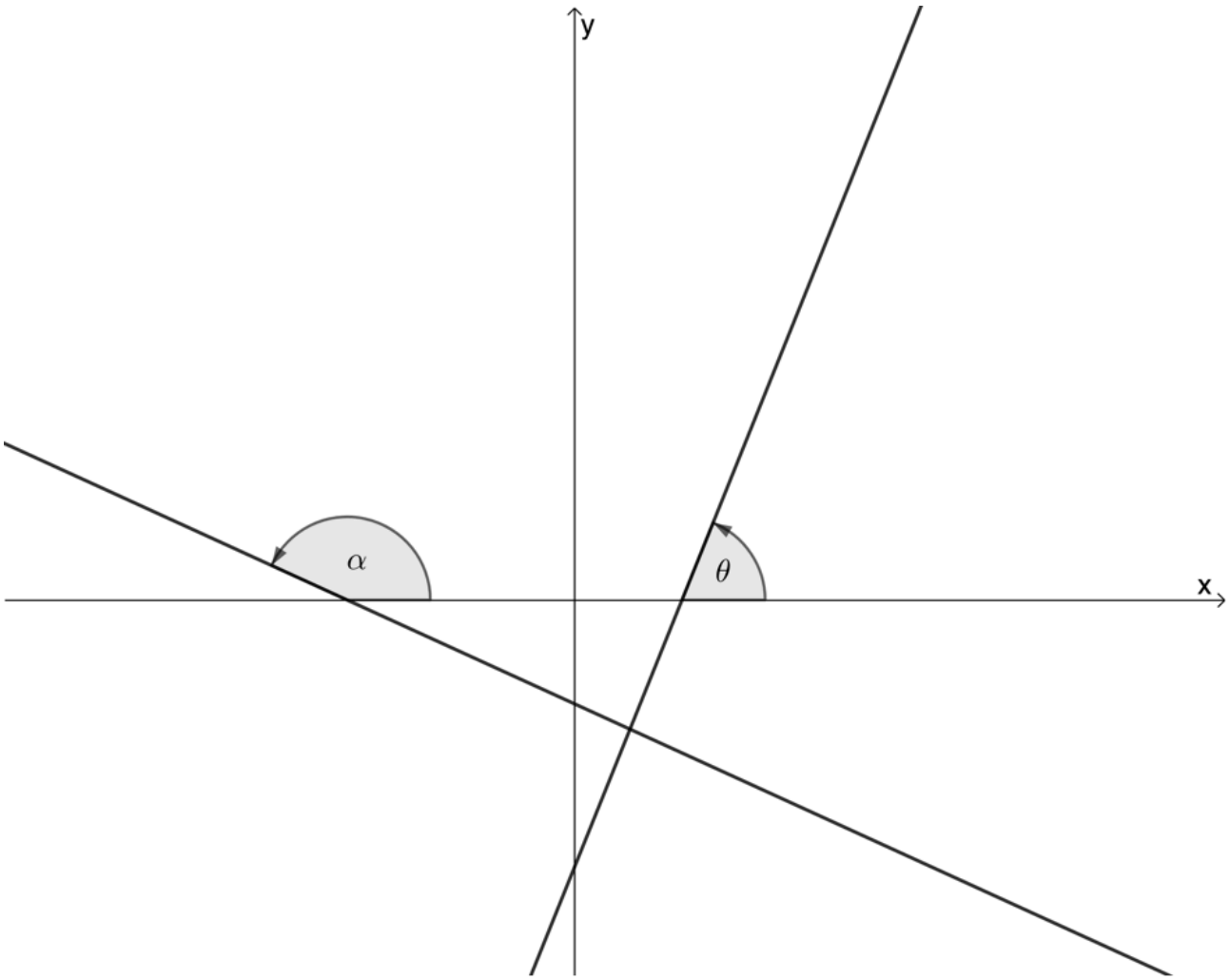


Figure 1: The angle of inclination

Notice how we always measure the angle of inclination in an anti-clockwise direction from the more positive side of the x-axis of where the line intersects with it.

This means that:

- the angle of inclination is always $0^\circ \leq \theta \leq 180^\circ$
- **positive** gradients have an angle of inclination $0^\circ < \theta < 90^\circ$
- **negative** gradients have an angle of inclination $90^\circ < \theta < 180^\circ$.

As the gradient of the straight line changes, so does its angle of inclination. If you have an internet connection visit this practical example.

Here you will find a slider with which to change the gradient of the line. Change the gradient to see how this affects the angle of inclination of the straight line.

We know that $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$ or the change in y divided by the change in x. However, if you look at Figure 2, you will see that, with respect to the angle of inclination, θ, Δy is the opposite side of the triangle and Δx is the adjacent side.

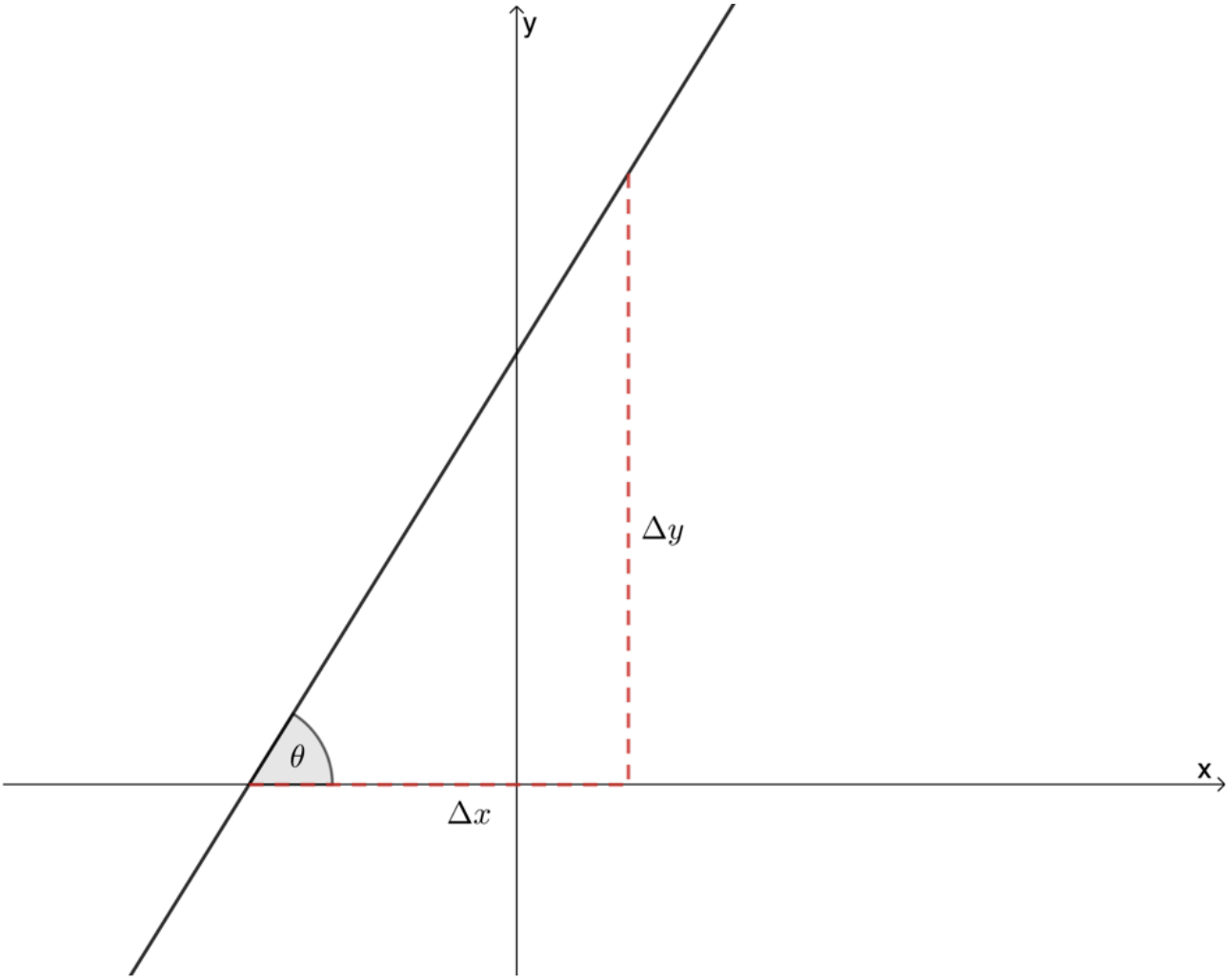


Figure 2:

Therefore, we can say that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m$. In other words, $\tan\theta = m$ for $0^\circ \leq \theta \leq 180^\circ$.

We know that the tangent function is positive in the first quadrant ($0^\circ < \theta < 90^\circ$) and negative in the second quadrant ($90^\circ < \theta < 180^\circ$). So, it makes sense that lines with an angle of inclination $0^\circ < \theta < 90^\circ$ have a positive gradient and lines with an angle of inclination $90^\circ < \theta < 180^\circ$ have a negative gradient.

Angle of inclination and gradient:
$m = \tan\theta$



Example 3.1

Determine the gradient of a straight line with an angle of inclination of:

1. 63°
2. 107°

Solution

We know that $m = \tan\theta$. Therefore, we need to use a calculator to determine the gradient in each case.
$\tan 63^\circ = 1.96$
$\tan 107^\circ = -3.27$

We can see that the angle of inclination $0^\circ < \theta < 90^\circ$ results in a positive gradient, while the angle of inclination $90^\circ < \theta < 180^\circ$ results in a negative gradient.



Example 3.2

Determine the angle of inclination of a straight line with a gradient of:

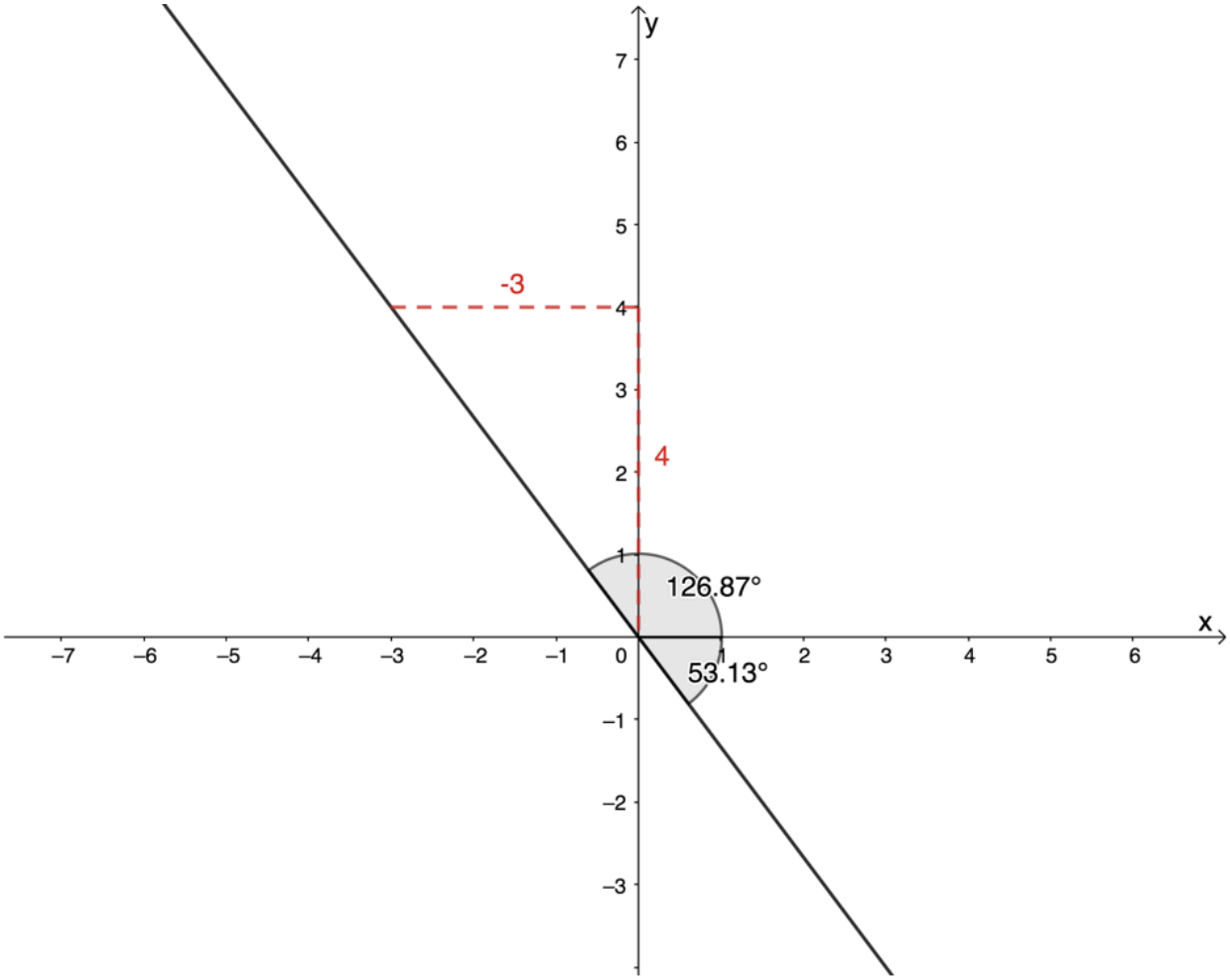
1. $m = \frac{3}{7}$
2. $m = -\frac{4}{3}$

Solution

We know that $m = \tan\theta$ but in this case, we have been given m and we need to use a calculator to determine the angle of inclination, θ. So, we need to use the $\tan^{-1}$ button on our calculator.

$$\tan\theta = \frac{3}{7}$$
$$\therefore \theta = 23.20^\circ$$
$$\tan\theta = -\frac{4}{3}$$
$$\therefore \theta = -53.13^\circ$$

In this case, the calculator gives an angle in the fourth quadrant. We need to add 180° to put the angle into the second quadrant.
$\theta = -53.13^\circ + 180^\circ = 126.87^\circ$



We can see that positive gradients give angles of inclination $0^\circ < \theta < 90^\circ$ while negative gradients give angles of inclination $90^\circ < \theta < 180^\circ$.

Exercise 3.1

1. Determine the gradient (correct to two decimal places) of straight lines with the following angles of inclination:
   a. 60°
   b. 50°
   c. 152°
   d. 0°
   e. 180°
   f. 90°
2. Determine the angle of inclination (to two decimal places) of the following straight lines:
   a. $3x + 2y - 6 = 0$

b. $\frac{1}{2}y - x = 3$

c. The line passing through $(2, -3)$ and $(4, 7)$

d. $y = 6$

e. $x = \sqrt{3}$

The full solutions are at the end of the unit.

Example 3.3

Determine the equation of the straight line passing through $(\frac{1}{2}, \frac{1}{4})$ and perpendicular to a line with an angle of inclination of $71.57°$.

Solution

The angle of inclination of the other line is $71.57°$. Therefore, we can find its gradient.

$\tan 71.57° = 3$

$m_1 = 3$. Therefore, $m_2 = -\frac{1}{3}$

$$y - y_1 = m(x - x_1)$$

$$\therefore y - \frac{1}{4} = -\frac{1}{3}\left(x - \frac{1}{2}\right)$$

$$\therefore y - \frac{1}{4} = -\frac{1}{3}x + \frac{1}{6}$$

$$\therefore y = -\frac{1}{3}x + \frac{2+3}{12}$$

$$y = -\frac{1}{3}x + \frac{5}{12}$$

Exercise 3.2

Determine the equation of the straight line perpendicular to the line with an angle of inclination of $165.964°$ and passing through the point $(-1, 2)$.

The full solution is at the end of the unit.

Example 3.4

Determine the acute angle (correct to one decimal place) between the line passing through $(-2, 4)$ and $(4, -2)$ and the straight line $y = 3x + 2$.

Solution

It will help to draw a sketch of the situation. We can see the angle of inclination of each line (θ and α) and that the acute angle between the two lines is β. If we find θ and α, we can find β using angles in a triangle.

$y = 3x + 2$. Therefore, $m = 3$.
$\tan\theta = 3$
$\therefore \theta = 71.57^\circ$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{-2 - 4} = \frac{6}{-6} = -1$$
$\tan\alpha = -1$
$\therefore \alpha = 135^\circ$

So, the angle next to α inside the triangle is 45° (angles on a straight line are supplementary).

$\beta = 180^\circ - 45^\circ - 71.57^\circ = 63.43^\circ$ (angles inside a triangle are supplementary).



Exercise 3.3

Determine the angle between the straight lines $y = -x + 3$ and $y - x = \frac{1}{2}$. What can you say about these two straight lines?

The full solution is at the end of the unit.

Summary

In this unit you have learnt the following:

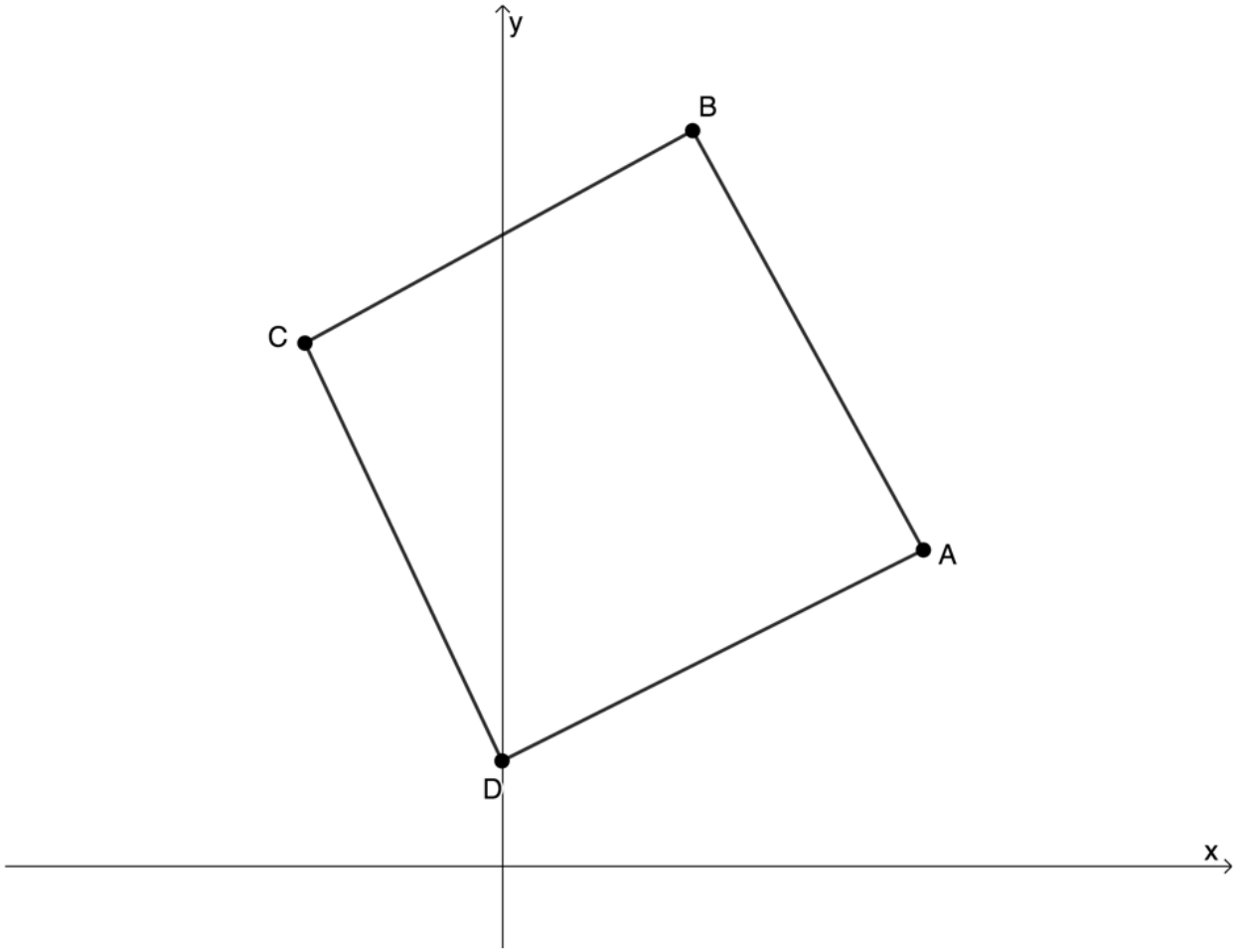
- The angle of inclination, θ, is the angle made between a straight line and the positive x-axis.
- The angle of inclination is always $0^\circ \leq \theta \leq 180^\circ$.
- Gradient $m = \tan\theta$.
- Positive gradients have an angle of inclination $0^\circ \leq \theta \leq 90^\circ$.
- Negative gradients have an angle of inclination $90^\circ \leq \theta \leq 180^\circ$.

Unit 3: Assessment

Suggested time to complete: 50 minutes

Question 1 is adapted from NC(V) Mathematics Level 3 Paper 2 November 2014 question 1.3

1. In the diagram below $A(4, 3)$, $B(2, 7)$, $C(-2, 5)$ and $D(0, 1)$ are four points on the Cartesian plane.



a. Calculate the length of BD (leaving the answer in the simplest surd form if necessary).

b. Show that $CA = BD$.

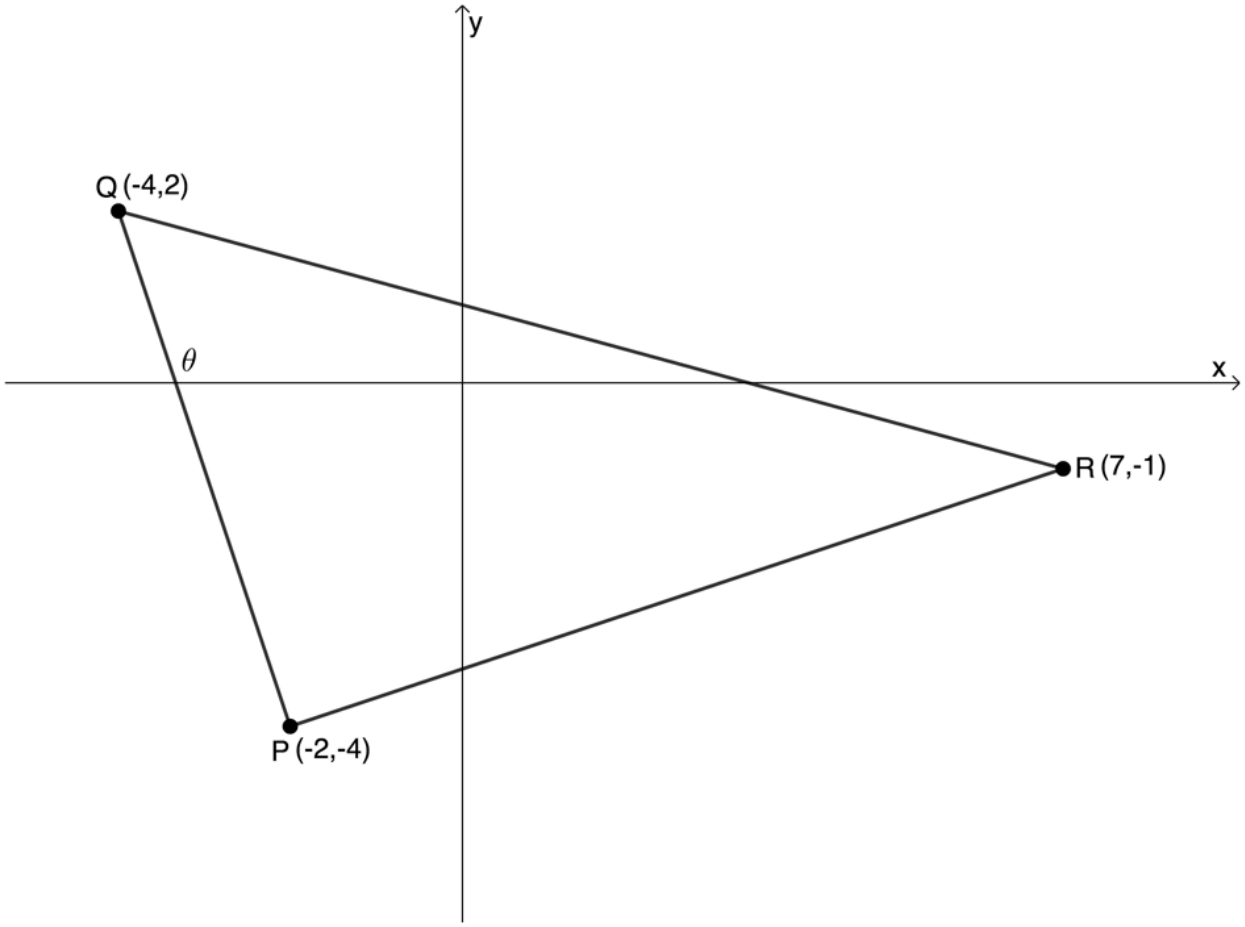
c. Calculate the co-ordinates of M , the midpoint of BD.

d. Prove that $AM \perp BD$.

e. What is the name given to quadrilateral $ABCD$? Give a reason for your answer.

Question 2 is adapted from NC(V) Mathematics Level 3 Paper 2 November 2016 question 1.3

2. In the diagram below, ΔPQR has vertices $P(-2, -4)$, $Q(-4, 2)$and $R(7, -1)$. The angle of inclination of QP is θ.



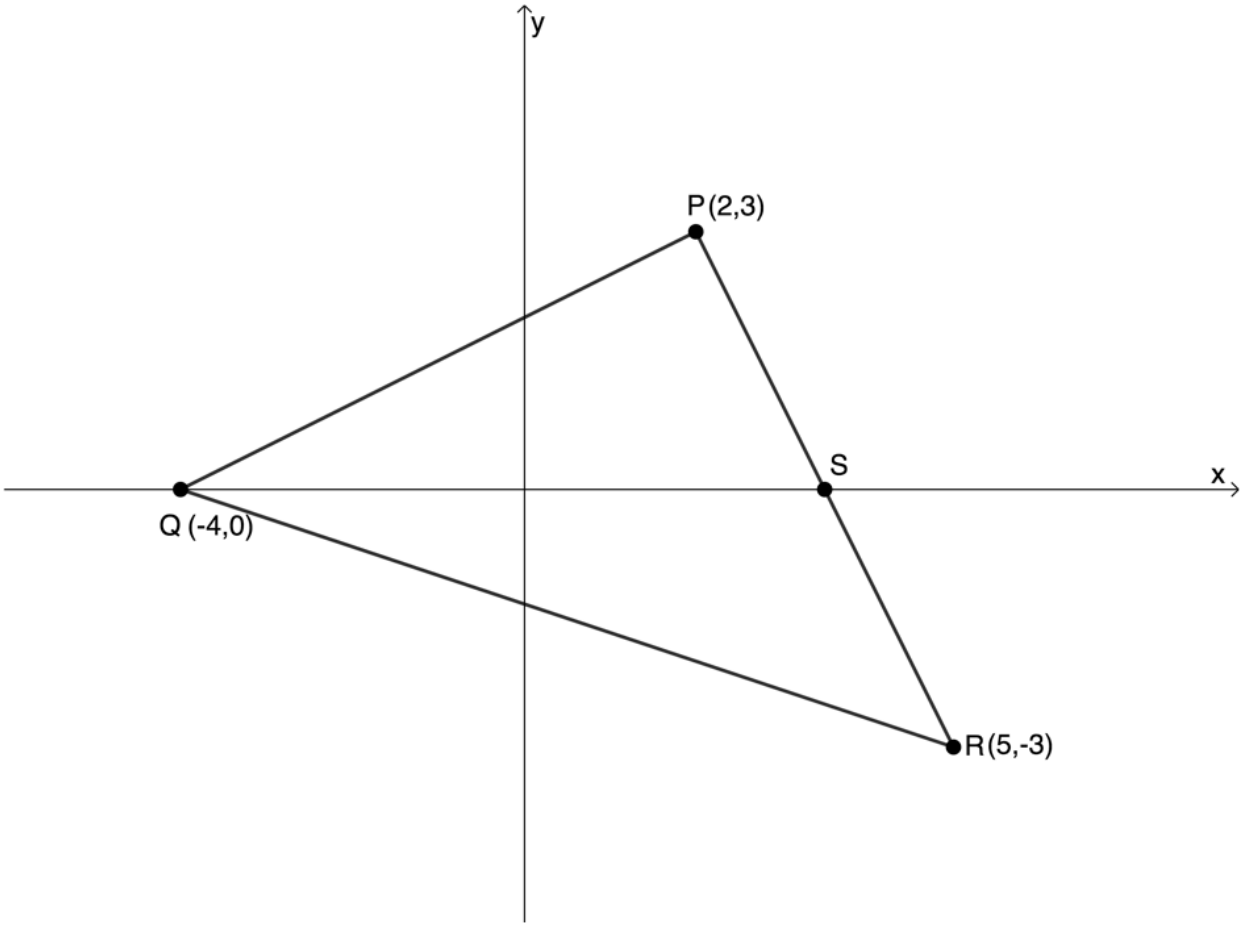
a. Determine the coordinates of M, the midpoint of QR.

b. Prove that $QP \perp PR$.

c. Calculate the size of θ to the nearest degree.

Question 3 is adapted from Everything Maths Grade 11 Exercise 4-9 question 3

3. $P(2,3)$, $Q(-4,0)$ and $R(5,-3)$ are the vertices of ΔPQR in the Cartesian plane. PR intersects the x-axis at S.



Determine the following:

a. The equation of the line PR.
b. The coordinates of point S.
c. The angle of inclination of PR (correct to two decimal places).
d. The gradient of line PQ.
e. $Q\hat{P}R$.
f. The equation of the line perpendicular to PQ and passing through the origin.
g. The mid-point M of QR.
h. The equation of the line parallel to PR and passing through point M.

Question 4 is adapted from Everything Maths Grade 11 Exercise 4-9 question 8

4. Consider triangle FGH with vertices $F(-1,3)$, $G(2,1)$ and $H(4,4)$.
   a. Sketch ΔFGH on the Cartesian plane.
   b. Show that ΔFGH is an isosceles triangle.
   c. Determine the equation of the line PQ, the perpendicular bisector of FH.
   d. Does G lie on the line PQ?
   e. Determine the equation of the line parallel to GH and passing through point F.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
   a. $\tan 60^\circ = 1.73$. Therefore, $m = 1.73$.
   b. $\tan 50^\circ = 1.19$. Therefore, $m = 1.19$.
   c. $\tan 152^\circ = -0.53$. Therefore, $m = -0.53$.
   d. $\tan 0^\circ = 0$. Therefore, $m = 0$.
   e. $\tan 180^\circ = 0$. Therefore, $m = 0$.
   f. $\tan 90^\circ$ is undefined. Therefore m is undefined, and the line is vertical.
2.
   a.
   $$3x + 2y - 6 = 0$$
   $$\therefore 2y = -3x + 6$$
   $$\therefore y = -\frac{3}{2}x + 3$$
   $$m = -\frac{3}{2}$$
   $$\tan\theta = -\frac{3}{2}$$
   $$\therefore \theta = -56.31^\circ$$
   But we need the second quadrant angle. Therefore, $\theta = -56.31^\circ + 180^\circ = 123.69^\circ$.
   b.
   $$\frac{1}{2}y - x = 3$$
   $$\therefore \frac{1}{2}y = x + 3$$
   $$\therefore y = 2x + 6$$
   $$m = 2$$
   $$\tan\theta = 2$$
   $$\therefore \theta = 63.43^\circ$$
   c.
   $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-3)}{4 - 2} = \frac{10}{2} = 5$$
   $$\tan\theta = 5$$
   $$\therefore \theta = 78.69^\circ$$
   d. $m = 0$. Therefore, $\theta = 0^\circ$
   e. m is undefined. Therefore $\theta = 90^\circ$

Back to Exercise 3.1

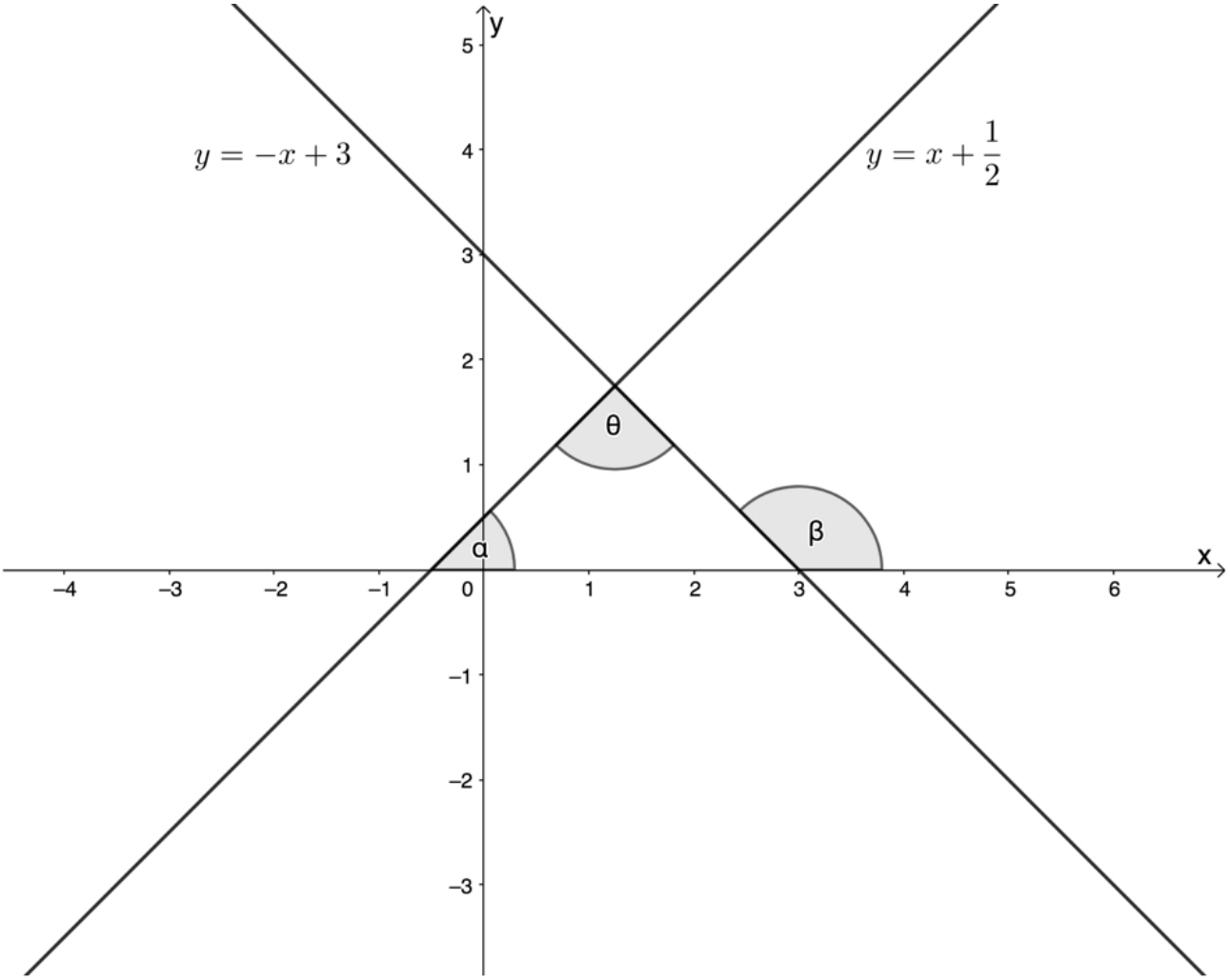
Exercise 3.2

$$m_1 = \tan 165.964^\circ$$
$$\therefore m_1 = -0.25 = -\frac{1}{4}$$

$m_2 = 4$
$y - y_1 = m(x - x_1)$
$\therefore y - 2 = 4\,(x - (-1))$
$\therefore y - 2 = 4(x + 1)$
$\therefore y - 2 = 4x + 4$
$\therefore y = 4x + 6$

Back to Exercise 3.2

Exercise 3.3



$y = -x + 3$
$m = -1$
$\tan\beta = -1$
$\therefore \beta = 135^\circ$
Therefore, the angle next to β inside the triangle is 45° (angles on straight line are supplementary).
$y - x = \frac{1}{2}$
$\therefore y = x + \frac{1}{2}$
$m = 1$
$\tan\alpha = 1$
$\therefore \alpha = 45^\circ$
$\theta = 180^\circ - 45^\circ - 45^\circ = 90^\circ$
The two straight lines are perpendicular.

Back to Exercise 3.3

Unit 3: Assessment

1.

a.

$$\begin{aligned} BD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0-2)^2 + (1-7)^2} \\ &= \sqrt{2^2 + 6^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

b.

$$\begin{aligned} CA &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2-4)^2 + (5-3)^2} \\ &= \sqrt{(-6)^2 + 2^2} \\ &= \sqrt{36 + 4} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

$CA = BD$

c.

$$\begin{aligned} M_{BD} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{2+0}{2}, \frac{7+1}{2}\right) \\ &= \left(\frac{2}{2}, \frac{8}{2}\right) \\ &= (1, 4) \end{aligned}$$

d. $A(4, 3)$, $M(1, 4)$

$$m_{AM} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{1-4} = -\frac{1}{3}$$

$$m_{BD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-7}{0-2} = \frac{6}{2} = 3$$

$m_{AM} \times m_{BD} = -1$. Therefore $AM \perp BD$.

e. $ABCD$ is a rhombus. The diagonals bisect at $90°$.

2.

a.

$$\begin{aligned} M_{QR} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{7-4}{2}, \frac{-1+2}{2}\right) \\ &= \left(\frac{3}{2}, \frac{1}{2}\right) \end{aligned}$$

b. $$m_{QP} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-2}{-2-(-4)} = \frac{-6}{-2+4} = -\frac{6}{2} = -3$$

$$m_{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-4)}{7-(-2)} = \frac{-1+4}{7+2} = \frac{3}{9} = \frac{1}{3}$$

$m_{QP} \times m_{PR} = -1$. Therefore, $QP \perp PR$.

c.

$$\tan\theta = m_{QP} = -3$$
$$\therefore \theta = 108^\circ$$

3.

a.

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$
$$\therefore \frac{y-3}{x-2} = \frac{-3-3}{5-2}$$
$$\therefore \frac{y-3}{x-2} = -\frac{6}{3} = -2$$
$$\therefore y-3 = -2(x-2)$$
$$\therefore y-3 = -2x+4$$
$$\therefore y = -2x+7$$

b. S is the x-axis intercept of the line segment PR.
x-intercept (let $y=0$):
$$0 = -2x+7$$
$$\therefore 2x = 7$$
$$\therefore x = \frac{7}{2}$$
$$S(\frac{7}{2}, 0)$$

c.

$$m_{PR} = -2 = \tan\theta$$
$$\therefore \theta = 116.57^\circ$$

d. $m_{PQ} = \frac{y_2-y_1}{x_2-x_1} = \frac{0-3}{-4-2} = \frac{-3}{-6} = \frac{1}{2}$

e. $m_{PR} = -2, m_{PQ} = \frac{1}{2}$
$m_{PR} \times m_{PQ} = -1$. Therefore, $P\hat{Q}R = 90^\circ$

f. $m_{PQ} = \frac{1}{2}$. Therefore, gradient of the line is $\frac{1}{2}$.
Equation of the line is $y = \frac{1}{2}x$

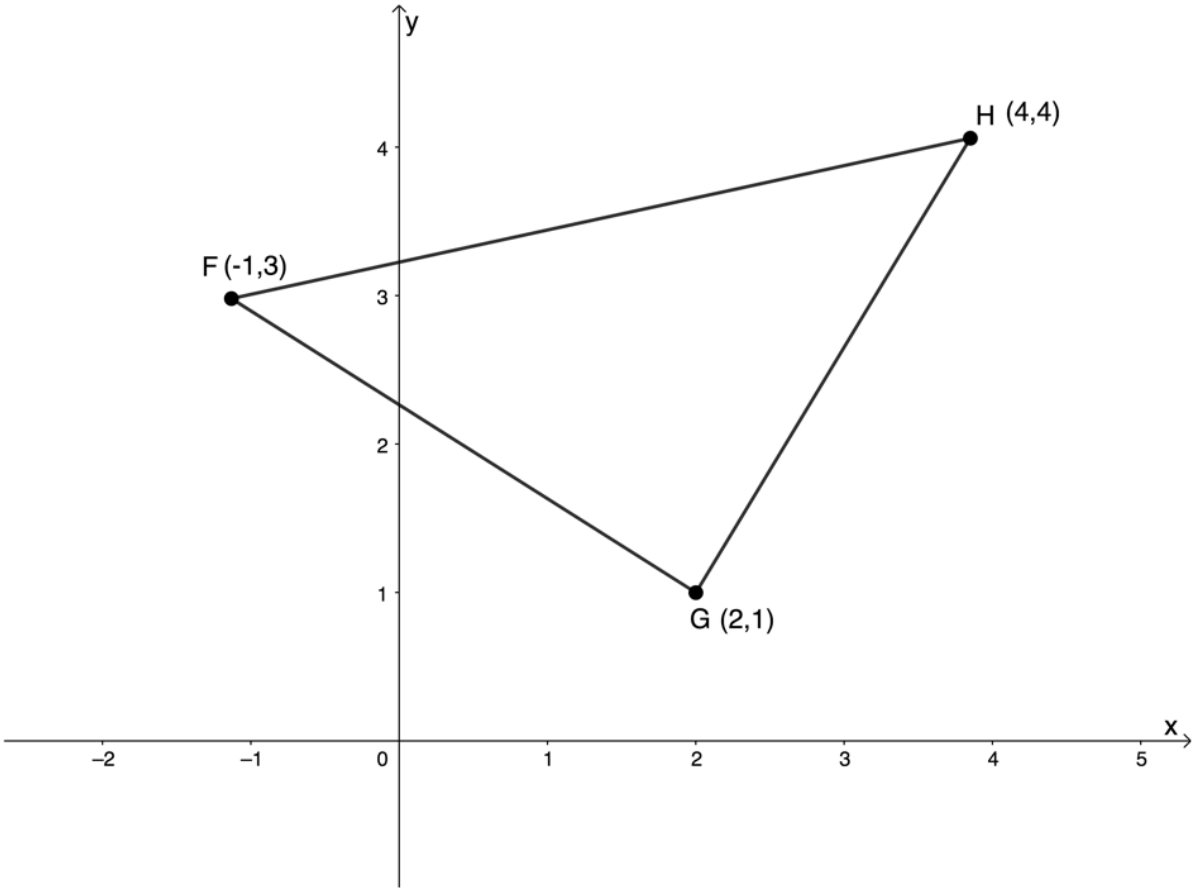
g.

$$M_{QR} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{5-4}{2}, \frac{-3+0}{2}\right)$$
$$= \left(\frac{1}{2}, -\frac{3}{2}\right)$$

h. $m_{PR} = -2, (\frac{1}{2}, -\frac{3}{2})$
$$y-y_1 = m(x-x_1)$$
$$\therefore y-\left(-\frac{3}{2}\right) = -2\left(x-\frac{1}{2}\right)$$
$$\therefore y+\frac{3}{2} = -2x+1$$
$$\therefore y = -2x-\frac{1}{2}$$

4. Consider triangle FGH with vertices $F(-1,3)$, $G(2,1)$ and $H(4,4)$.

a.



b.

$$\begin{aligned} FG &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - (-1))^2 + (1 - 3)^2} \\ &= \sqrt{(3)^2 + (2)^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} GH &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - 1)^2 + (4 - 2)^2} \\ &= \sqrt{(3)^2 + (2)^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} \end{aligned}$$

$FG = GH$. Therefore, ΔFGH is an isosceles triangle.

c.

$$\begin{aligned} M_{FH} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{4 - 1}{2}, \frac{4 + 3}{2}\right) \\ &= \left(\frac{3}{2}, \frac{7}{2}\right) \end{aligned}$$

$$m_{FH} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3}{4 - (-1)} = \frac{1}{5}$$

$m_{PQ} = -5$ and PQ passes through $M(\frac{3}{2}, \frac{7}{2})$.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - \frac{7}{2} = -5\left(x - \frac{3}{2}\right)$$
$$\therefore y - \frac{7}{2} = -5x + \frac{15}{2}$$
$$\therefore y = -5x + \frac{22}{2}$$
$$y = -5x + 11$$

d. $G(2,1)$. If G lies on PQ, G will satisfy the equation of PQ.
$y = -5x + 11$
Substitute the x-coordinate from $G(2,1)$ into the equation of PQ:
$$\begin{aligned} \text{y} &= -2(5) + 11 \\ &= -10 + 11 \\ &= 1 \end{aligned}$$
The x- and y-coordinates of G satisfy the equation of PQ. Therefore G lies on PQ.

e. $m_{GH} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{4-2} = \frac{3}{2}$
Therefore, gradient of the line passing through $F(-1,3)$ is $\frac{3}{2}$.
$$y - y_1 = m(x - x_1)$$
$$\therefore y - 3 = \frac{3}{2}(x - (-1))$$
$$\therefore y - 3 = \frac{3}{2}x + \frac{3}{2}$$
$$\therefore y = \frac{3}{2}x + \frac{9}{2}$$

Back to Unit 3: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example3.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise3.3A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA4a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

SUBJECT OUTCOME X

SPACE, SHAPE AND MEASUREMENT: SOLVE PROBLEMS BY CONSTRUCTING AND INTERPRETING TRIGONOMETRIC MODELS

Subject outcome

Subject outcome 3.3: Solve problems by constructing and interpreting trigonometric models

Learning outcomes

- Derive and use the values of the trigonometric functions (in surd form where applicable) of 30°, 45° and 60°.
- Use the reduction formulae and special angles to solve trigonometric expressions and prove equations in all four quadrants (without the use of a calculator) for the following functions:
  - $\sin(90^\circ \pm \theta)$; $\cos(90^\circ \pm \theta)$
  - $\sin(180^\circ \pm \theta)$; $\cos(180^\circ \pm \theta)$; $\tan(180^\circ \pm \theta)$
  - $\sin(360^\circ \pm \theta)$; $\cos(360^\circ \pm \theta)$; $\tan(360^\circ \pm \theta)$
- Use the following trigonometric identities to simplify expressions and prove equations:
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
  - $\sin^2\theta + cos^2\theta = 1$
- Solve trigonometric equations (with the use of a calculator) involving reduction formulae using special triangles for the three trigonometric functions in all four quadrants.
  - $\sin(90^\circ \pm \theta)$; $\cos(90^\circ \pm \theta)$
  - $\sin(180^\circ \pm \theta)$; $\cos(180^\circ \pm \theta)$; $\tan(180^\circ \pm \theta)$
  - $\sin(360^\circ \pm \theta)$; $\cos(360^\circ \pm \theta)$; $\tan(360^\circ \pm \theta)$
- Apply the sine, cosine and area rules.
- Solve problems in two dimensions by using the sine, cosine and area rules by interpreting given geometric and trigonometric models.

Unit 1 outcomes

By the end of this unit you will be able to:

- Simplify trigonometric ratios involving the angles $30°$, $45°$ and $60°$.

Unit 2 outcomes

By the end of this unit you will be able to:

- Apply reduction formulae for function values of $(90° \pm \theta)$.
- Apply reduction formulae for function values of $(180° \pm \theta)$.
- Apply reduction formulae for function values of $(360° \pm \theta)$.
- Simplify trigonometric ratios involving any combination of reduction formulae.

Unit 3 outcomes

By the end of this unit you will be able to:

- Use the quotient identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to prove equations and simplify expressions.
- Use the square identity $\sin^2\theta + \cos^2\theta = 1$ to prove equations and simplify expressions.

Unit 4 outcomes

By the end of this unit you will be able to:

- Solve trigonometric equations using a general solution.

Unit 5 outcomes

By the end of this unit you will be able to:

- Apply the area rule in 2-D triangles.

Unit 6 outcomes

By the end of this unit you will be able to:

- Use the sine rule to find an unknown length of a triangle.
- Use the sine rule to find an unknown angle of a triangle.

Unit 7 outcomes

By the end of this unit you will be able to:

- Use the cosine rule to find the length of an unknown side of a triangle.
- Use the cosine rule to find an unknown angle of a triangle.

Unit 8 outcomes

By the end of this unit you will be able to:

- Solve real problems by applying the area, sine and cosine rules.

Unit 1: Special angles

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Simplify trigonometric ratios involving the angles $30°$, $45°$ and $60°$.

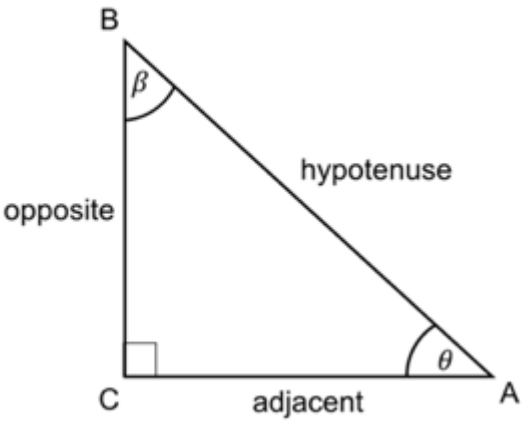
What you should know

Before you start this unit, make sure you can:

- Define the three basic trigonometric ratios of sine, cosine and tangent. Refer to level 2 subject outcome 3.6 unit 1 if you need help with this.

Introduction

We know by now that the trigonometric, or trig, ratios of sine, cosine and tangent define the ratios of sides in a right-angled triangle in relation to its angles. In the following triangle, for example, the three basic trig ratios are:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AB}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AC}{AB}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjascent}} = \frac{BC}{AC}$$

Figure 1: The three basic trigonometric ratios of sine, cosine and tangent

Most of the time, we need a calculator to work out what these ratios are. However, there are some special angles that give us 'nice' answers. You may have encountered these special angles in previous units. These angles come up so often that it is very worthwhile just remembering their trig ratios. This unit is all about these special angles.

The special angles

There are three special angles. They are $30°$, $45°$ and $60°$. They are special because they provide complementary pairs of angles ($30°$ and $60°$, and $45°$ and $45°$) for right-angled triangles. Trig ratios can easily be read off right-angled triangles, which can also be solved using Pythagoras' Theorem.

Activity 1.1: Special angles

Time required: 15 minutes

What you need:

- a pen or pencil
- a blank piece of paper
- a ruler
- a protractor

What to do:

In this activity we are going to discover why the angles of $30°$, $45°$ and $60°$ are considered 'special angles'.

1. On your piece of paper, draw a right-angled triangle with a $30°$ angle in it making sure that the length of the side opposite the $30°$ angle is a whole number.
2. Measure the length of the hypotenuse of this triangle.
3. Write down the value of $\sin 30°$.
4. Calculate, using the theorem of Pythagoras what the length of the side adjacent to the $30°$ angle is.
5. Write down the values of $\cos 30°$ and $\tan 30°$.
6. Now write down what $\sin 60°$, $\cos 60°$ and $\tan 60°$ are. Do you need to draw a new triangle?
7. On your piece of paper, draw another right-angled triangle with a $45°$ angle in it, making sure that the length of the side opposite this $45°$ angle is a whole number.
8. What is the size of the other angle?
9. Measure, or work out using the theorem of Pythagoras, what the lengths of the other two sides are and write down the values of $\sin 45°$, $\cos 45°$ and $\tan 45°$.

What did you find?

1. Let's assume that we drew a triangle like this where the side opposite the $30°$ angle was $10\ \text{cm}$ (see Figure 2).

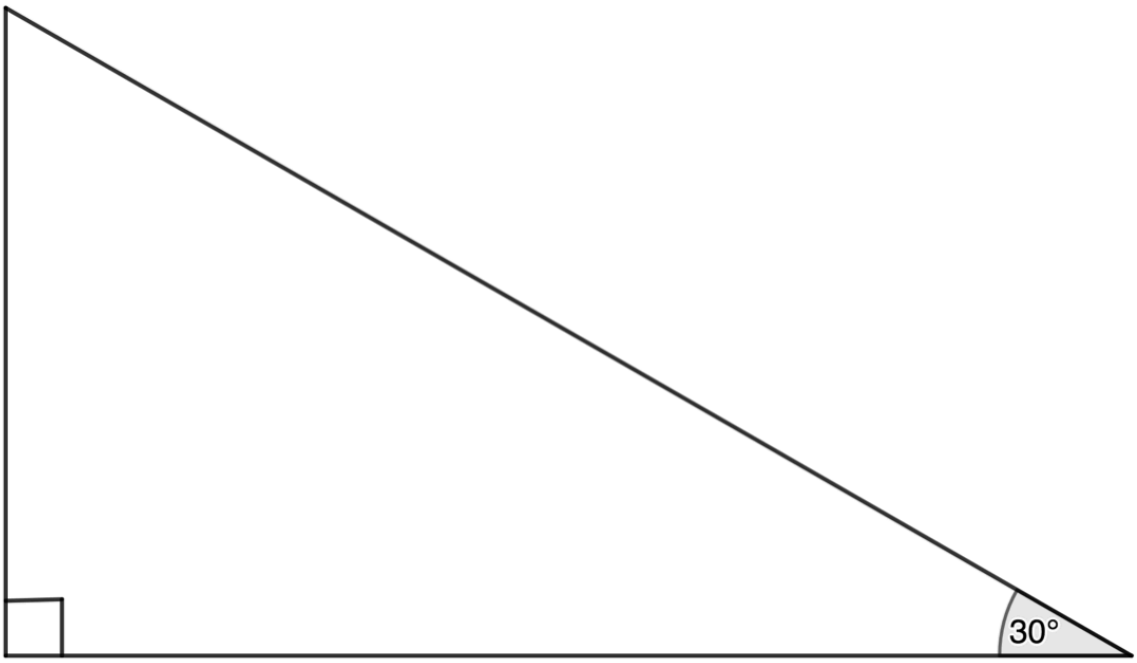


Figure 2: Right-angled triangle with a 30° angle

2. You should find that the hypotenuse is exactly double the length of the side opposite the 30° angle. In our case, it is $20\ \text{cm}$.
3. $\sin 30^\circ = \frac{1}{2}$. Remember, it does not matter how big your triangle is. The length of the side opposite the angle and the hypotenuse are always in the **same ratio**. When the angle is 30°, this ratio is $\frac{1}{2}$.
4. Using Pythagoras, the length of the side adjacent to the 30° angle is $\sqrt{3}$ or some multiple of $\sqrt{3}$ depending on how big your triangle is.
5. $\cos 30^\circ = \frac{\sqrt{3}}{2}$. Remember, it does not matter how big your triangle is. The length of the side adjacent the angle and the hypotenuse are always in the same ratio. When the angle is 30°, this ratio is $\frac{\sqrt{3}}{2}$.

   $\tan 30^\circ = \frac{1}{\sqrt{3}}$. Remember, it does not matter how big your triangle is. The length of the side adjacent the angle and the hypotenuse are always in the **same ratio**. When the angle is 30°, this ratio is $\frac{1}{\sqrt{3}}$.
6. The other angle in our triangle is 60° so we can just use this as our reference angle (see Figure 3). Now

   $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$ and $\tan 60^\circ = \frac{\sqrt{3}}{1}$.

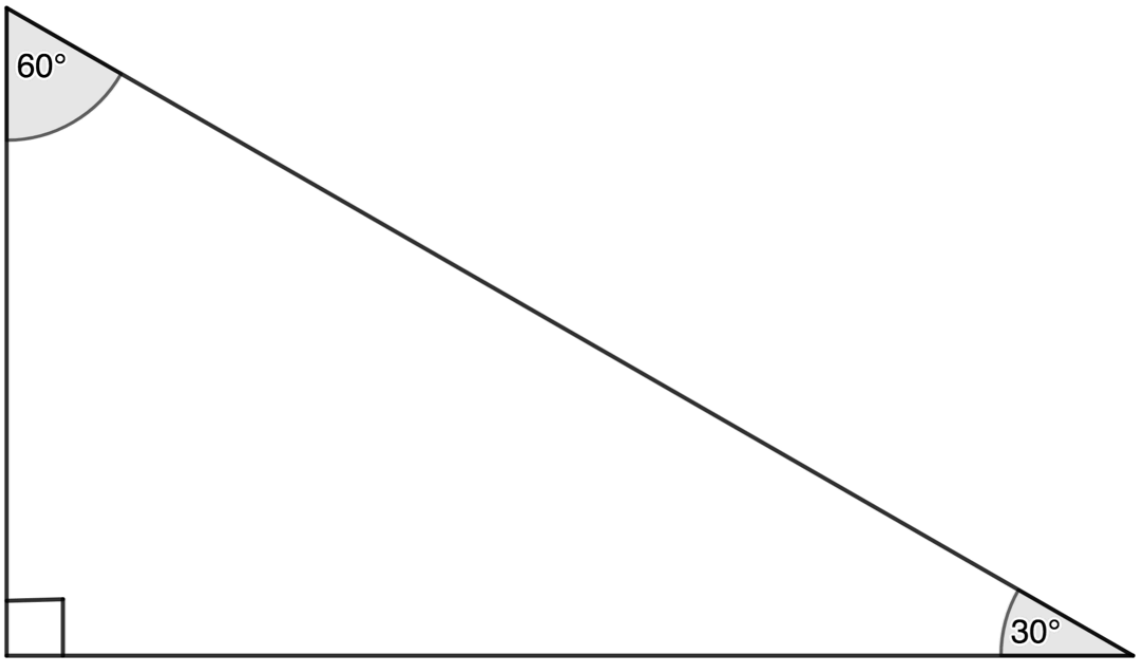


Figure 3: The other angle in the triangle is 60°

7. Let's assume that we drew a triangle like this where the side opposite the 45° angle was $10\ \text{cm}$ (see Figure 4).

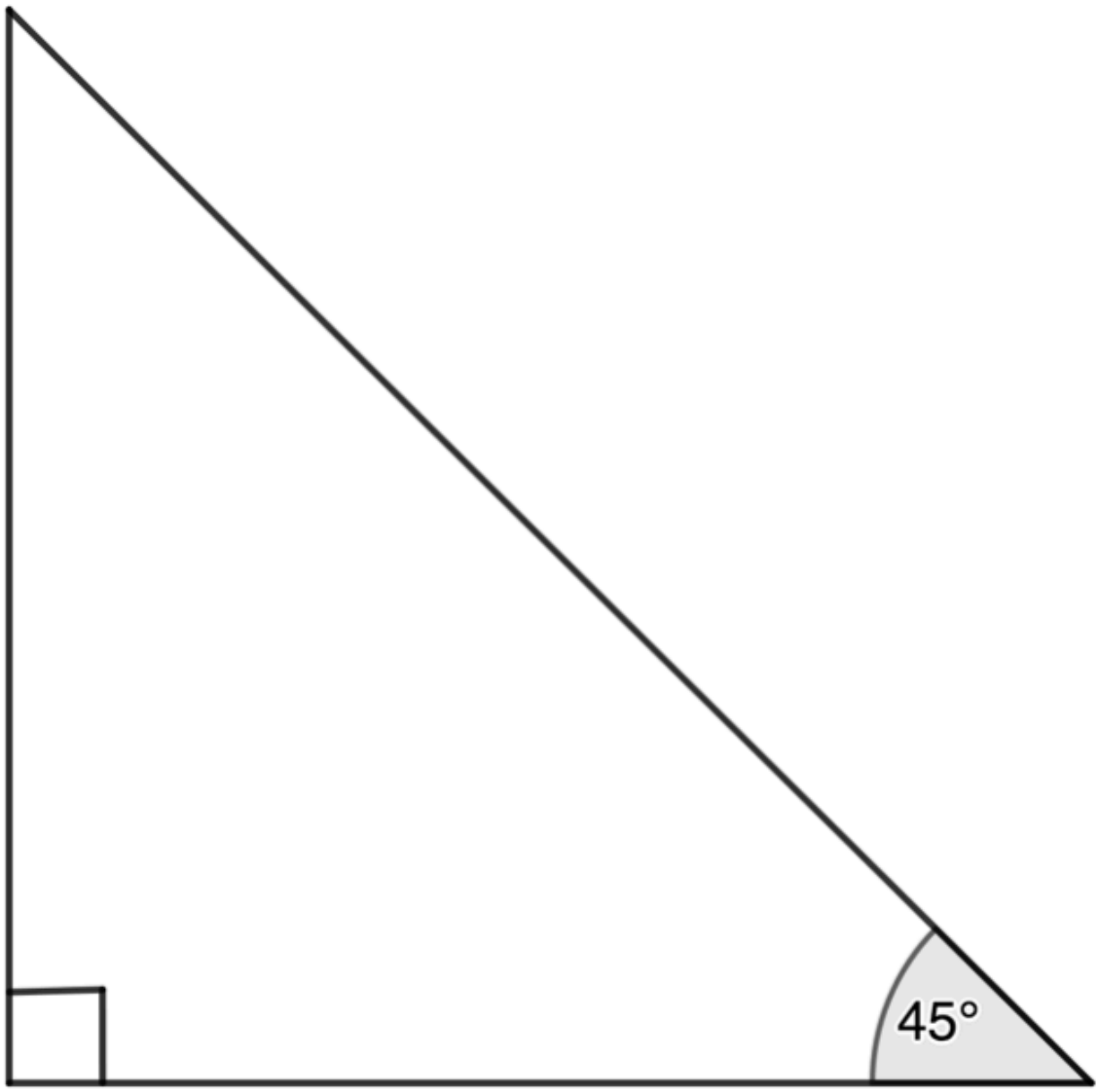
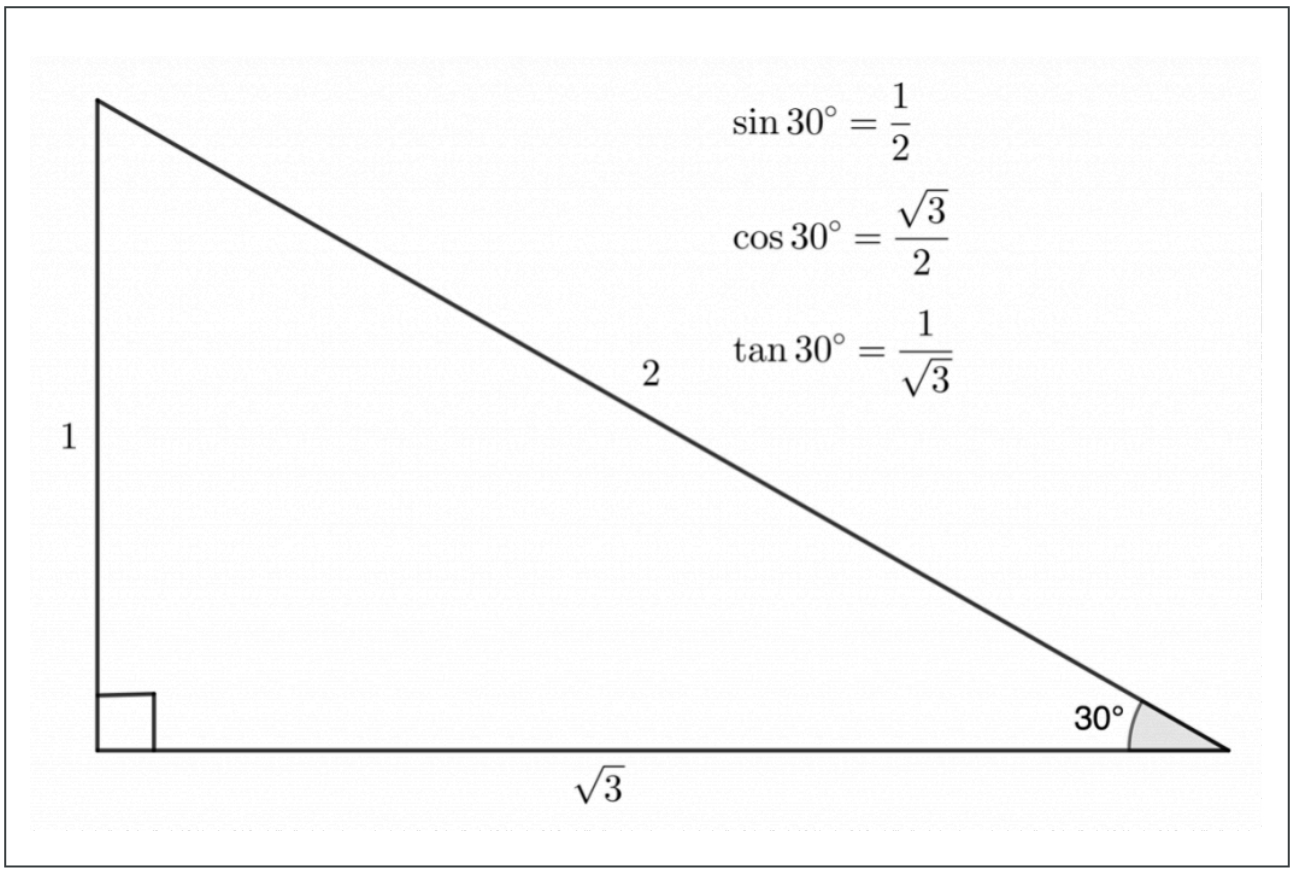


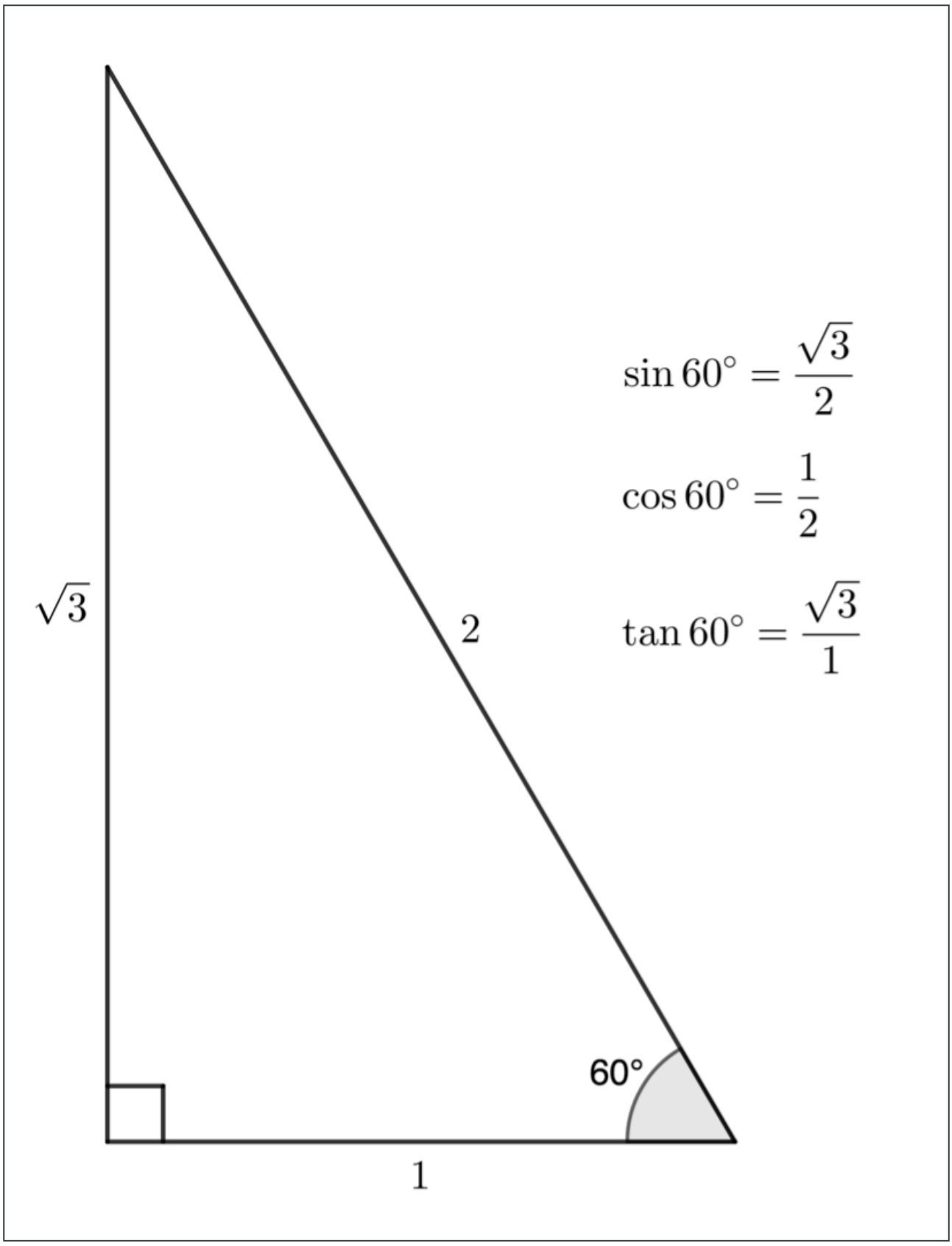
Figure 4: Right-angled triangle with a $45°$ angle

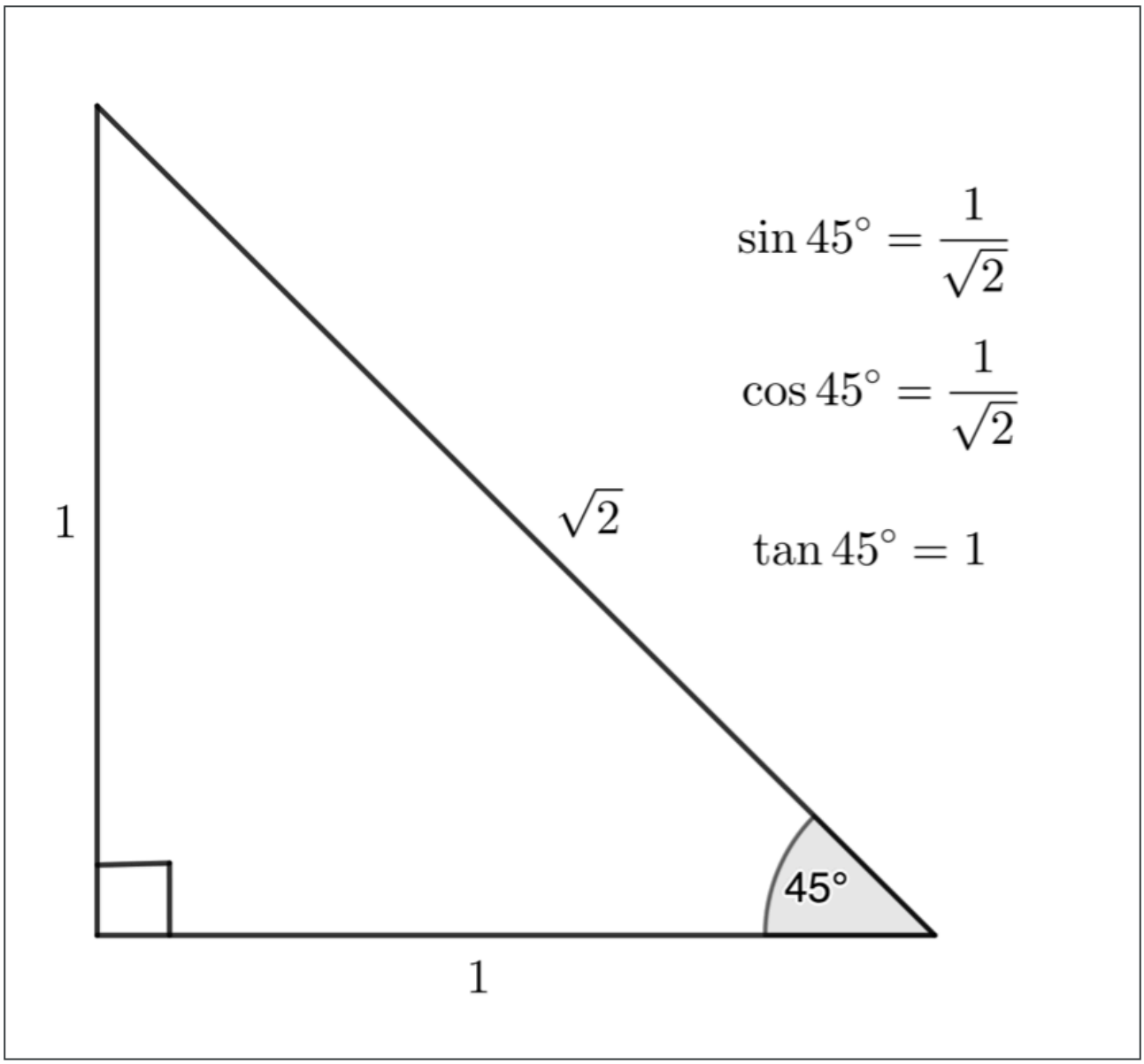
8. The other angle is also $45°$.
9. You will find that the three trig ratios are $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$ and $\tan 45° = 1$.

We have discovered the three basic trig ratios for the angles $30°$, $45°$ and $60°$. Remember that it does not matter how big the triangles are. The sides will always be in the same ratio. This ratio depends only on the angle.

Special angles:


$\sin 30° = \frac{1}{2}$
$\cos 30° = \frac{\sqrt{3}}{2}$
$\tan 30° = \frac{1}{\sqrt{3}}$
2
1
30°
$\sqrt{3}$


$\sqrt{3}$
2
1
60°
$\sin 60° = \frac{\sqrt{3}}{2}$
$\cos 60° = \frac{1}{2}$
$\tan 60° = \frac{\sqrt{3}}{1}$



Simplify trig expressions with special angles

Note

If you ever come across a question requiring the calculation of values of trig ratios 'without a calculator', you know that you will be dealing with the trig ratios of these three special angles.

Example 1.1

Simplify the following trig expression without a calculator, leaving your answer in simplest surd form: $\sin 45° + \sin 30°$.

Solution

We are told that we may not use a calculator. Therefore, we need to use the special angle ratios.

$$\begin{aligned}\sin 45^\circ + \sin 30^\circ &= \frac{1}{\sqrt{2}} + \frac{1}{2} \\ &= \frac{2+\sqrt{2}}{2\sqrt{2}} \qquad \text{We prefer not to have surds in the denominator, so multiply by } \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}\times\left(2+\sqrt{2}\right)}{\sqrt{2}\times 2\sqrt{2}} \\ &= \frac{2\sqrt{2}+2}{4}\end{aligned}$$

Note

For greater accuracy, the denominators of fractions should be rational, rather than irrational numbers. The value of irrational numbers (such as $\sqrt{2}$) can only be estimated, whereas the exact value of rational numbers is known. When we multiply by $\frac{\sqrt{2}}{\sqrt{2}}$ in the above example, we say that we are 'rationalising the denominator'.



Example 1.2

Simplify the following trig expression without a calculator, leaving your answer in simplest surd form: $\dfrac{\cos 45^\circ + \sin 45^\circ}{\tan 45^\circ}$.

Solution

$$\begin{aligned}\frac{\cos 45^\circ + \sin 45^\circ}{\tan 45^\circ} &= \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1} \\ &= \frac{2}{\sqrt{2}} \qquad \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}} \text{ to rationalise the denominator} \\ &= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{2}}{2}\end{aligned}$$



Example 1.3

Simplify the following trig expression without a calculator, leaving your answer in simplest surd form: $\frac{\sin 30^\circ \cos 30^\circ}{\tan 30^\circ} + \frac{\sin 60^\circ \cos 60^\circ}{\tan 60^\circ}$.

Solution

$$\begin{aligned}
\frac{\sin 30^\circ \cos 30^\circ}{\tan 30^\circ} + \frac{\sin 60^\circ \cos 60^\circ}{\tan 60^\circ} &= \frac{\frac{1}{2} \times \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{3}}} + \frac{\frac{\sqrt{3}}{2} \times \frac{1}{2}}{\frac{\sqrt{3}}{1}} \\
&= \frac{\frac{\sqrt{3}}{4}}{\frac{1}{\sqrt{3}}} + \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{1}} \\
&= \frac{\sqrt{3}}{4} \times \frac{\sqrt{3}}{1} + \frac{\sqrt{3}}{4} \times \frac{1}{\sqrt{3}} \\
&= \frac{3}{4} + \frac{1}{4} \\
&= 1
\end{aligned}$$

Exercise 1.1

Simplify the following trig expression without a calculator, leaving your answer in simplest surd form:

1. $\sin 30^\circ + \tan 60^\circ$
2. $\frac{\sin^\circ 60}{\cos^\circ 60}$
3. $\sin^2 30^\circ + \cos^2 30^\circ$
4. $\sqrt{\sin^2 45^\circ \cdot \cos^2 45^\circ}$
5. $\frac{\sin 45^\circ}{\cos 45^\circ} + \tan 45^\circ$

The full solutions are at the end of the unit.

Finding special angles

In the previous section, we were asked to simplify expressions containing trig ratios of the special angles. If we discover that $\sin \theta = \frac{1}{2}$, for example, we can work in reverse to state that $\theta = 30^\circ$.

Example 1.4

Given the following diagram, determine the values of α and θ.

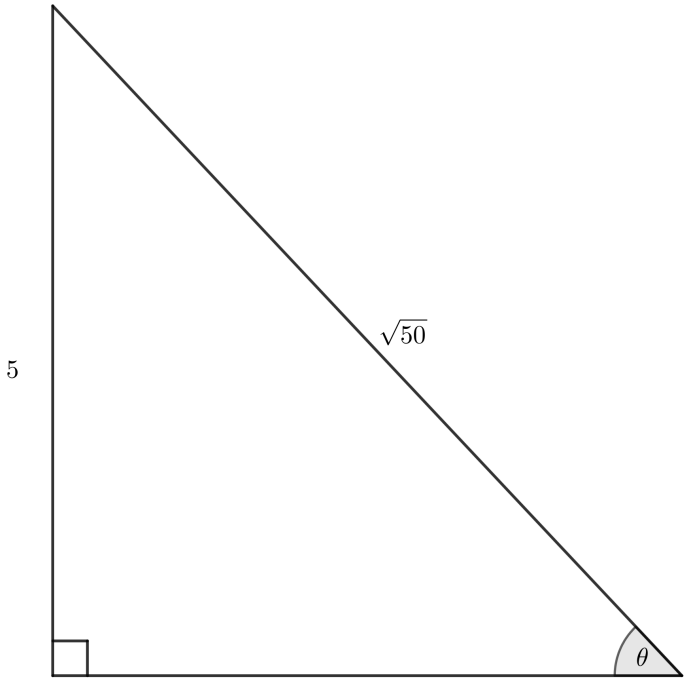
Solution

Let's start with θ as our angle. With respect to θ, we know that $\cos\theta = \dfrac{3\sqrt{3}}{6\sqrt{3}} = \dfrac{1}{2}$. Therefore, $\theta = 60^\circ$.

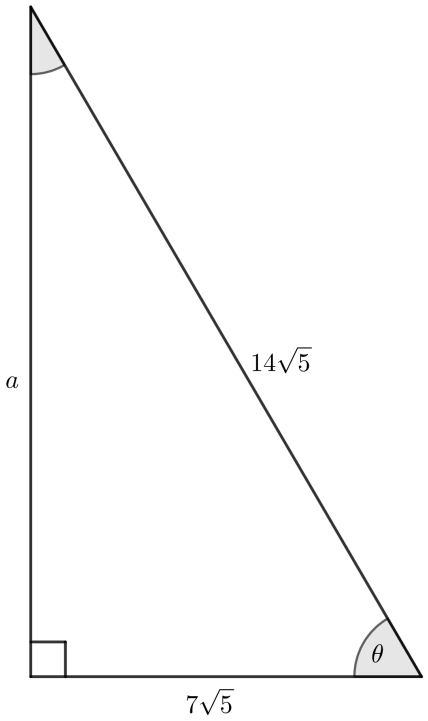
Because the angles inside a triangle are supplementary (they add up to 180°), we know that $\alpha = 180^\circ - 90^\circ - 60^\circ = 30^\circ$.

Exercise 1.2

1. Determine the value of θ without a calculator:



2. Using the following diagram:



a. Determine θ without a calculator.

b. Find:

   i. $\sin\theta$

   ii. $\tan\theta$

c. Determine the length of a.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to determine any of the basic trig ratios without a calculator if the angle involved is one of the special angles.
- How to determine the angle, if the sides of a triangle are in any of the special angle ratios.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. Simplify the following trig expressions without a calculator, leaving your answer in simplest surd form:

   a. $\sin 45^\circ + \tan 30^\circ$

b. $\sqrt{1-\sin^2 45^\circ}$

c. $\dfrac{\sin^2 30^\circ}{\tan^2 60^\circ \times \cos^2 60^\circ}$

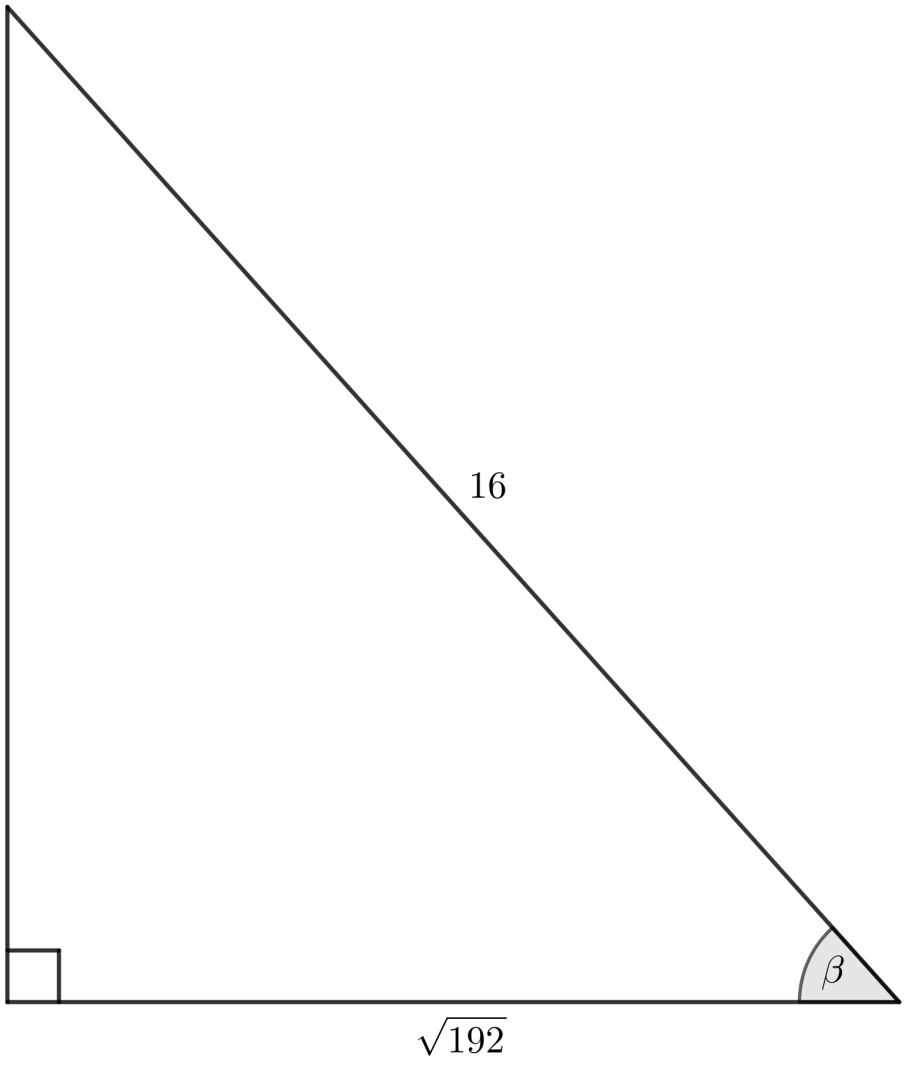
d. $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ + \tan 45^\circ$

2. Prove, without a calculator, that $\dfrac{\sin 30^\circ \cos 30^\circ \tan 45^\circ}{\sin 60^\circ} = \dfrac{1}{2}$.

3. Prove that $\dfrac{\sin 60^\circ . \cos 60^\circ}{\cos 45^\circ . \sin 45^\circ} = \dfrac{\sqrt{3}}{2}$.

4. Prove, without a calculator, that $\sin(60^\circ) . \tan 60^\circ + \cos 45^\circ . \sin 45^\circ = 2$.

5. Using the diagram:



a. Determine β without using a calculator.

b. Determine the value of $\dfrac{\tan \beta + \sin \beta}{\cos^2(90^\circ - \beta)}$ without using a calculator.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $$\begin{aligned}\sin 30^\circ + \tan 60^\circ &= \frac{1}{2} + \frac{\sqrt{3}}{1} \\ &= \frac{1+2\sqrt{3}}{2}\end{aligned}$$

2. $$\begin{aligned}\frac{\sin 60^\circ}{\cos 60^\circ} &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= \frac{\sqrt{3}}{2} \times \frac{2}{1} \\ &= \sqrt{3}\end{aligned}$$

3. $$\begin{aligned}\sin^2 30^\circ + \cos^2 30^\circ &= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= \frac{1}{4} + \frac{3}{4} \\ &= 1\end{aligned}$$

4. $$\begin{aligned}\sqrt{\sin^2 45^\circ \cdot \cos^2 45^\circ} &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{1}{2} \cdot \frac{1}{2}} \\ &= \sqrt{\frac{1}{4}} \\ &= \pm\frac{1}{2}\end{aligned}$$

5. $$\begin{aligned}\frac{\sin 45^\circ}{\cos 45^\circ} + \tan 45^\circ &= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} + 1 \\ &= 2\end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1.

$$\begin{aligned}\sin\theta &= \frac{5}{\sqrt{50}} \\ &= \frac{5}{\sqrt{25\times 2}} \\ &= \frac{5}{5\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \\ \therefore \theta &= 45^\circ\end{aligned}$$

2.

a.

$$\begin{aligned}\cos\theta &= \frac{7\sqrt{5}}{14\sqrt{5}} = \frac{1}{2} \\ \therefore \theta &= 60^\circ\end{aligned}$$

b.

i. $\sin\theta = \sin 60^\circ = \frac{\sqrt{3}}{2}$

ii. $\tan\theta = \tan 60^\circ = \frac{\sqrt{3}}{1}$

c.

$$\begin{aligned}\tan\theta &= \frac{\sqrt{3}}{1} = \frac{a}{7\sqrt{5}} \\ \therefore a &= 7\sqrt{3}\sqrt{5} \\ &= 7\sqrt{15}\end{aligned}$$

Note: You could also have used $\sin\theta = \sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{a}{14\sqrt{5}}$, etc.

Back to Exercise 1.2

Unit 1: Assessment

1.

a.

$$\begin{aligned}\sin 45^\circ + \tan 30^\circ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}\sqrt{3}} \\ &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}\sqrt{3}} \times \frac{\sqrt{2}\sqrt{3}}{\sqrt{2}\sqrt{3}} \\ &= \frac{3\sqrt{2}+2\sqrt{3}}{6}\end{aligned}$$

b.

$$\begin{aligned}\sqrt{1-\sin^2 45^\circ} &= \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} \\ &= \sqrt{1-\frac{1}{2}} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \quad \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

c.

$$\begin{aligned}\frac{\sin^2 30^\circ}{\tan^2 60^\circ \times \cos^2 60^\circ} &= \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{1}\right)^2 \times \left(\frac{1}{2}\right)^2} \\ &= \frac{\frac{1}{4}}{\frac{3}{1} \times \frac{1}{4}} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{4} \times \frac{4}{3} \\ &= \frac{1}{3}\end{aligned}$$

d.

$$\begin{aligned}\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ + \tan 45^\circ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} + 1 \\ &= \frac{3}{4} - \frac{1}{4} + 1 \\ &= \frac{1}{2} + 1 \\ &= \frac{3}{2}\end{aligned}$$

2.

$$\begin{aligned}\text{LHS} &= \frac{\sin 30^\circ \cos 30^\circ \tan 45^\circ}{\sin 60^\circ} \\ &= \frac{\frac{1}{2} \times \frac{\sqrt{3}}{2} \times 1}{\frac{\sqrt{3}}{2}} \\ &= \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}} \\ &= \frac{\sqrt{3}}{4} \times \frac{2}{\sqrt{3}} \\ &= \frac{1}{2} \\ &= \text{RHS}\end{aligned}$$

3.

$$\begin{aligned}\text{LHS} &= \frac{\sin 60^\circ . \cos 60^\circ}{\cos 45^\circ . \sin 45^\circ} \\ &= \frac{\frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} \\ &= \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2}} \\ &= \frac{\sqrt{3}}{4} \times \frac{2}{1} \\ &= \frac{\sqrt{3}}{2} = \text{RHS}\end{aligned}$$

4.

$$\begin{aligned}\text{LHS} &= \sin 60^\circ . \tan 60^\circ + \cos 45^\circ . \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{1} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \\ &= \frac{3}{2} + \frac{1}{2} \\ &= 2 = \text{RHS}\end{aligned}$$

5.

a.

$$\begin{aligned}\cos \beta &= \frac{\sqrt{192}}{16} = \frac{\sqrt{64 \cdot 3}}{16} = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2} \\ \therefore \beta &= 30^\circ\end{aligned}$$

b.

$$
\begin{aligned}
\frac{\tan\beta+\sin\beta}{\cos^2(90^\circ-\beta)} &= \frac{\tan 30^\circ+\sin 30^\circ}{\cos^2 60^\circ} \\
&= \frac{\frac{1}{\sqrt{3}}+\frac{1}{2}}{\left(\frac{1}{2}\right)^2} \\
&= \frac{\frac{2+\sqrt{3}}{2\sqrt{3}}}{\frac{1}{4}} \\
&= \frac{2+\sqrt{3}}{2\sqrt{3}} \times \frac{4}{1} \\
&= \frac{8+4\sqrt{3}}{2\sqrt{3}} \quad \text{Multiply by } \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{8\sqrt{3}+12}{6} \\
&= \frac{4\left(2\sqrt{3}+3\right)}{6} \\
&= \frac{2\left(2\sqrt{3}+3\right)}{3}
\end{aligned}
$$

Back to Unit 1: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- 30degree © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- 60degree © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- 45degree © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise1.2Q1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise1.2Q2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 2: Use reduction formulae to simplify trig ratios

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Apply reduction formulae for function values of $(90^\circ \pm \theta)$.
- Apply reduction formulae for function values of $(180^\circ \pm \theta)$.
- Apply reduction formulae for function values of $(360^\circ \pm \theta)$.
- Simplify trigonometric ratios involving any combination of reduction formulae.

What you should know

Before you start this unit, make sure you can:

- Define the three basic trigonometric ratios of sine, cosine and tangent.
- Use the CAST diagram to determine in which quadrants the three trigonometric ratios are positive or negative.

Introduction

We have said many times that one of the unique properties of the trigonometric functions is that they are **periodic**. They repeat themselves. We know that the period of the sine and cosine functions are 360° and the period of the tangent function is 180°.

A good way to understand why the trig functions are periodic is to consider the unit circle, something we have done on numerous occasions. Each time we complete one revolution of the unit circle, we are basically starting at 0° again and so the behaviour of the trig functions simply repeat themselves.

If you need a reminder of this and have an internet connection, then spend some time playing with the following interactive simulations for the sine and cosine functions. In each case, you will find a slider that will allow you to change the angle between -720° and 720°.

- Sine simulation
- Cosine simulation

For each function, remind yourself how the point on the unit circle, as the angle with the positive x-axis changes, produces the function's graph in each case. Also note how the functions repeat themselves every $360°$.

One of the consequences of this **periodicity** is that there are specific rules we can use to reduce the trig ratios of angles greater than $90°$ to their equivalents with angles between $0°$ and $90°$.

Reduce sine, cosine and tangent

Let's begin by investigating how we can reduce the sine of angles in the second, third and fourth quadrants to their equivalent acute or first quadrant ratios.

Activity 2.1: Reduce sine

Time required: 20 minutes

What you need:

- an internet connection

What to do:

For this activity, you will need a working Internet connection.

1. Open this interactive sine function simulation with a slider to change the angle from $0°$ to $360°$ If you have access to the internet, visit this interactive simulation.
   As you change the angle, you can see that the angles in the unit circle and on the graph change. You will also notice that the angles within each of the four unit circle quadrants have also been marked on the graph.
2. Move the slider slowly from $0°$ to $180°$ and notice each time the function value is 0.5. From unit 1 you should already know what angle results in a sine ratio of $\frac{1}{2}$.
3. When $\sin\theta = \frac{1}{2}$ in the second quadrant, what angle does the radius of the unit circle make with the positive x-axis? What angle does it make with the negative x-axis? What is the relationship between these two angles?
4. What can you say about the values of $\sin 30°$ and $\sin 150°$?

5. Does this relationship hold for other angles in the first and second quadrants? What value of θ in the second quadrant gives the same value as $\sin 45°$? What value of θ in the second quadrant gives the same value as $\sin 60°$? What is the size of the angle between the radius and the negative x-axis in each case?
6. Can you write a general expression for this relationship that is true for any angle for $0° \leq \theta \leq 90°$?
7. Now continue to move the slider into the third and fourth quadrants. What angles result in a function value of $\sin\theta = -\frac{1}{2}$?
8. What can you say about the values of $\sin 30°$ and $\sin 210°$ and $\sin 330°$?
9. Can you write general rules for these relationships that are true for $0° \leq \theta \leq 90°$ that are similar to the one you wrote in response to question 6? In each case, think about what angle the radius of the unit circle makes with the negative and positive x-axis respectively?

What did you find?

1. The interactive simulation at the link provided is shown in Figure 1.

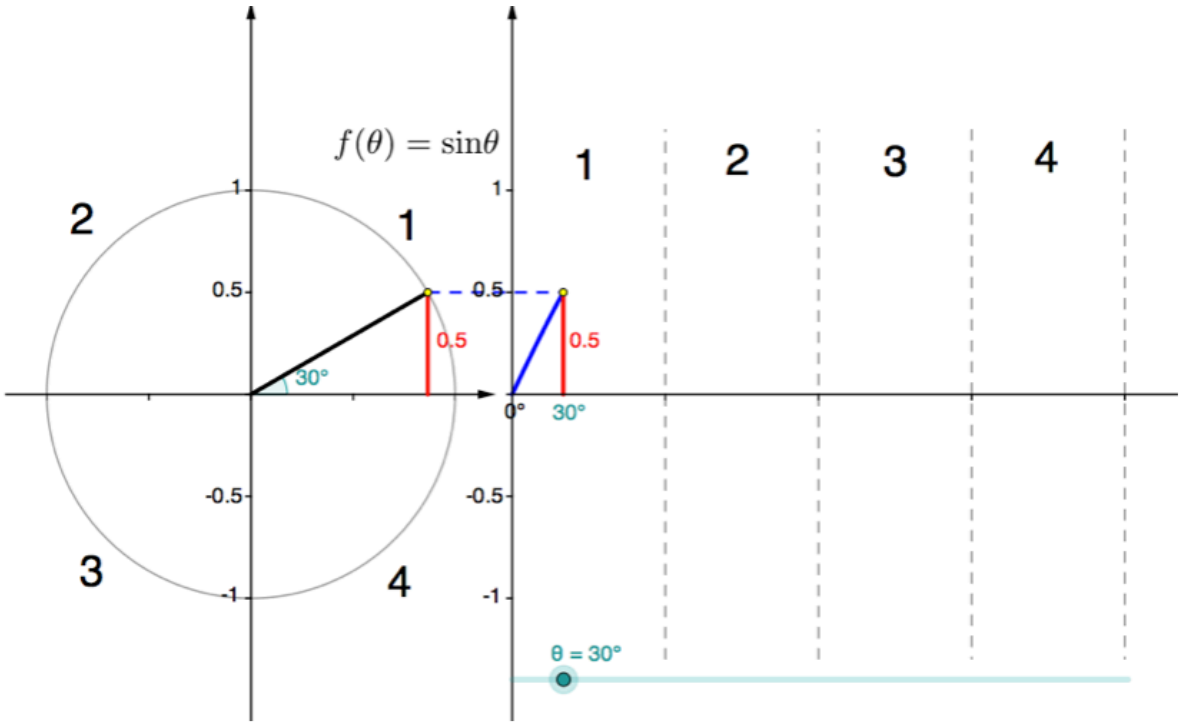


Figure 1: Interactive sine function simulation

2. $\sin\theta = \frac{1}{2}$ at two angles: $30°$ (see Figure 1) and $150°$ (see Figure 2).

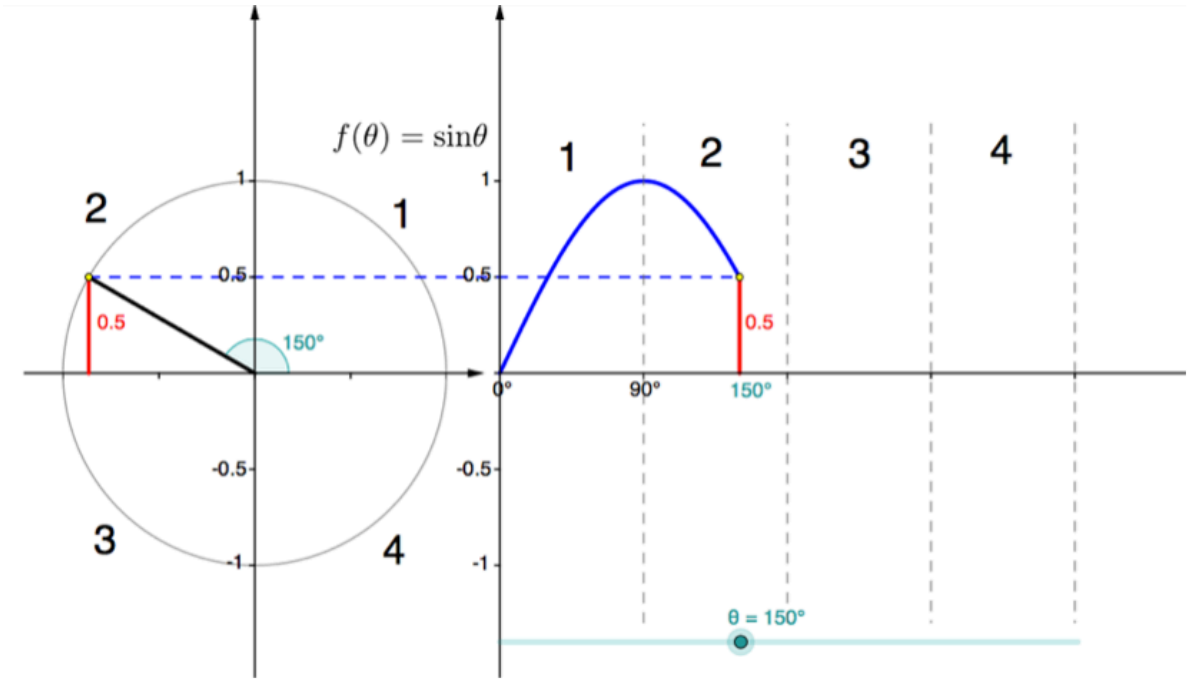


Figure 2: $\sin 150^\circ = \frac{1}{2}$

3. The radius makes an angle of 150° with the positive x-axis but an angle of 30° with the negative x-axis. Therefore, the angle made with the negative x-axis is $180^\circ - 150^\circ = 30^\circ$.
4. $\sin 30^\circ = \sin 150^\circ = 0.5$
5. $\sin 45^\circ = \sin 135^\circ$ (see Figure 3). The size of the angle between the radius and negative x-axis is 45°.

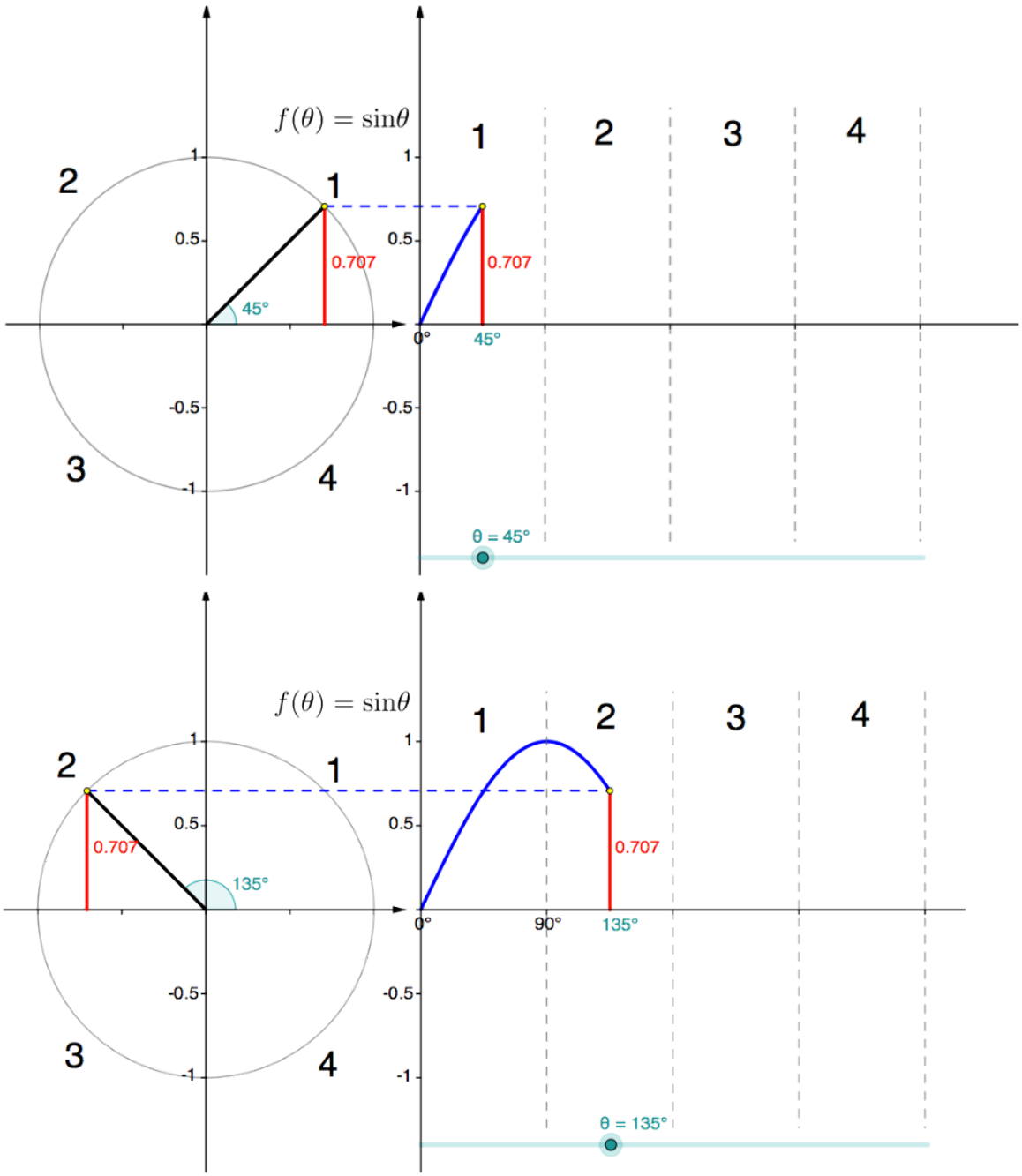


Figure 3: $\sin 45^\circ = \sin 135^\circ$

$\sin 60^\circ = \sin 120^\circ$ (see Figure 4). The size of the angle between the radius and negative x-axis is 60°.

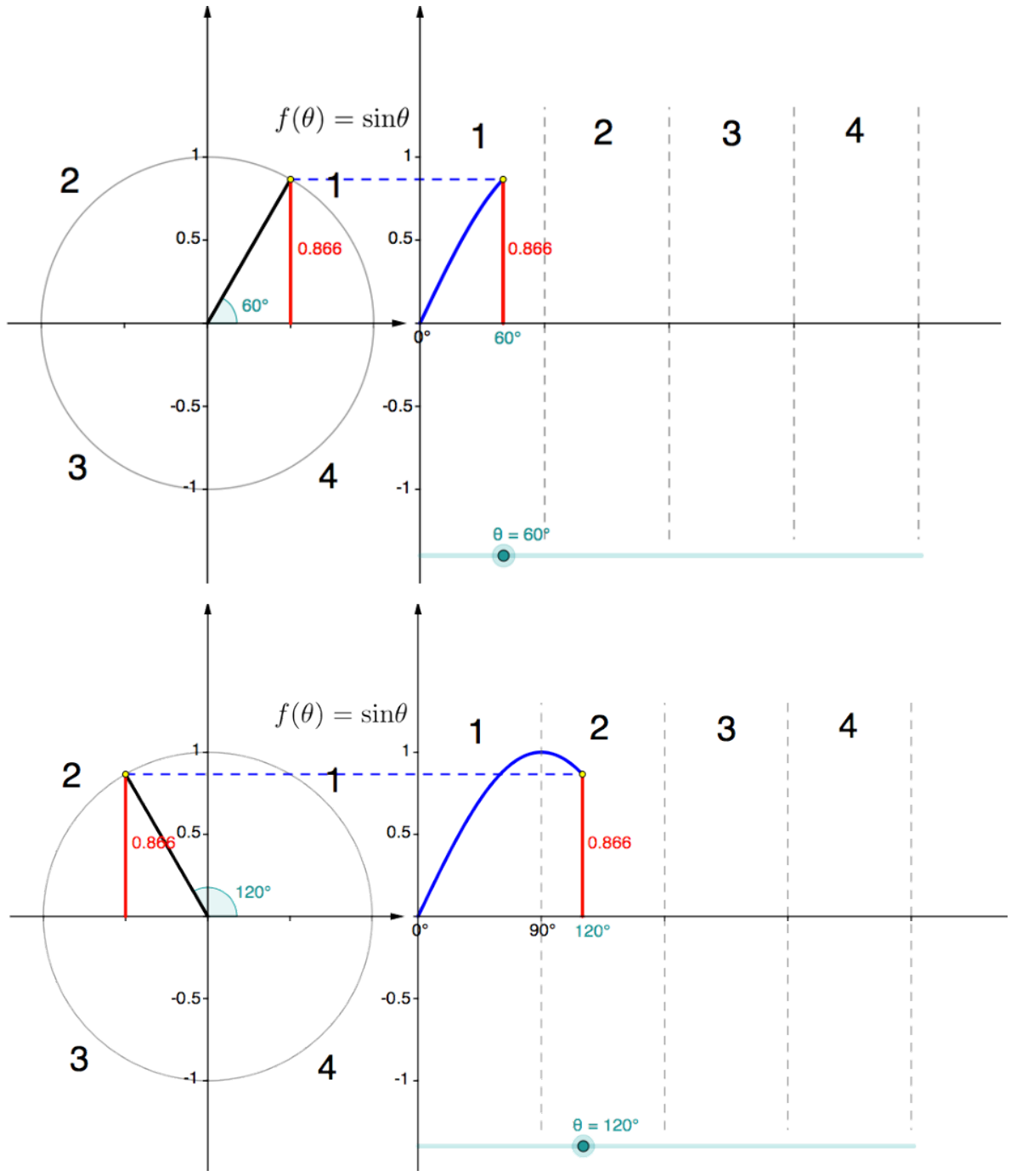


Figure 4: $\sin 60^\circ = \sin 120^\circ$

6. $\sin\theta = \sin(180^\circ - \theta)$
7. $\sin 210^\circ = -\frac{1}{2}$ and $\sin 330^\circ = -\frac{1}{2}$
8. $\sin 30^\circ = -\sin 210^\circ$ or $-\sin 30^\circ = \sin 210^\circ$ (see Figure 5)

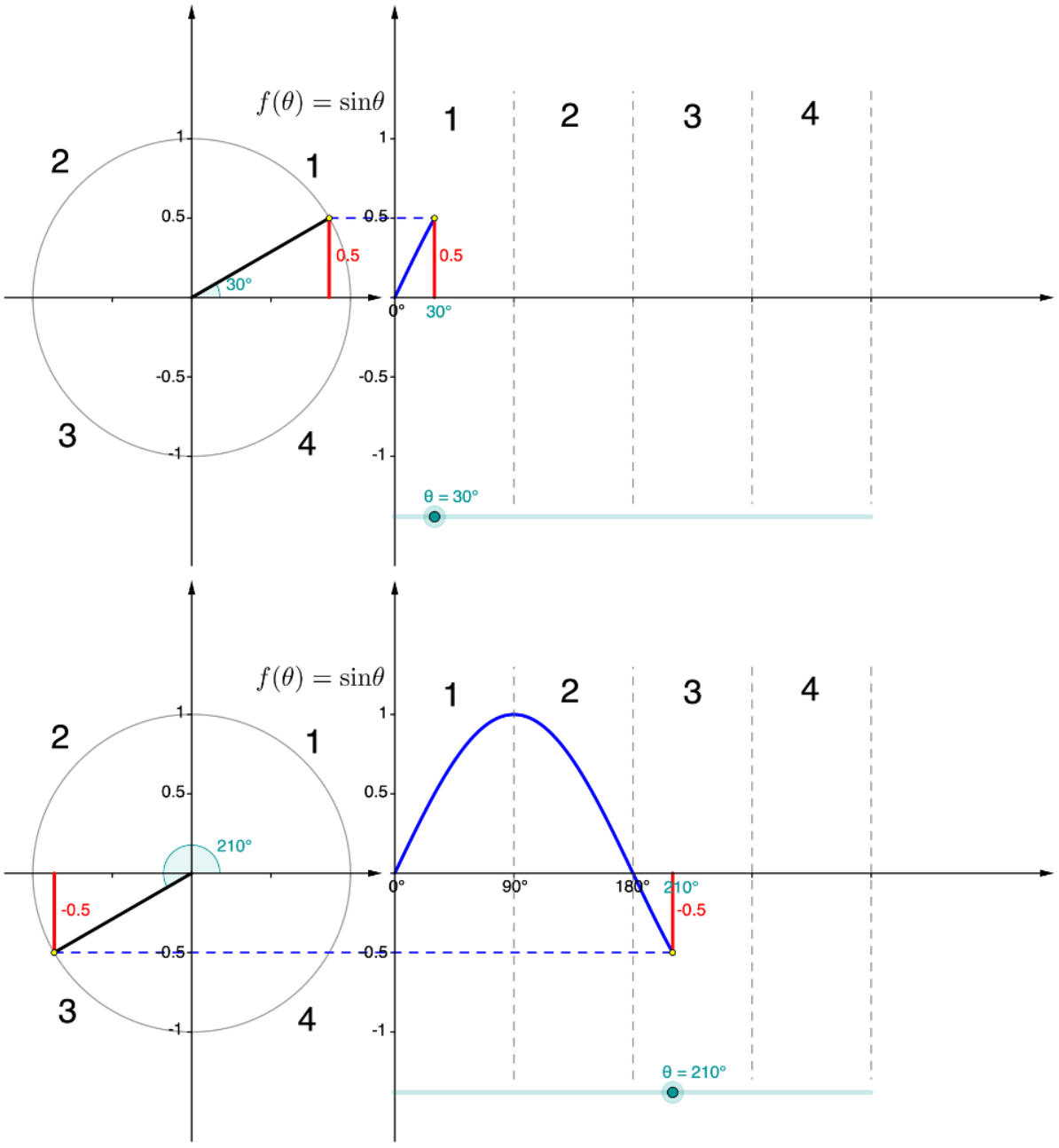


Figure 5: $\sin 30^\circ = -\sin 210^\circ$ *or* $-\sin 30^\circ = \sin 210^\circ$

$\sin 30^\circ = -\sin 330^\circ$ or $-\sin 30^\circ = \sin 330^\circ$ (see Figure 6)

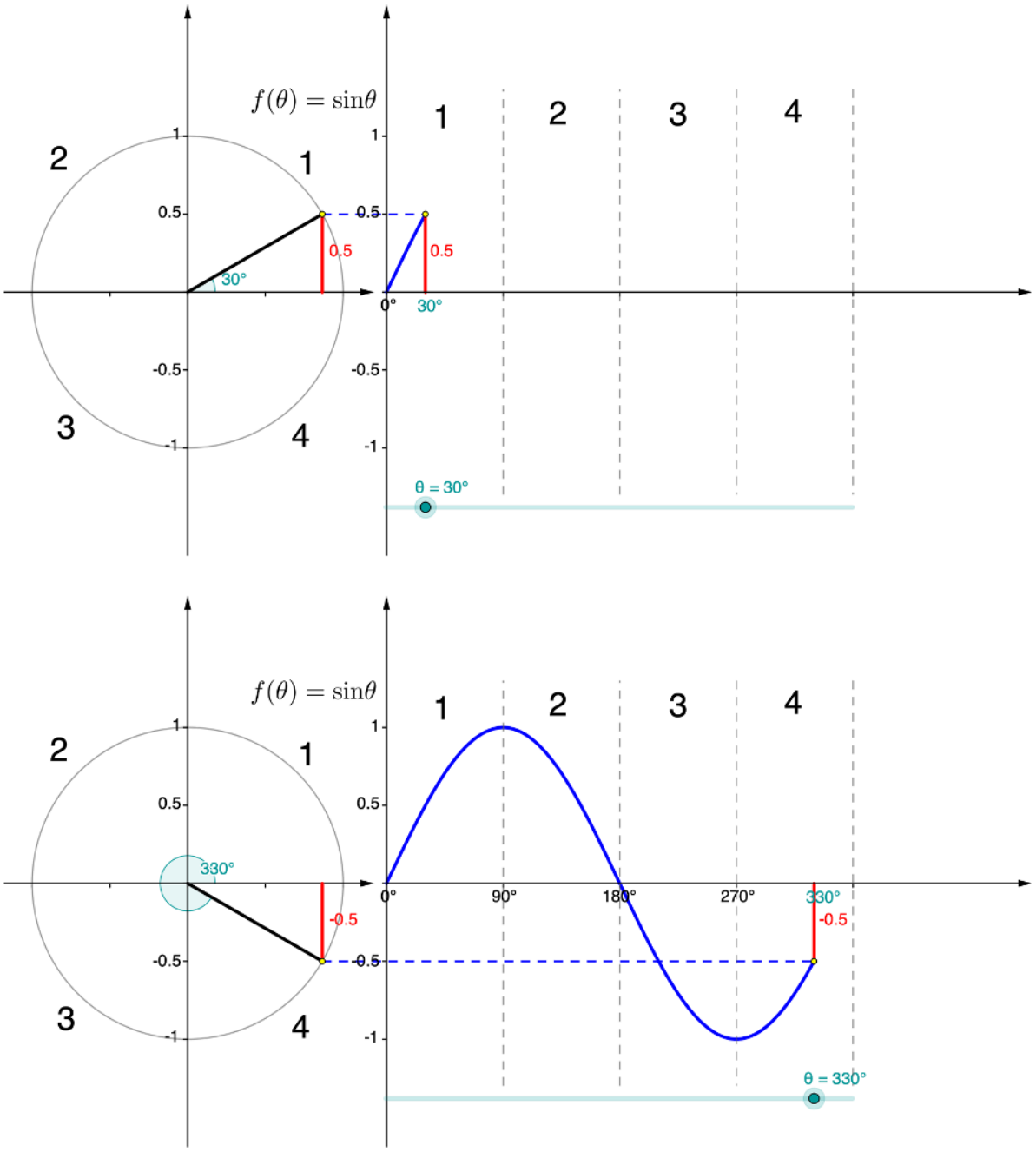


Figure 6: $\sin 30^\circ = -\sin 330^\circ$ *or* $-\sin 30^\circ = \sin 330^\circ$

9. $\sin\theta = -\sin(180^\circ + \theta)$ or
   $\sin\theta = -\sin(360^\circ - \theta)$ or $\sin\theta = -\sin(-\theta)$

We have ways to rewrite the sine of angles that are between 90° and 360° (angles that are in the second, third or fourth quadrants) in terms of the equivalent ratio of an acute angle in the first quadrant.

We saw that:

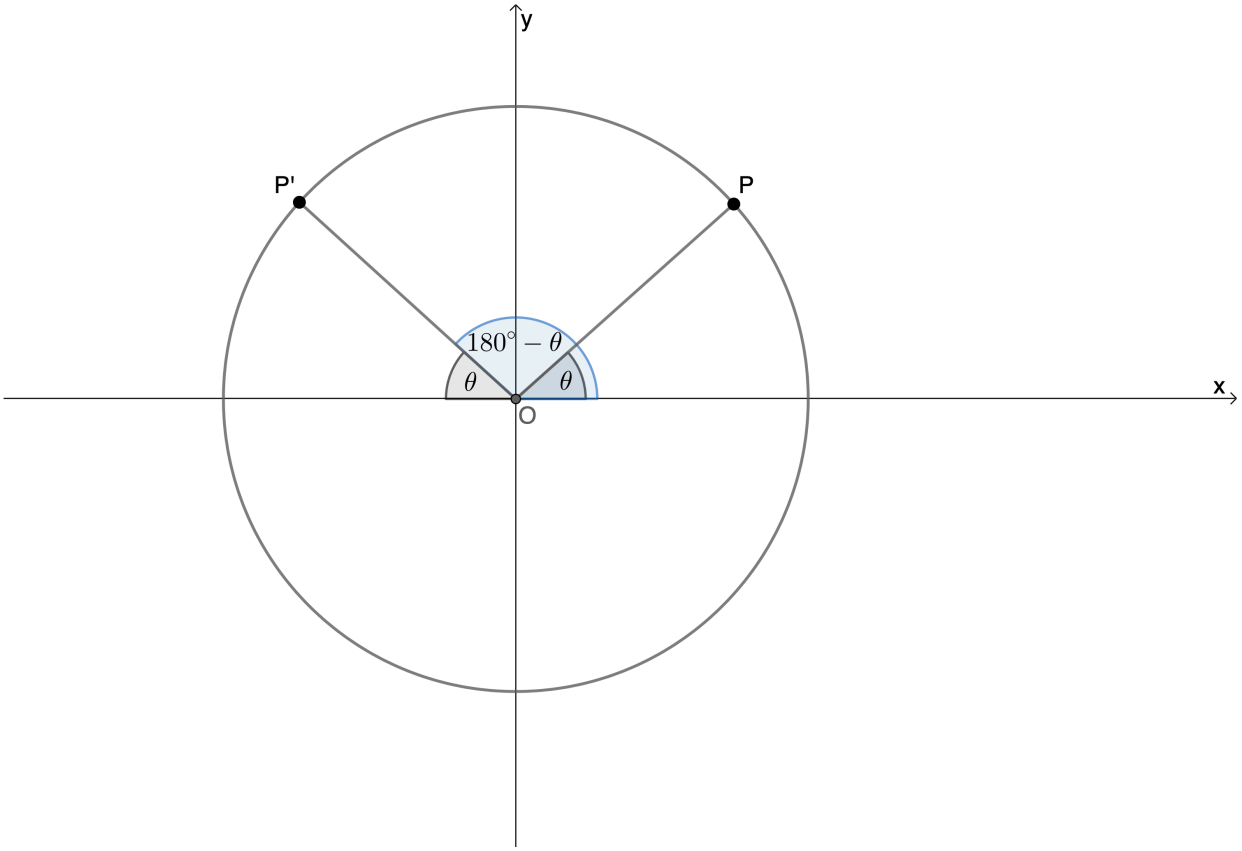
- $\sin\theta = \sin(180^\circ - \theta)$ the graph shows that sine is positive in the first and second quadrants.
- $\sin\theta = -\sin(180^\circ + \theta)$ the graph shows that that sine is negative in the third quadrant, hence we need

the negative sign.

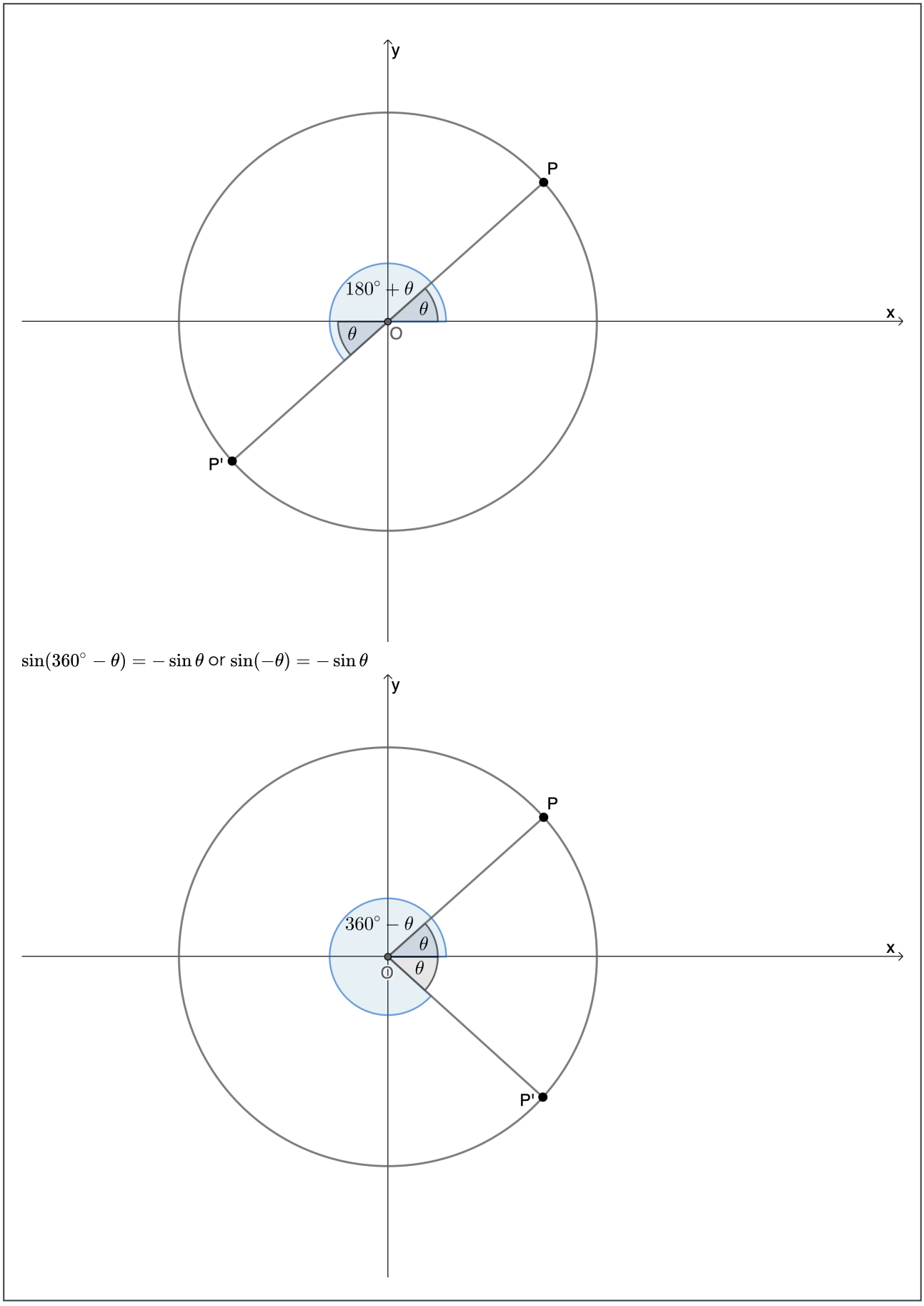
- $\sin\theta = -\sin(360^\circ - \theta)$ or $\sin\theta = -\sin(-\theta)$ the graph shows that sine is negative in the fourth quadrant, hence we need the negative sign. We can write the acute angle in the fourth quadrant either as $(360^\circ - \theta)$ or just $(-\theta)$.

Reduction rules for sine:

$\sin(180^\circ - \theta) = \sin\theta$



$\sin(180^\circ + \theta) = -\sin\theta$



$\sin(360^\circ - \theta) = -\sin\theta$ or $\sin(-\theta) = -\sin\theta$

The same kind of reduction rules exist for the cosine and tangent functions. If you have an internet connection, satisfy yourself that this is true by exploring the cosine function simulation.

If you have access to the internet, visit this interactive simulation.

While you explore, see if you can write down the same set of reduction rules as above but this time for cosine. Pay careful attention to the quadrants in which cosine is negative and positive.

Reduction rules for cosine:

- $\cos(180^\circ - \theta) = -\cos\theta$
- $\cos(180^\circ + \theta) = -\cos\theta$
- $\cos(360^\circ - \theta) = \cos\theta$ or $\cos(-\theta) = \cos\theta$

Can you write the same set of reduction rules for tangent? If you have an internet connection have a look at and the tangent function simulation.

f you have access to the internet, visit this interactive simulation.

Pay careful attention to the quadrants in which tangent is negative and positive.

Reduction rules for tangent:

- $\tan(180^\circ - \theta) = -\tan\theta$
- $\tan(180^\circ + \theta) = \tan\theta$
- $\tan(360^\circ - \theta) = -\tan\theta$ or $\tan(-\theta) = -\tan\theta$

Example 2.1

Simplify the following trig ratios to ratios of an acute angle:

1. $\sin 300^\circ$
2. $\cos 213^\circ$
3. $\tan 117^\circ$

Solution

1.
$$\sin 300^\circ = \sin(360^\circ = 60^\circ)$$
$$= -\sin 60^\circ \qquad 300^\circ \text{ is in the fourth quadrant where sin in negative}$$

2.
$$\cos 213^\circ = \cos(180^\circ + 33^\circ)$$
$$= -\cos 33^\circ \qquad 213^\circ \text{ is in the third quadrant where cosine is negative}$$

3.
$$\tan 117^\circ = \tan(180^\circ - 63^\circ)$$
$$= -\tan 63^\circ \qquad 117^\circ \text{ is in the second quadrant where tangent is negative}$$



Example 2.2

Simplify the following to ratios of an acute angle:

$$\frac{\sin 163^\circ}{\cos 197^\circ} + \tan 17^\circ$$

Solution

$$\frac{\sin 163^\circ}{\cos 197^\circ} + \tan 17^\circ = \frac{\sin(180^\circ - 17^\circ)}{\cos(180^\circ + 17^\circ)} + \tan 17^\circ$$
$$= \frac{\sin 17^\circ}{-\cos 17^\circ} + \tan 17^\circ$$



Example 2.3

Determine the value of the following expression without a calculator:
$\tan 150^\circ \sin 210^\circ - \cos 210^\circ$

Solution

Remember that 'without a calculator' means that you are expected to use special angles.

$\tan 150^\circ \sin 210^\circ - \cos 210^\circ = \tan(180^\circ - 30^\circ)\sin(180^\circ + 30^\circ) - \cos(180^\circ + 30^\circ)$

Write each of the angles with a special angle

$= (-\tan 30^\circ)(-\sin 30^\circ) - (-\cos 30^\circ)$

Apply the reduction rules being careful about signs

$= \left(-\frac{1}{\sqrt{3}}\right) \times \left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$

$= \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}$

$= \frac{1+\sqrt{3}}{2\sqrt{3}}$

$= \frac{1+\sqrt{3}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{3}+3}{6}$

Exercise 2.1

1. Simplify the following to ratios of an acute angle:
   a. $\cos 257^\circ$
   b. $\tan 198^\circ$
   c. $\tan 162^\circ$
   d. $\tan(-18^\circ)$
   e. $\sin 241^\circ$
   f. $\frac{\sin 67^\circ}{\tan 293^\circ} + \cos 113^\circ$
2. Determine the value of the following expressions without a calculator:
   a. $\frac{\tan^2 225^\circ - \sin(-30^\circ)\cos(-30^\circ)}{\tan 300^\circ}$
   b. $\left(\frac{\sin 210^\circ}{\cos 210^\circ}\right)^2 + 2\tan 210^\circ$

The full solutions are at the end of the unit.

Co-function reductions

In unit 1 you might have noticed that $\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$ and that $\sin 45^\circ = \frac{1}{\sqrt{2}} = \cos 45^\circ$. Can you see a pattern? Can you write a general rule for this relationship between sine and cosine in any right-angled triangle?

We can see from Figure 7 that if the angle at C is θ, then the angle at A must be $(90^\circ - \theta)$ (the angles

inside a triangle are supplementary). Therefore, if $\sin\theta = \frac{c}{b}$, $\cos(90^\circ - \theta) = \frac{c}{b}$ and $\sin\theta = \cos(90^\circ - \theta)$. Also, $\cos\theta = \sin(90^\circ - \theta)$.

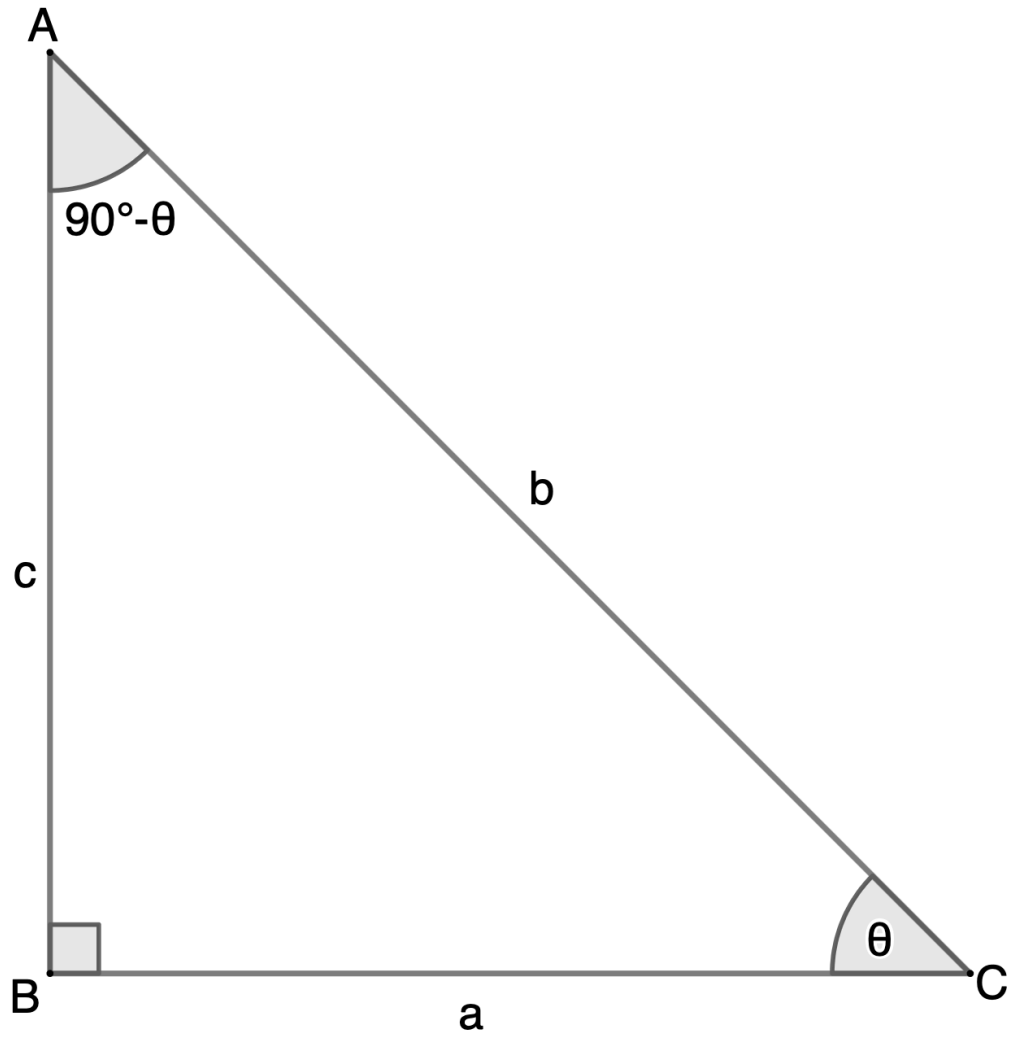


Figure 7: $\sin\theta = \frac{c}{b} = \cos(90^\circ - \theta)$

On the Cartesian plane, we can see that the angles θ and $90^\circ - \theta$ create two identical triangles such that $\sin\theta = \cos(90^\circ - \theta)$ and $\cos\theta = \sin(90^\circ - \theta)$ (see Figure 8).

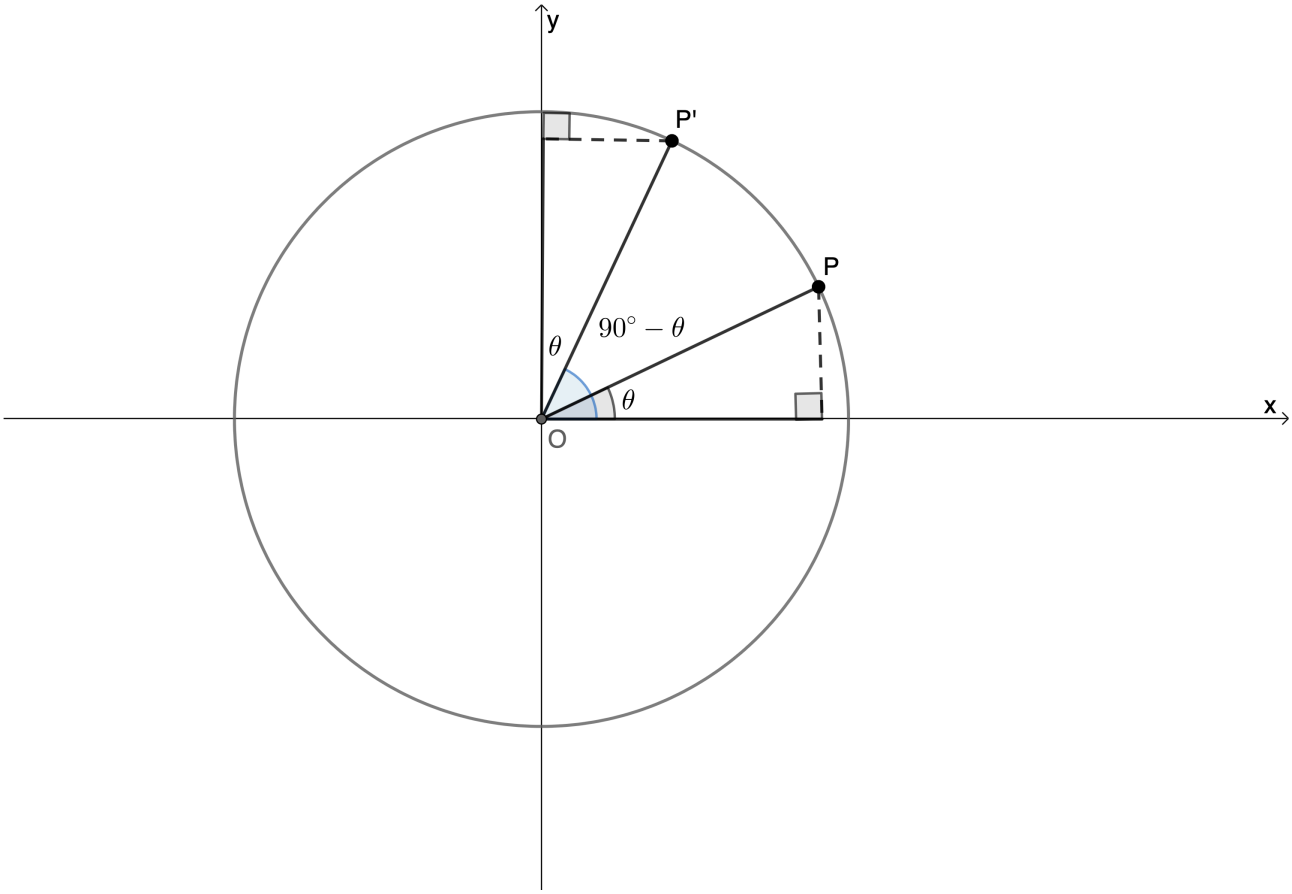


Figure 8: $\sin\theta = \cos(90^\circ - \theta)$ *and* $\cos\theta = \sin(90^\circ - \theta)$

Because of this relationship, we say that sine and cosine are **co-functions**. This is why cosine is called cosine. It is the co-function of sine!

Did you know?

The co-function of tangent is called cotangent or cot for short. If $\tan\theta = \frac{y}{x}$ then $\cot\theta = \frac{x}{y}$. In other words, $\cot\theta = \frac{1}{\tan\theta}$. The cotangent function is not part of the NC(V) level 3 curriculum.

Does this relationship between sine and cosine extend outside the first quadrant? In other words, is $\sin(90^\circ + \theta) = \cos\theta$? Have a look at Figure 9.

Here we can see that θ and $90^\circ + \theta$ create two identical triangles such that $\sin(90^\circ + \theta) = \cos\theta$ and $\cos(90^\circ + \theta) = -\sin\theta$. Notice that the distance P from the y-axis is a negative value on the x-axis, so cosine is negative in the second quadrant, and we need the negative sign).

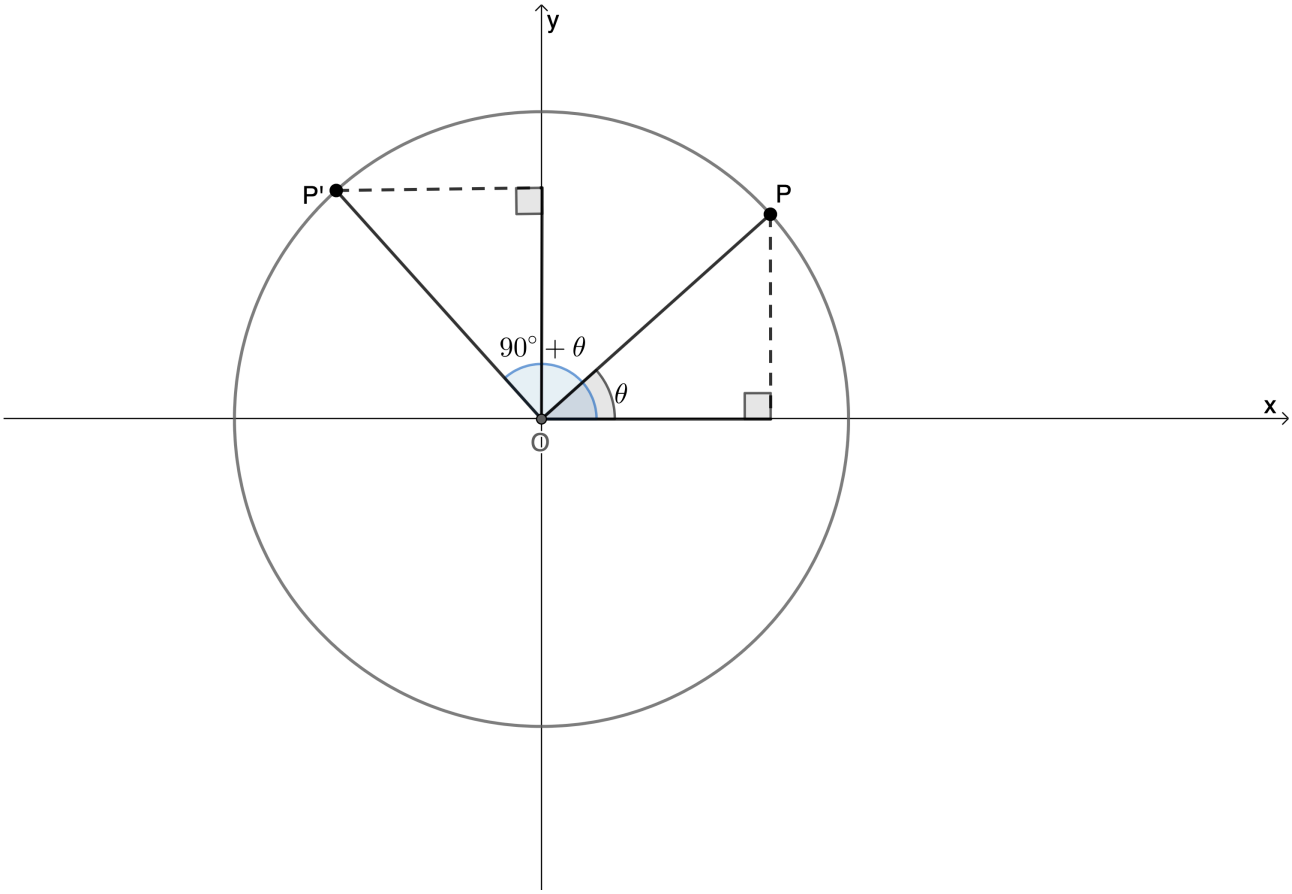


Figure 9: $\sin(90^\circ + \theta) = \cos\theta$ *and* $\cos(90^\circ + \theta) = -\sin\theta$

The same is true for angles in the third and fourth quadrants where:

- $\sin(270^\circ - \theta) = -\cos\theta$ and $\cos(270^\circ - \theta) = -\sin\theta$ (sine and cosine are both negative in the third quadrant)
- $\sin(270^\circ + \theta) = -\cos\theta$ and $\cos(270^\circ + \theta) = \sin\theta$ (only sine is negative in the fourth quadrant).

Note: The co-function reduction rules for $270^\circ \pm \theta$ are not officially part of the NC(V) level 3 curriculum.

Co-function reduction rules:

- $\sin(90^\circ - \theta) = \cos\theta$
- $\sin(90^\circ + \theta) = \cos\theta$
- $\cos(90^\circ - \theta) = \sin\theta$
- $\cos(90^\circ + \theta) = -\sin\theta$ cosine is negative in the second quadrant hence the need for the negative sign

Example 2.4

Write the following as ratios of 20°:

1. $\sin 70^\circ$
2. $\cos 110^\circ$
3. $\sin 250^\circ$
4. $\cos(\theta - 90^\circ)$

Solutions

1. $\begin{aligned}\sin 70^\circ &= \sin(90^\circ - 20^\circ) \\ &= \cos 20^\circ\end{aligned}$
2. $\begin{aligned}\cos 110^\circ &= \cos(90^\circ + 20^\circ) \\ &= -\sin 20^\circ \quad \text{Don't forget the negative sign; cosine is negative in the second quadrant}\end{aligned}$
3. We could answer this as follows:
   $\sin 250^\circ = \sin(270^\circ - 20^\circ) = -\cos 20^\circ$ However, the $270^\circ \pm \theta$ co-function reduction rules are not part of the curriculum, so we answer it as follows:
   $\begin{aligned}\sin 250^\circ &= \sin(180^\circ + 70^\circ) \\ &= -\sin 70^\circ \\ &= -\sin(90^\circ - 20^\circ) \\ &= -\cos 20^\circ\end{aligned}$
4. $\begin{aligned}\cos(\theta - 90^\circ) &= \cos(-90^\circ + \theta) \\ &= \cos(-(90^\circ - \theta)) \quad \text{The angle } -(90^\circ - \theta) \text{ is in the fourth quadrant where cosine is positive} \\ &= \cos(90^\circ - \theta) \\ &= \sin\theta\end{aligned}$

Example 2.5

Express the following with all ratios in terms of θ alone and simplify:

$$\frac{\cos(\theta - 90^\circ)\ \cos(720^\circ + \theta)\ \tan(\theta - 360^\circ)}{\sin^2(\theta + 360^\circ)\ \cos(\theta + 90^\circ)}$$

Solution

Let's break this question down and look at each trig ratio separately first.

- $\begin{aligned}\cos(\theta - 90^\circ) &= \cos(-90^\circ + \theta) \\ &= \cos(-(90^\circ - \theta)) \quad -(90^\circ - \theta) \text{ is in the fourth quadrant where cosine is positive} \\ &= \cos(90^\circ - \theta) \\ &= \sin\theta\end{aligned}$
  - $\cos(720^\circ + \theta) = \cos\theta$ because 720° is just two full revolutions or two full periods

- $\tan(\theta - 360^\circ) = \tan\theta$ $\quad -360^\circ$ is one full revolution or two full periods
- $\sin^2(\theta + 360^\circ) = \sin^2\theta$ $\quad 360^\circ$ is one full revolution or one full period
- $\cos(\theta + 90^\circ) = \cos(90^\circ + \theta) = -\sin\theta$

Therefore:

$$\frac{\cos(\theta - 90^\circ)\cos(720^\circ + \theta)\tan(\theta - 360^\circ)}{\sin^2(\theta + 360^\circ)\cos(\theta + 90^\circ)} = \frac{\sin\theta\cos\theta\tan\theta}{\sin^2\theta(-\sin\theta)}$$

Now we can simplify the expression.

$$\frac{\sin\theta\ \cos\theta\ \tan\theta}{\sin^2\theta(-\sin\theta)} = \frac{\cancel{\sin\theta}\ \cos\theta\ \tan\theta}{\sin^2\theta(-\cancel{\sin\theta})}$$
$$= -\frac{\cos\theta\tan\theta}{\sin^2\theta}$$

Exercise 2.2

1. Simplify the following to a ratio of 38°:
   a. $\sin 52^\circ$
   b. $\sin 128^\circ$
   c. $\cos 128^\circ$
   d. $\sin 308^\circ$
2. Simplify the following expressions:
   a. $\dfrac{\cos(90^\circ + \theta)\sin(90^\circ + \theta)}{\sin(90^\circ - \theta)\sin(180^\circ + \theta)}$
   b. $\dfrac{2\sin(90^\circ - x)\sin(90^\circ + x)}{\cos^2(90^\circ + x) + \cos(180^\circ + x)}$
3. Determine the value of the following expression without a calculator:
   $\dfrac{\tan 225^\circ - \sin^2 300^\circ}{\cos(90^\circ + 30^\circ)}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- The reduction rules for sine, cosine and tangent:
  - $\sin(180^\circ - \theta) = \sin\theta$
  - $\sin(180^\circ + \theta) = -\sin\theta$
  - $\sin(360^\circ - \theta) = \sin\theta$ or $\sin(-\theta) = \sin\theta$
  - $\cos(180^\circ - \theta) = -\cos\theta$
  - $\cos(180^\circ + \theta) = -\cos\theta$

- $\cos(360^\circ - \theta) = \cos\theta$ or $\cos(-\theta) = \cos\theta$
- $\tan(180^\circ - \theta) = -\tan\theta$
- $\tan(180^\circ + \theta) = \tan\theta$
- $\tan(360^\circ - \theta) = -\tan\theta$ or $\tan(-\theta) = -\tan\theta$.

- The co-function reduction rules for sine and cosine for $90^\circ \pm \theta$:
  - $\sin(90^\circ - \theta) = \cos\theta$
  - $\sin(90^\circ + \theta) = \cos\theta$
  - $\cos(90^\circ - \theta) = \sin\theta$
  - $\cos(90^\circ + \theta) = -\sin\theta$.

Unit 2: Assessment

Suggested time to complete: 25 minutes

1. Write the following as a function of an angle less than 45°:
   a. $\sin 167^\circ$
   b. $\tan 214^\circ$
   c. $\cos(-236^\circ)$
2. If $\cos^2\theta = \frac{16}{25}$ and $0^\circ \leq \theta \leq 90^\circ$, calculate the following:
   a. $\cos\theta$
   b. $\tan(180^\circ - \theta)$
   c. $\cos^2(180^\circ + \theta) + \sin^2(180^\circ + \theta)$
   d. $\frac{\sin(-\theta)\cos(180^\circ + \theta)\cos(90^\circ + \theta)}{\tan(360^\circ - \theta)\sin(90^\circ + \theta)}$
3. Prove that $\frac{\sin 210^\circ \cos 150^\circ \tan 315^\circ}{\sin 120^\circ} = \frac{1}{2}$.
4. Determine the value of the following without a calculator:
   a. $\frac{\cos(-30^\circ)}{\tan 150^\circ} + \sin 330^\circ$
   b. $\sin 780^\circ - (\cos 315^\circ)(\sin 405^\circ)$
   c. $\sqrt{2}\tan 150^\circ . \cos 225^\circ . \sin 240^\circ$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
   a.

$$\begin{aligned}\cos 257^\circ &= \cos(180^\circ + 77^\circ) \\ &= -\cos 77^\circ\end{aligned}$$

b.

$$\begin{aligned}\tan 198^\circ &= \tan(180^\circ + 18^\circ) \\ &= \tan 18^\circ\end{aligned}$$

c.

$$\begin{aligned}\tan 162^\circ &= \tan(180^\circ - 18^\circ) \\ &= -\tan 18^\circ\end{aligned}$$

d. $\tan(-18^\circ) = -\tan 18^\circ$

e.

$$\begin{aligned}\sin 241^\circ &= \sin(180^\circ + 61^\circ) \\ &= -\sin 61^\circ\end{aligned}$$

f.

$$\begin{aligned}\frac{\sin 67^\circ}{\tan 293^\circ} + \cos 113^\circ &= \frac{\sin 67^\circ}{\tan(360^\circ - 67^\circ)} + \cos(180^\circ - 67^\circ) \\ &= \frac{\sin 67^\circ}{-\tan 67^\circ} - \cos 67^\circ\end{aligned}$$

2.

a.

$$\begin{aligned}\frac{\tan^2 225^\circ - \sin(-30^\circ)\cos(-30^\circ)}{\tan 300^\circ} &= \frac{\tan^2(180^\circ + 45^\circ) + \sin 30^\circ \cos 30^\circ}{\tan(360^\circ - 60^\circ)} \\ &= \frac{\tan^2 45^\circ + \sin 30^\circ \cos 30^\circ}{-\tan 60^\circ} \\ &= \frac{1^2 + \frac{1}{2} \times \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{1}} \\ &= \frac{1 + \frac{\sqrt{3}}{4}}{-\sqrt{3}} \\ &= \frac{\frac{4 + \sqrt{3}}{4}}{-\sqrt{3}} \\ &= \frac{4 + \sqrt{3}}{4} \times \frac{1}{-\sqrt{3}} \\ &= \frac{4 + \sqrt{3}}{-4\sqrt{3}} \quad \text{Multiply by } \frac{\sqrt{3}}{\sqrt{3}} \\ &= -\frac{4\sqrt{3} + 3}{12}\end{aligned}$$

b.

$$\begin{aligned}\left(\frac{\sin 210^\circ}{\cos 210^\circ}\right)^2 + 2\tan 210^\circ &= \left(\frac{\sin(180^\circ + 30^\circ)}{\cos(180^\circ + 30^\circ)}\right)^2 + 2\tan(180^\circ + 30^\circ) \\ &= \left(\frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\right)^2 + 2 \times \frac{1}{\sqrt{3}} \\ &= \left(\frac{1}{2} \times \frac{2}{\sqrt{3}}\right)^2 + \frac{2}{\sqrt{3}} \\ &= \left(\frac{2}{2\sqrt{3}}\right)^2 + \frac{2}{\sqrt{3}} \\ &= \frac{4}{12} + \frac{2}{\sqrt{3}} \\ &= \frac{\sqrt{3} + 6}{3\sqrt{3}} \\ &= \frac{3 + 6\sqrt{3}}{9} \\ &= \frac{3\left(1 + 2\sqrt{3}\right)}{9} \\ &= \frac{1 + 2\sqrt{3}}{3}\end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.
   a. $\sin 52^\circ = \sin(90^\circ - 38^\circ) = \cos 38^\circ$
   b. $\sin 128^\circ = \sin(90^\circ + 38^\circ) = \cos 38^\circ$
   c. $\cos 128^\circ = \cos(90^\circ + 38^\circ) = -\sin 38^\circ$
   d. $\sin 308^\circ = \sin(360^\circ - 52^\circ) = -\sin 52^\circ = -\sin(90^\circ - 38^\circ) = -\cos 38^\circ$
2.
   a.
   $$\begin{aligned}\frac{\cos(90^\circ + \theta)\sin(90^\circ + \theta)}{\sin(90^\circ - \theta)\sin(180^\circ + \theta)} &= \frac{-\sin\theta\cos\theta}{\cos\theta(-\sin\theta)} \\ &= 1\end{aligned}$$
   b.
   $$\frac{2\sin(90^\circ - x)\sin(90^\circ + x)}{\cos^2(90^\circ + x) + \cos(180^\circ + x)} \qquad \left[\cos^2(90^\circ + x) = \cos(90^\circ + x)\cos(90^\circ + x) = (-\sin x)(-\sin x) = \sin^2 x\right]$$
   $$\begin{aligned}&= \frac{2\cos x \cos x}{\sin^2 x - \cos x} \\ &= \frac{2\cos^2 x}{\sin^2 x - \cos x}\end{aligned}$$
3.

$$\begin{aligned}\frac{\tan 225^\circ - \sin^2 300^\circ}{\cos(90^\circ + 30^\circ)} &= \frac{\tan(180^\circ + 45^\circ) - \sin^2(360^\circ - 60^\circ)}{-\sin 30^\circ} \\ &= \frac{\tan 45^\circ - \sin^2 60^\circ}{-\sin 30^\circ} \qquad \left[\sin^2(360^\circ - 60^\circ) = \sin(360^\circ - 60^\circ) \times \sin(360^\circ - 60^\circ)\right] \\ &= \frac{1 - \left(\frac{\sqrt{3}}{2}\right)^2}{-\frac{1}{2}} \\ &= \frac{1 - \frac{3}{4}}{-\frac{1}{2}} \\ &= \frac{\frac{1}{4}}{-\frac{1}{2}} \\ &= -\frac{1}{4} \times \frac{2}{1} \\ &= -\frac{1}{2}\end{aligned}$$

Back to Exercise 2.2

Unit 2: Assessment

1.

a. $\sin 167^\circ = \sin(180^\circ - 13^\circ) = \sin 13^\circ$

b. $\tan 214^\circ = \tan(180^\circ + 34^\circ) = \tan 34^\circ$

c. $\cos(-236^\circ) = \cos(-236^\circ + 360^\circ) = \cos 124^\circ = \cos(90^\circ + 34^\circ) = -\sin 34^\circ$

2. $\cos^2\theta = \frac{16}{25}$ and $0^\circ \leq \theta \leq 90^\circ$

To complete the question, we need to find the missing side of the triangle.

$y^2 + 4^2 = 5^2$

$\therefore y^2 = 25 - 16 = 9$

$\therefore y = 3$

a. $\cos\theta = \frac{4}{5}$

b. $\tan(180^\circ - \theta) = -\tan\theta = -\frac{3}{4}$

c.
$$\begin{aligned}\cos^2(180^\circ + \theta) + \sin^2(180^\circ + \theta) &= \cos^2\theta + \sin^2\theta \\ &= \frac{16}{25} + \frac{9}{25} \\ &= 1\end{aligned}$$

d.

$$\begin{aligned}\frac{-\sin\theta\cos(180^\circ+\theta)\cos(90^\circ+\theta)}{\tan(360^\circ-\theta)\sin(90^\circ+\theta)} &= \frac{-\sin\theta(-\cos\theta)(-\sin\theta)}{(-\tan\theta)\cos\theta} \\ &= \frac{\sin^2\theta\cos\theta}{\tan\theta\cos\theta} \\ &= \frac{\frac{9}{25}}{\frac{3}{4}} \\ &= \frac{9}{25}\times\frac{4}{3} \\ &= \frac{12}{25}\end{aligned}$$

3.

$$\begin{aligned}\text{LHS} &= \frac{\sin 210^\circ\cos 150^\circ\tan 315^\circ}{\sin 120^\circ} \\ &= \frac{\sin(180^\circ+30^\circ)\cos(180^\circ-30^\circ)\tan(360^\circ-45^\circ)}{\sin(180^\circ-60^\circ)} \\ &= \frac{(-\sin 30^\circ)(-\cos 30^\circ)(-\tan 45^\circ)}{\sin 60^\circ} \\ &= \frac{-\frac{1}{2}\cdot-\frac{\sqrt{3}}{2}\cdot-1}{\frac{\sqrt{3}}{2}} \\ &= \frac{-\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}} \\ &= -\frac{\sqrt{3}}{4}\times\frac{2}{\sqrt{3}} \\ &= -\frac{1}{2} = \text{RHS}\end{aligned}$$

4.

a.

$$\begin{aligned}\frac{\cos(-30^\circ)}{\tan 150^\circ}+\sin 330^\circ &= \frac{\cos 30^\circ}{\tan(180^\circ-30^\circ)}+\sin(360^\circ-30^\circ) \\ &= \frac{\cos 30^\circ}{-\tan 30^\circ}-\sin 30^\circ \\ &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{\sqrt{3}}}-\frac{1}{2} \\ &= \frac{\sqrt{3}}{2}\times\left(-\frac{\sqrt{3}}{1}\right)-\frac{1}{2} \\ &= -\frac{3}{2}-\frac{1}{2} \\ &= -2\end{aligned}$$

b.

$$\begin{aligned}\sin 780^\circ - (\cos 315^\circ)(\sin 405^\circ) &= \sin(720^\circ + 60^\circ) - (\cos(360^\circ - 45^\circ))\,(\sin(360^\circ + 45^\circ)) \\ &= \sin(2 \times 360^\circ + 60^\circ) - (\cos(360^\circ - 45^\circ))\,(\sin(360^\circ + 45^\circ)) \\ &= \sin 60^\circ - \cos 45^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2} \\ &= \frac{\sqrt{3} - 1}{2}\end{aligned}$$

c.

$$\begin{aligned}\sqrt{2}\tan 150^\circ . \cos 225^\circ . \sin 240^\circ &= \sqrt{2}\tan(180^\circ - 30^\circ) . \cos(180^\circ + 45^\circ) . \sin(180^\circ + 60^\circ) \\ &= \sqrt{2}(-\tan 30^\circ) . (-\cos 45^\circ) . (-\sin 60^\circ) \\ &= \sqrt{2}\left(-\frac{1}{\sqrt{3}}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right) \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{\sqrt{6}}{2\sqrt{6}} \\ &= -\frac{1}{2}\end{aligned}$$

Back to Unit 2: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © DHET is licensed under a CC BY (Attribution) license
- figure4 © DHET is licensed under a CC BY (Attribution) license
- figure5 © DHET is licensed under a CC BY (Attribution) license
- figure6 © DHET is licensed under a CC BY (Attribution) license
- 180-theta © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- 180+theta © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- 360-theta © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure7 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure8 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure9 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 3: Prove trig identities

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use the quotient identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to prove equations and simplify expressions.
- Use the square identity $\sin^2\theta + \cos^2\theta = 1$ to prove equations and simplify expressions.

What you should know

Before you start this unit, make sure you can:

- Define the three basic trigonometric ratios of sine, cosine and tangent.

Introduction

A mathematical identity is an equality relating one expression to another expression, such that both expressions produce the same value for all values of the variables within a certain range of validity.

An example of an algebraic identity is $(a+b)^2 = a^2 + 2ab + b^2$. This might not seem important or meaningful. However, identities can become very important when one is trying to prove that some, possibly unrelated, thing is true. Once an identity is established, it can be used again and again without having to be proven each time. Mathematical identities, therefore, behave as reusable statements of truth.

There are many different trigonometric identities. A statement such as $\cos(90^\circ + \theta) = -\sin\theta$ can be considered a trig identity. Once we have shown it to be true once, we can simply assume that it is true in other circumstances.

There are two very important identities that you need to know about and that we will cover in this unit. These are the **quotient identity** and the **square identity**.

The quotient and square identities

The quotient and square identities are very easy to prove. We will prove each of them in turn and then see how we can use these identities to help us simplify more complicated expressions.

The quotient identity

Have a look at the circle, with radius 5, on the Cartesian plane in Figure 1. It has point $P(4,3)$ on its circumference.

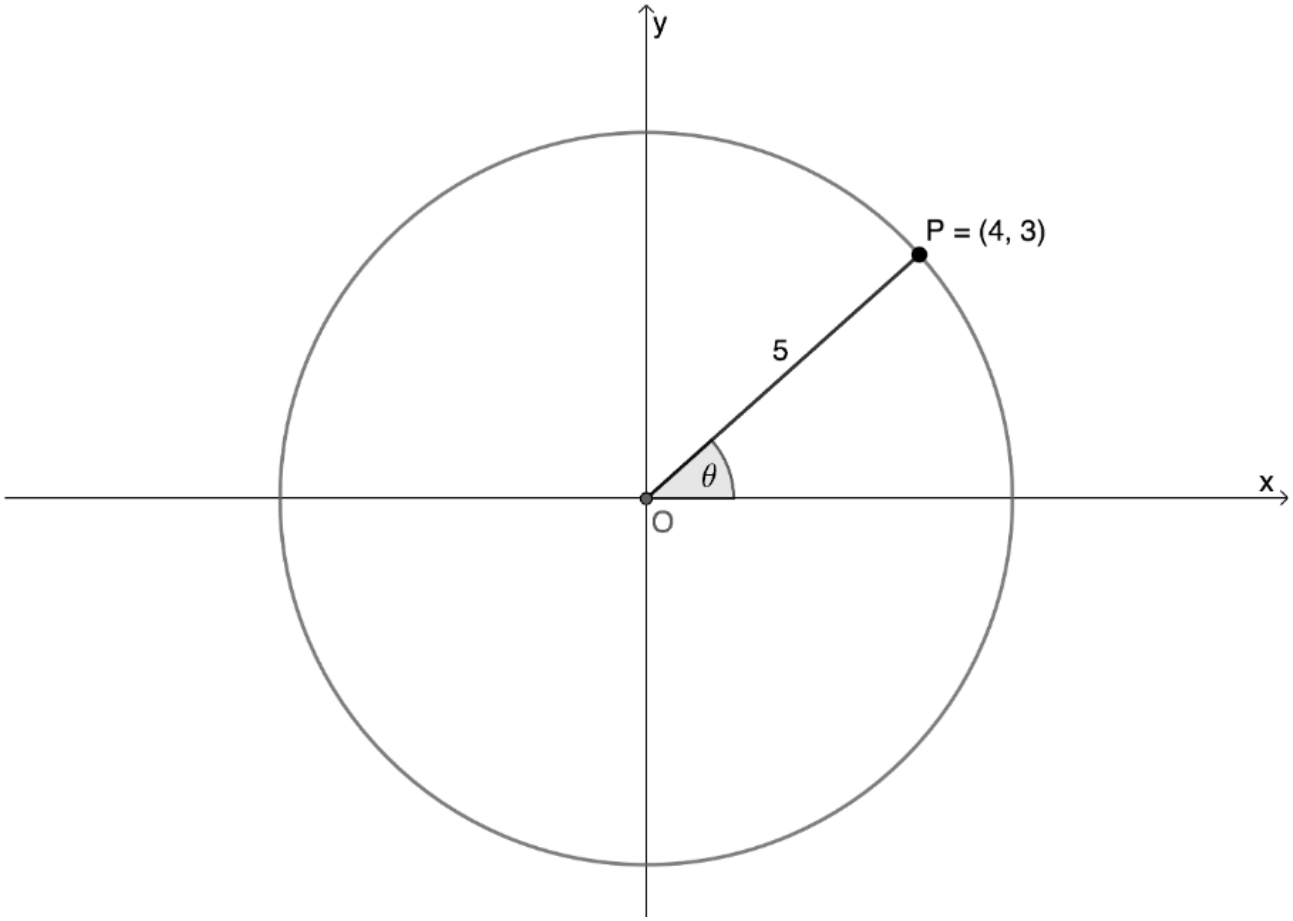


Figure 1: Circle centre origin and radius 5

What is the value of:

- $\sin\theta$?
- $\cos\theta$?
- $\tan\theta$?

Now, what is the value of $\frac{\sin\theta}{\cos\theta}$?

What general statement can we write that relates $\sin\theta$, $\cos\theta$ and $\tan\theta$?

In Figure 1, we know that:

- $\sin\theta = \frac{y}{r} = \frac{3}{5}$
- $\cos\theta = \frac{x}{r} = \frac{4}{5}$
- $\tan\theta = \frac{y}{x} = \frac{3}{4}$

We can see that $\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4} = \tan\theta$. The equation $\frac{\sin\theta}{\cos\theta} = \tan\theta$ is the **quotient identity** and is true for all values of θ.

In general, $\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta$.

Quotient identity:

$\frac{\sin\theta}{\cos\theta} = \tan\theta, \theta \in \mathbb{R}$

Now that we know that $\frac{\sin\theta}{\cos\theta} = \tan\theta$ is true for all values of θ, we can manipulate it in various ways. We now also know that $\sin\theta = \tan\theta \times \cos\theta$ and $\frac{\sin\theta}{\tan\theta} = \cos\theta$.

Take note!

The quotient identity is another way to explain why the tangent function has vertical asymptotes at $90°$ and $270°$ (for $0° \leq \theta \leq 360°$). $\cos 90° = 0$ and $\cos 270° = 0$. Therefore, at $\theta = 90°$ and $\theta = 270°$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is undefined.

The square identity

Let's use the same circle in Figure 1 to prove the square identity. What is the value of:

- $\sin^2\theta$
- $\cos^2\theta$
- $\sin^2\theta + \cos^2\theta$?

Can you prove that $\sin^2\theta + \cos^2\theta = 1$ is always true? Try this before you read on.

Here is the general proof:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 \\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ &= \frac{y^2 + x^2}{r^2} \qquad \text{But } x^2 + y^2 = r^2 \text{ by Pythagoras} \\ &= \frac{r^2}{r^2} \\ &= 1\end{aligned}$$

Did you know?

The square identity is also sometimes called the Pythagorean identity because it follows from $x^2 + y^2 = r^2$.

We can manipulate the square identity to get the following, which are also useful:

- $\sin^2\theta = 1 - \cos^2\theta$
- $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ (the sign depends on which quadrant the angle is in)
- $\cos^2\theta = 1 - \sin^2\theta$
- $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ (the sign depends on which quadrant the angle is in)

Using the quotient and square identities

We use the quotient and square identities to solve equations, simplify complicated expressions or prove other identities. Doing this can take some practise. The more you practise the better you will get at it.

Example 3.1

Simplify $\tan^2\theta \times \cos^2\theta$.

Solution

Whenever you see tangent, it is usually a good idea to use the quotient identity to replace this with $\frac{\sin\theta}{\cos\theta}$.

$$\begin{aligned}\tan^2\theta \times \cos^2\theta &= (\tan\theta)^2 \times \cos^2\theta \qquad \text{Remember } \tan^2\theta = \tan\theta \times \tan\theta \\ &= \left(\frac{\sin\theta}{\cos\theta}\right)^2 \times \cos^2\theta \\ &= \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \times \frac{\cancel{\cos^2\theta}}{1} \\ &= \sin^2\theta\end{aligned}$$

We could use the square identity to replace $\sin^2\theta$ with $1 - \cos^2\theta$ but this would be a more complicated expression.

Example 3.2

Simplify $\frac{1}{\cos^2\theta} - \tan^2\theta$.

Solution

Start by using the quotient identity to replace $\tan^2\theta$.

$$\begin{aligned}\frac{1}{\cos^2\theta} - \tan^2\theta &= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{1 - \sin^2\theta}{\cos^2\theta}\end{aligned}$$

Now we have an expression in the numerator that can be simplified using the square identity.

$\sin^2\theta + \cos^2\theta = 1$

$\therefore \cos^2\theta = 1 - \sin^2\theta$

$$\frac{1-\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta}$$
$$= 1$$

Example 3.3

Prove that $\dfrac{\cos x}{1+\sin x} = \dfrac{1-\sin x}{\cos x}$.

Solution

This is a different kind of question to Examples 3.1 and 3.2. Here we need to prove that the given left-hand side (LHS) is equal to the given right-hand side (RHS). We can either simplify one side to be exactly the same as the other or we can simplify both sides to the exact same expression. In either case, start with one side of the equality and simplify it as far as possible to see where this gets you.

$$\text{LHS} = \frac{\cos x}{1+\sin x}$$
$$= \frac{\cos x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$

Multiplying the fraction by this convenient form of 1 means that we can create a difference of two squares $(1+\sin x)(1-\sin x)$ in the denominator which will give us $1-\sin^2 x$. So now we can use the square identity.

$$\text{LHS} = \frac{\cos x}{1+\sin x}$$
$$= \frac{\cos x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$
$$= \frac{\cos x(1-\sin x)}{(1+\sin x)(1-\sin x)}$$
$$= \frac{\cos x(1-\sin x)}{1-\sin^2 x} \quad \left[\sin^2 x + \cos^2 = 1 \therefore 1 - \sin^2 x = \cos^2 x\right]$$
$$= \frac{\cancel{\cos x}\,(1-\sin x)}{\cos \cancel{^1} x}$$
$$= \frac{1-\sin x}{\cos x} = \text{RHS}$$

We have proven that the LHS is equal to the RHS.

Exercise 3.1

1. Reduce the following to a single trig ratio:

a. $\dfrac{\sin x}{\tan x}$

b. $1 - \cos\alpha \sin\alpha \tan\alpha$

c. $\tan^2\theta\cos^2\theta + \tan^2\theta\sin^2\theta$

2. Prove the following identities:

a. $(1 + \sin x)[1 + \sin(-x)] = \cos^2 x$

b. $\cos^2\alpha(1 + \tan^2\alpha) = 1$

c. $\dfrac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \dfrac{\sin\theta + 1}{\tan\theta}$

The full solutions are at the end of the unit.

Using the identities to prove and/or simplify other statements does take some practise. Here are some useful strategies you can apply:

Take note!

Strategies to prove identities:

- Change all ratios to sine and cosine. Use the quotient identity to do this. Very often, you will create terms that can 'cancel'.
- Choose the more complicated side of the equality to simplify. Often this will simplify down to the other side of the equality.
- Look out for opportunities to factorise, especially common factors, difference of squares and sometimes quadratics.
- If this does not happen, then try and simplify the other side of the equality as well.
- A square root sign or a squared term usually indicates that you need to use the square identity.
- Try replacing 1 with $\sin^2\theta + \cos^2\theta$.
- Try multiplying fractions by convenient forms of 1 such as $\dfrac{1 + \cos\theta}{1 + \cos\theta}$ to try and create a square identity that can be replaced with 1.

Summary

In this unit you have learnt the following:

- How to prove trig identities by simplifying the LHS and/or the RHS to show that the LHS of the expression is equal to the RHS of the expression.

Unit 3: Assessment

Suggested time to complete: 75 minutes

1. Simplify the following expressions down to a single trig ratio:
   a. $\dfrac{1-\cos^2 x}{\tan^2 x} + 2\sin^2 x$
   b. $\dfrac{1-\cos^2 \beta}{\tan\beta \sin\beta}$
   c. $\tan x \cos x \sin(-x)$
   d. $\left(\dfrac{\sin^2\theta}{1-\sin^2\theta}\right) - \tan^2\theta$
2. Simplify $\dfrac{\cos(90^\circ - x)\cos(360^\circ - x)\tan x}{\sin(90^\circ - x)\cos(180^\circ - x)\tan(180^\circ + x)}$.
3. Prove the following identities:
   a. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$
   b. $\cos x - \cos^3 x = \cos x \sin^2 x$
   c. $\dfrac{1+\sin^2\theta}{\cos^2\theta} = \dfrac{2\tan^2\theta}{\sin^2\theta} - 1$
   d. $\left(\dfrac{1}{\cos x} + \tan x\right)\left(\dfrac{1}{\cos x} - \tan x\right) = 1$
   e. $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$
   f. $(1-\sin^2 x)(1+\tan^2 x) = 1$
   g. $\dfrac{\cos^2 x - \sin^2 x}{1-\tan^2 x} = \cos^2 x$
   h. $\dfrac{2\sin x \cos x}{\sin x + \cos x} + \dfrac{1}{\sin x + \cos x} = \sin x + \cos x$
   i. $\dfrac{\cos^2\alpha}{\sin\alpha} + \cos\alpha\tan\alpha = \dfrac{1}{\sin\alpha}$
   j. $\dfrac{(1+\tan^2 x)\sin x}{(1-\tan x)} = \dfrac{\tan x}{\cos x - \sin x}$

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
   a.

$$\frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}}$$
$$= \frac{\sin x}{1} \times \frac{\cos x}{\sin x}$$
$$= \cos x$$

b.

$$1 - \cos\alpha \sin\alpha \tan\alpha = 1 - \cancel{\cos\alpha} \sin\alpha \frac{\sin\alpha}{\cancel{\cos\alpha}}$$
$$= 1 - sin^2\alpha$$
$$= \cos^2\alpha$$

c.

$$\tan^2\theta\cos^2\theta + \tan^2\theta\sin^2\theta = \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta + \frac{\sin^2\theta}{\cos^2\theta} \cdot \sin^2\theta$$
$$= \frac{\sin^2\theta \cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} + \frac{\sin^2\theta\sin^2\theta}{\cos^2\theta}$$
$$= \sin^2\theta + \frac{\sin^4\theta}{\cos^2\theta}$$
$$= \frac{\sin^2\theta\cos^2\theta + \sin^4\theta}{\cos^2\theta}$$ Take a common factor of $\sin^2\theta$ out of the numerator
$$= \frac{\sin^2\theta(\cos^2\theta + \sin^2\theta)}{\cos^2\theta}$$ Use the square identity
$$= \frac{\sin^2\theta}{\cos^2\theta}$$ Use the quotient identity
$$= \tan^2\theta$$

2.

a.

$$\text{LHS} = (1 + \sin x)[1 + \sin(-x)]$$
$$= (1 + \sin x)(1 - \sin x)$$
$$= 1 - \sin^2 x$$
$$= \cos^2 x = \text{RHS}$$

b.

$$\text{LHS} = \cos^2\alpha(1 + \tan^2\alpha)$$
$$= \cos^2\alpha\left(1 + \frac{\sin^2\alpha}{\cos^2\alpha}\right)$$ Add terms in brackets over the LCD of $\cos^2\alpha$
$$= \cos^2\alpha\left(\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}\right)$$ Use the square identity
$$= \cos^2\alpha\left(\frac{1}{\cos^2\alpha}\right)$$
$$= 1 = \text{RHS}$$

c.

$$\text{LHS} = \frac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta}$$ Factorise the numerator (diff of squares) and denominator (common factor)
$$= \frac{(\sin\theta + 1)\cancel{(\sin\theta - 1)}}{\tan\theta\cancel{(\sin\theta - 1)}}$$
$$= \frac{\sin\theta + 1}{\tan\theta}$$

Back to Exercise 3.1

Unit 3: Assessment

1.

a.

$$\begin{aligned}\frac{1-\cos^2 x}{\tan^2 x}+2\sin^2 x &= \frac{\sin^2 x}{\tan^2 x}+2\sin^2 x \\ &= \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}} \\ &= \frac{\cancel{\sin^2 x}}{1}\times\frac{\cos^2 x}{\cancel{\sin^2 x}}+2\sin^2 x \\ &= \cos^2 x+2\sin^2 x \\ &= 1-\sin^2 x+2\sin^2 x \\ &= 1+\sin^2 x\end{aligned}$$

b.

$$\begin{aligned}\frac{1-\cos^2\beta}{\tan\beta\sin\beta} &= \frac{\sin^{\cancel{2}}\beta}{\tan\beta\cancel{\sin\beta}} \\ &= \frac{\sin\beta}{\tan\beta} \quad \left[\frac{\sin\beta}{\cos\beta}=\tan\beta\therefore\frac{\sin\beta}{\tan\beta}=\cos\beta\right] \\ &= \cos\beta\end{aligned}$$

c.

$$\begin{aligned}\tan x\cos x\sin(-x) &= \tan x\cos x(-\sin x) \\ &= \frac{\sin x}{\cancel{\cos x}}\cdot\cancel{\cos x}\cdot(-\sin x) \\ &= -\sin^2 x\end{aligned}$$

d.

$$\begin{aligned}\left(\frac{\sin^2\theta}{1-\sin^2\theta}\right)-\tan^2\theta &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)-\tan^2\theta \\ &= \tan^2\theta-\tan^2\theta \\ &= 0\end{aligned}$$

2.

$$\begin{aligned}\frac{\cos(90^\circ-x)\cos(360^\circ-x)\tan x}{\sin(90^\circ-x)\cos(180^\circ-x)\tan(180^\circ+x)} &= \frac{\sin x\cancel{\cos x}\cancel{\tan x}}{\cancel{\cos x}(-\cos x)\cancel{\tan x}} \\ &= -\frac{\sin x}{\cos x} \\ &= -\tan x\end{aligned}$$

3.

a.

$$\begin{aligned}\text{LHS} &= (\sin x+\cos x)^2 \\ &= (\sin x+\cos x)(\sin x+\cos x) \\ &= \sin^2 x+\sin x\cos x+\sin x\cos x+\cos^2 x \\ &= \sin^2+\cos^2 x+2\sin x\cos x \\ &= 1+2\sin x\cos x=\text{RHS}\end{aligned}$$

b.

$$\begin{aligned}\text{LHS} &= \cos x-\cos^3 x \\ &= \cos(1-\cos^2 x) \\ &= \cos x\sin^2 x=\text{RHS}\end{aligned}$$

c.

$$\begin{aligned}\text{LHS} &= \frac{1+\sin^2\theta}{\cos^2\theta} \\ &= \frac{1+(1-\cos^2\theta)}{\cos^2\theta} \\ &= \frac{2-\cos^2\theta}{\cos^2\theta} \\ &= \frac{2}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} \\ &= \frac{2}{\cos^2\theta} - 1 \\ &= \frac{2}{\frac{\sin^2\theta}{\tan^2\theta}} - 1 \\ &= \frac{2}{1} \times \frac{\tan^2\theta}{\sin^2\theta} - 1 \\ &= \frac{2\tan^2\theta}{\sin^2\theta} - 1 = \text{RHS}\end{aligned}$$

d.

$$\begin{aligned}\text{LHS} &= \left(\frac{1}{\cos x} + \tan x\right)\left(\frac{1}{\cos x} - \tan x\right) \\ &= \frac{1}{\cos^2 x} - \tan^2 x \\ &= \frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{1-\sin^2 x}{\cos^2 x} \\ &= \frac{\cos^2 x}{\cos^2 x} \\ &= 1 = \text{RHS}\end{aligned}$$

e.

$$\begin{aligned}\text{LHS} &= \sin^4\theta - \cos^4\theta \\ &= (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) \\ &= (\sin^2\theta - \cos^2\theta) \\ &= \left((1-\cos^2\theta) - \cos^2\theta\right) \\ &= 1 - 2\cos^2\theta = \text{RHS}\end{aligned}$$

f.

$$\begin{aligned}\text{LHS} &= (1-\sin^2 x)(1+\tan^2 x) \\ &= \cos^2 x(1+\tan^2 x) \\ &= \cos^2 x + \cos^2 x\tan^2 x \\ &= \cos^2 x + \cancel{\cos^2 x} \cdot \frac{\sin^2 x}{\cancel{\cos^2 x}} \\ &= \cos^2 x + \sin^2 x \\ &= 1 = \text{RHS}\end{aligned}$$

g.

$$
\begin{aligned}
\text{LHS} &= \frac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} \\
&= \frac{\cos^2 x - \sin^2 x}{1 - \dfrac{\sin^2 x}{\cos^2 x}} \\
&= \frac{\cos^2 x - \sin^2 x}{\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}} \\
&= \frac{\cancel{\cos^2 x - \sin^2 x}}{1} \times \frac{\cos^2 x}{\cancel{\cos^2 x - \sin^2 x}} \\
&= \cos^2 x = \text{RHS}
\end{aligned}
$$

h.

$$
\begin{aligned}
\text{LHS} &= \frac{2\sin x \cos x}{\sin x + \cos x} + \frac{1}{\sin x + \cos x} \\
&= \frac{2\sin x \cos x + 1}{\sin x + \cos x} \\
&= \frac{2\sin x \cos x + \sin^2 x + \cos^2}{\sin x + \cos x} \\
&= \frac{\sin^2 x + 2\sin x \cos x + \cos^2}{\sin x + \cos x} \qquad \text{Factorise the quadratic in the numerator} \\
&= \frac{(\sin x + \cos x)^{\cancel{2}}}{\cancel{\sin x + \cos x}} \\
&= \sin x + \cos x = \text{RHS}
\end{aligned}
$$

i.

$$
\begin{aligned}
\text{LHS} &= \frac{\cos^2 \alpha}{\sin \alpha} + \cos \alpha \tan \alpha \\
&= \frac{\cos^2 \alpha}{\sin \alpha} + \cancel{\cos \alpha} \times \frac{\sin \alpha}{\cancel{\cos \alpha}} \\
&= \frac{\cos^2 \alpha}{\sin \alpha} + \sin \alpha \\
&= \frac{\cos^2 \alpha + \sin^2 \alpha}{\sin \alpha} \\
&= \frac{1}{\sin \alpha} = \text{RHS}
\end{aligned}
$$

j.

$$
\begin{aligned}
\text{LHS} &= \frac{(1+\tan^2 x)\sin x}{(1-\tan x)} \\
&= \frac{\left(1+\dfrac{\sin^2 x}{\cos^2 x}\right)\sin x}{\left(1-\dfrac{\sin x}{\cos x}\right)} \\
&= \frac{\left(\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}\right)\sin x}{\left(\dfrac{\cos x-\sin x}{\cos x}\right)} \\
&= \frac{\left(\dfrac{1}{\cos^2 x}\right)\sin x}{\left(\dfrac{\cos x-\sin x}{\cos x}\right)} \\
&= \frac{\sin x}{\cos \cancel{x} x} \times \frac{\cancel{\cos x}}{(\cos x-\sin x)} \\
&= \frac{\sin x}{\cos x(\cos x-\sin x)} \\
&= \frac{\tan x}{\cos x-\sin x} = \text{RHS}
\end{aligned}
$$

Back to Unit 3: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 4: Find the general solution to trig equations

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve trigonometric equations using a general solution.

What you should know

Before you start this unit, make sure you can:

- Define the three basic trigonometric ratios of sine, cosine and tangent.
- Solve basic linear equations.
- Calculate with the special angles. Refer to unit 1 of this subject outcome if you need help with this.
- Use the trig reduction formulae. Refer to unit 2 of this subject outcome if you need help with this.
- Draw and work with the CAST diagram to determine in which quadrants each of the trig ratios are positive or negative. Refer to level 2 subject outcome 3.6 unit 1 if you need help with this.

Introduction

If $\sin\theta = \frac{1}{2}$, what is θ? If you said that $\theta = 30^\circ$ because of your knowledge of special angles from unit 1, you would be absolutely correct and yet still not completely correct! How can this be?

Remember that the trig functions are **periodic** – they repeat themselves over and over again. We know already, for example, that $\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$ as well. But the function repeats itself again every 360°. Therefore, there are infinitely many solutions to $\sin\theta = \frac{1}{2}$.

If you have an internet connection, visit this simulation to see all the solutions to $\sin\theta = \frac{1}{2}$ for the interval $-720^\circ \leq \theta \leq 720^\circ$.

You will find that there are eight in total.

All of these solutions are shown in Figure 1.

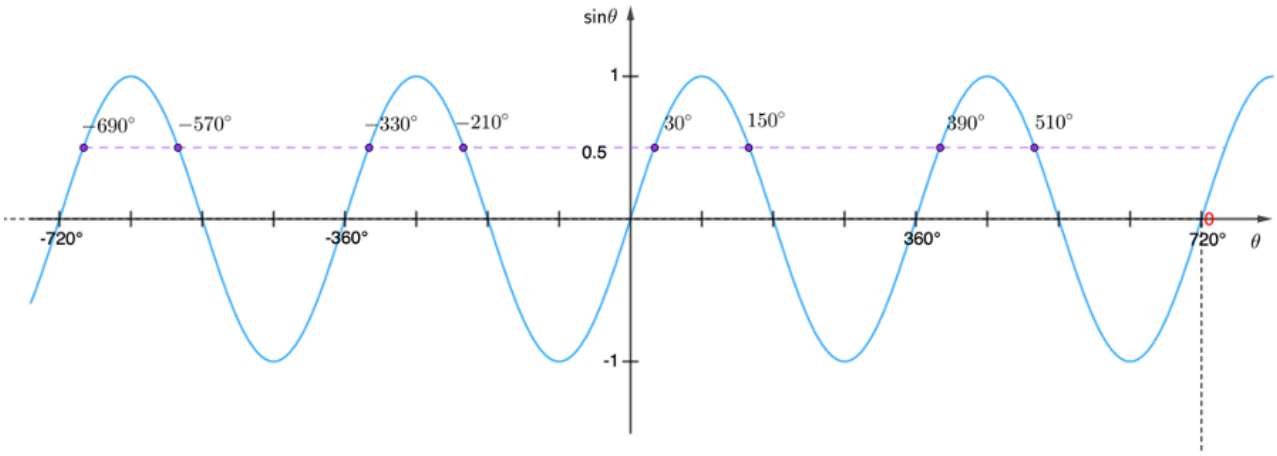


Figure 1: Solutions for $\sin\theta = \frac{1}{2}$ *for* $-720° \leq \theta \leq 720°$

Therefore, it is not good enough to say that the solution to $\sin\theta = \frac{1}{2}$ is $\theta = 30°$. Nor can we list all the possible solutions. We need a way to give a **general solution** that can generate all the possible solutions for $\theta \in \mathbb{R}$.

The general solution for $\sin\theta$

Let's look at the solutions for θ for $\sin\theta = \frac{1}{2}$ for the interval $-720° \leq \theta \leq 720°$ again.

$-690°, -570°, -330°, -210°, 30°, 150°, 390°, 510°$. Can you see any pattern? Look back at Figure 1 to help you.

What if you arrange the solutions in two columns like this:

$-690°$	$-570°$
$-330°$	$-210°$
$30°$	$150°$
$390°$	$510°$

Can you see the pattern now? Hopefully you can see that each solution in each column is separated from the next by $360°$, the period of the sine function. Look at the first column again. If we use $30°$ (the answer that a calculator would give us) as the starting angle or the **reference angle**, look at how we can generate all the other solutions.

$30° - 2 \times 360° - 690°$
$30° - 1 \times 360° = -330°$
$30°$
$30° + 1 \times 360° = 390°$

We can do the same for the second column of solutions using $180° - 30° = 150°$ as the starting angle.

$150° - 2 \times 360° = -570°$
$150° - 1 \times 360° = -210°$
$150°$
$150° + 1 \times 360° = 510°$

So, with just the two starting angles 30° and 150°, we can generate every other solution to $\sin\theta = \frac{1}{2}$ by adding integer ($\mathbb{Z}$) multiples of the period of the sine function i.e. 360°.

We write the full or **general solution** to $\sin\theta = \frac{1}{2}$ as follows:

$\theta = 30^\circ + k.360^\circ \text{ or } \theta = 150^\circ + k.360^\circ, k \in \mathbb{Z}$



Take note!

We call the answer we get for θ, (usually on a calculator) and from which we generate all the other possible solutions, the **reference angle**.



Example 4.1

Determine the general solution for $\sin\theta = 0.35$.

Solution

Step 1: Use a calculator to determine the reference angle

$\sin\theta = 0.35$

$\therefore \theta = 20.5^\circ$

Note: Unless told otherwise, we usually round the reference angle to one decimal place.

Step 2: Use the CAST diagram to determine any other possible solutions

$\sin\theta = 0.35$. In other words, sine is positive. Sine is positive in the first and second quadrants. We already have the first quadrant solution (the reference angle of $\theta = 20.5^\circ$). We need to find the second quadrant solution. We know that $\sin(180^\circ - \theta) = \sin\theta$. Therefore, the second quadrant solution is $180^\circ - 20.5^\circ = 159.5^\circ$.

Step 3: Generate the general solution

$\theta = 20.5^\circ + k.360^\circ \text{ or } \theta = 159.5^\circ + k.360^\circ, k \in \mathbb{Z}$

Step 4: Check your general solution

It is always a good idea to check that your final solutions satisfy the original equation. Choose a random value for k.

$k = -2$:

$\theta = 20.5^\circ - 2 \times 360^\circ \text{ or } \theta = 159.5^\circ - 2 \times 360^\circ$

$\therefore \theta = -699.5^\circ \text{ or } \theta = -560.5^\circ$

$\sin(-699.5^\circ) = 0.35$

$\sin(-560.5^\circ) = 0.35$

Our general solution is correct.



Example 4.2

Solve for θ if $4\sin\theta = -3$ for the interval $[-180^\circ, 180^\circ]$.

Solution

$$4\sin\theta = -3$$
$$\therefore \sin\theta = -\frac{3}{4} = -0.75$$

Step 1: Use a calculator to determine the reference angle
When we have a negative ratio, we ignore the sign when finding the reference angle.
$\sin\theta = 0.75$
$\therefore \theta = 48.6^\circ$

Step 2: Use the CAST diagram to determine any other possible solutions
Our equation is $\sin\theta = -0.75$. In other words, sine is negative. Sine is negative in the third and fourth quadrants. Our reference angle is $\theta = 48.6^\circ$.

Third quadrant: $\sin(180^\circ + \theta) = -\sin\theta$. Therefore, the third quadrant solution is $180^\circ + 48.6^\circ = 228.6^\circ$.

Fourth quadrant: $\sin(360^\circ - \theta) = -\sin\theta$. Therefore, the fourth quadrant solution is $360^\circ - 48.6^\circ = 311.4^\circ$.

Step 3: Generate the general solution
$\theta = 228.6^\circ + k.360^\circ$ or $\theta = 311.4^\circ + k.360^\circ, k \in \mathbb{Z}$

Step 4: Generate the solution for the specific range
In this instance, we were not asked for the general solution but only for the solutions in $[-180^\circ, 180^\circ]$. We need to use our general solution to generate solutions that fall within this range by adding or subtracting multiples of 360°.

$\theta = 228.6^\circ - 1 \times 360^\circ = -131.4^\circ$ or $\theta = 311.4^\circ - 1 \times 360^\circ = -48.6^\circ$

Step 5: Check your general solution
It is always a good idea to check that your final solutions satisfy the original equation.
$\sin(-131.4^\circ) = -0.75$
$\sin(-48.6^\circ) = -0.75$



Exercise 4.1

1. Determine the general solution for $2\sin\theta = \sqrt{3}$.
2. Solve for x if $5\sin x = -2$ and $0^\circ \leq x \leq 360^\circ$.

The full solutions are at the end of the unit.

The general solution for $\cos\theta$ and $\tan\theta$

The general solution for $\cos\theta = y$ is basically the same as that for sine. Remember that cosine also has a period of 360°. The only difference occurs when you find the quadrants in which cosine is either positive or negative.

The general solution for $\tan\theta = y$ is basically the same as that for sine except that the period of tangent is

$180°$. The other difference occurs when you find the quadrants in which tangent is either positive or negative.

Example 4.3

Determine the general solution for $3\cos x = \sin 14°$.

Solution

$$3\cos x = \sin 14°$$
$$\therefore \cos x = \frac{\sin 14°}{3}$$

Step 1: Use a calculator to determine the reference angle
$$\cos x = \frac{\sin 14°}{3}$$
$$\therefore x = 85.4°$$

Step 2: Use the CAST diagram to determine any other possible solutions
Our equation is $\cos x = \frac{\sin 14°}{3}$. Because $\sin 14° > 0$ we know that $\cos x > 0$. Cosine is positive in the first and fourth quadrants. Our reference angle is $\theta = 85.4°$.

First quadrant: $\theta = 85.4°$

Fourth quadrant: $\cos(360° - \theta) = \cos\theta$
$360° - 85.4° = 274.6°$.

Step 3: Generate the general solution
$\theta = 85.4° + k.360°$ or $\theta = 274.6° + k.360°, k \in \mathbb{Z}$

Step 4: Check your general solution
$k = 2$:
$\theta = 85.4° + 2.360° = 805.4°$ or $\theta = 274.6° + 2.360° = 994.6°$
$$\cos 805.4° = \frac{\sin 14°}{3}$$
$$\sin 994.6° = \frac{\sin 14°}{3}$$

Example 4.4

Solve for α if $\tan\alpha = -5.2$ and $0° \leq \alpha \leq 360°$.

Solution

Step 1: Use a calculator to determine the reference angle
$\tan\alpha = 5.2$ Remember, when finding the reference angle we always use the positive ratio
$\therefore \alpha = 79.1°$

Step 2: Use the CAST diagram to determine any other possible solutions
Our equation is $\tan\alpha = -5.2$. $\tan\alpha < 0$.

Second quadrant: $\tan(180^\circ - \theta) = -\tan\theta$. Therefore, the second quadrant solution is $180^\circ - 79.1^\circ = 100.9^\circ$.

Fourth quadrant: $\tan(360^\circ - \theta) = -\tan\theta$. Therefore, the fourth quadrant solution is $360^\circ - 79.1^\circ = 280.9^\circ$.

Step 3: Generate the general solution
$\alpha = 100.9^\circ + k.180^\circ$ or $\alpha = 280.9^\circ + k.180^\circ, k \in \mathbb{Z}$

Note: Because the period of tangent is 180°, the general solution includes integer multiples of 180° . Therefore, it is also possible to generate the general solution from only one angle. $280.9^\circ = 100.9^\circ + 1 \times 180^\circ$.

The simplest general solution is thus $\alpha = 100.9^\circ + k.180^\circ, k \in \mathbb{Z}$

Step 4: Generate the solution for the specific range
In this instance, we were not asked for the general solution but only for the solutions in$0^\circ \leq \alpha \leq 360^\circ$. We need to use our general solution to generate solutions that fall within this range by adding or subtracting multiples of 180°.

$\alpha = 100.9^\circ + 0 \times 180^\circ = 100.9^\circ$or $\alpha = 100.9^\circ + 1 \times 180^\circ = 280.9^\circ$

Step 5: Check your general solutions
$\tan 100.9^\circ = -5.2$
$\tan 280.9^\circ = -5.2$



Exercise 4.2

1. Determine the general solution for $5 - 3\tan\theta = 0$.
2. Solve for x if $3\cos x - 1 = -2$ for the interval $[0^\circ, 360^\circ]$.

The full solutions are at the end of the unit.

Solve more complicated trig equations

In level 3 subject outcome 2.1 units 7, 9 and 11, we learnt about the functions $y = a\sin kx$, $y = a\cos kx$ and $y = a\tan kx$. We saw, for example, that the period of the function $y = \sin 2x$ was no longer 360°. The period was actually $\frac{360^\circ}{2} = 180^\circ$.

Note

If you have an internet connection, visit this interactive simulation.

Here you will see the solutions to the equation $\sin kx = c$ where you can control the values of k and c.

How many solutions are there to $\sin x = 0.5$? How many solutions are there to $\sin 2x = 0.5$?
Figure 2 shows the solutions to $\sin x = 0.5$. As you would expect, these are $x = 30^\circ$ and $x = 150^\circ$ for the interval $[0^\circ, 360^\circ]$.

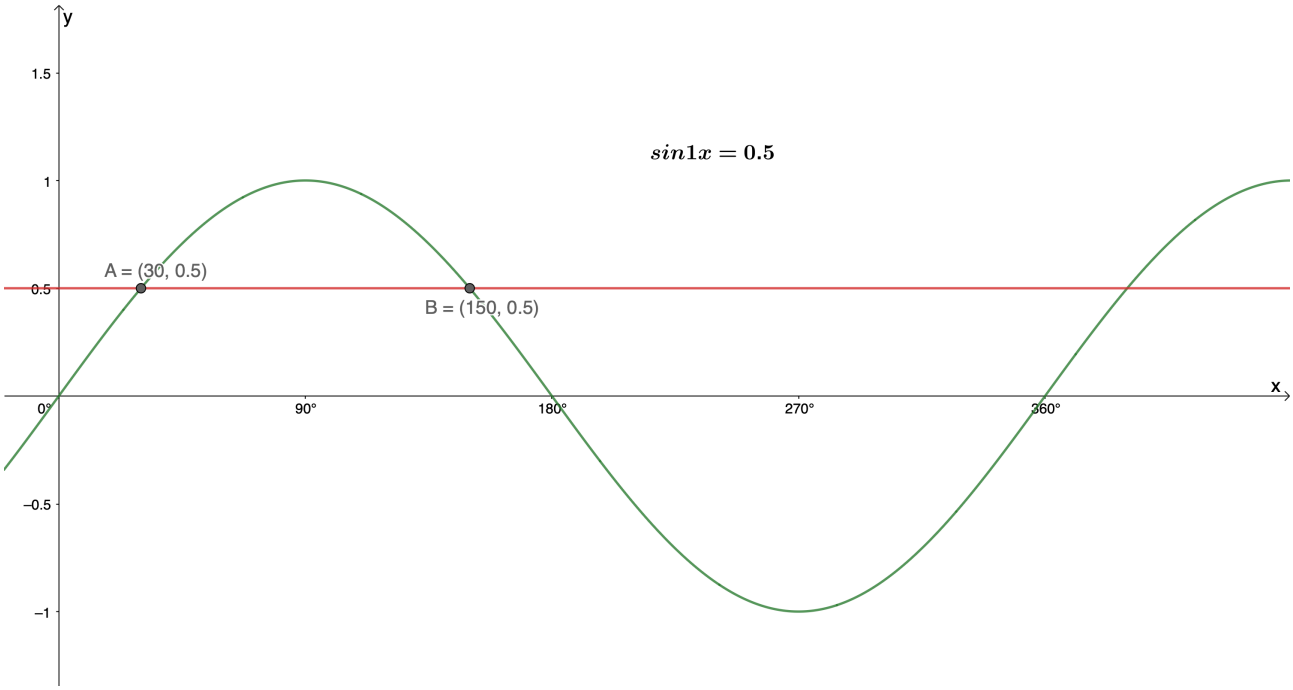


Figure 2: Solution to $\sin x = 0.5$ *for* $[0^\circ, 360^\circ]$

Now look at the solutions to $\sin 2x = 0.5$ in Figure 3. There are three important things to notice.

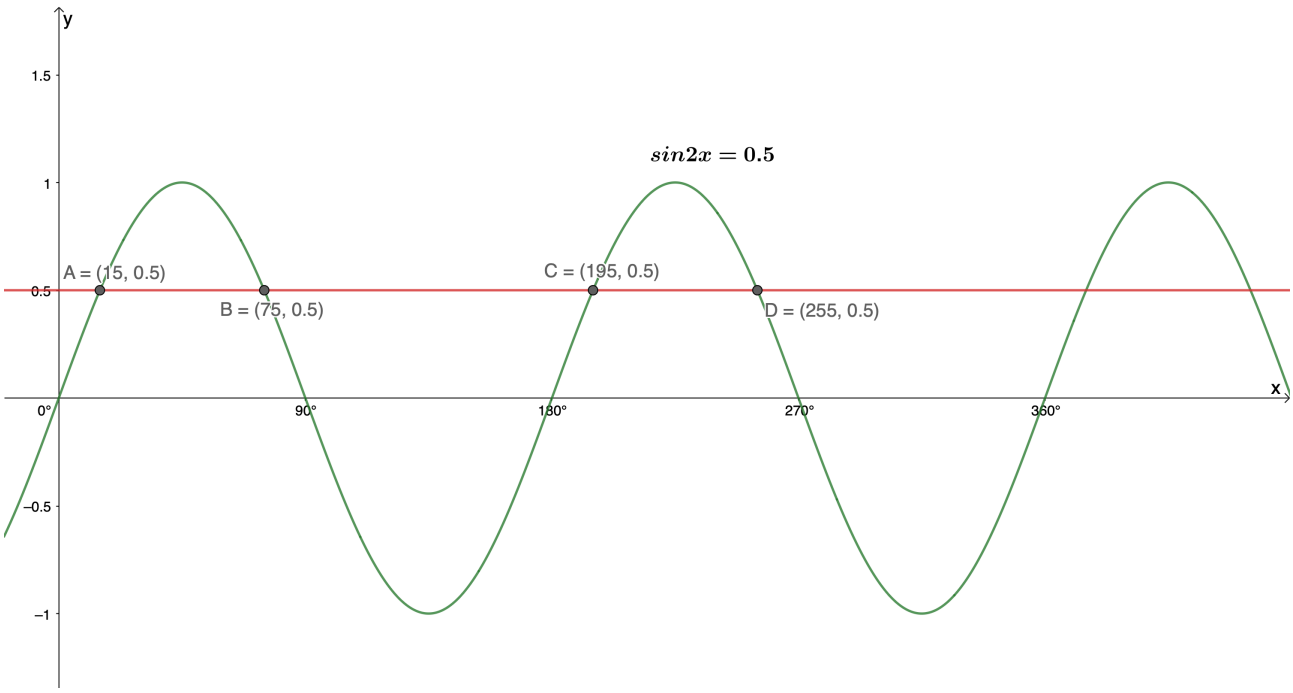


Figure 3: Solutions to $\sin 2x = 0.5$ *for* $[0^\circ, 360^\circ]$

1. The period of the graph has halved and the graph repeats twice in the interval $[0^\circ, 360^\circ]$. Therefore, there are double the number of solutions in the same interval.
2. The solution $x = 30^\circ$ from before is now $x = \frac{30^\circ}{2} = 15^\circ$. The solution $x = 150^\circ$ from before is now $x = \frac{150^\circ}{2} = 75^\circ$.
3. Solutions to $\sin x = 0.5$ are separated by integer multiples of 360°. Solutions to $\sin 2x = 0.5$ are seperated by interger multiples of $\frac{360^\circ}{2} = 180^\circ$.

Therefore, when we solve equations such as $\sin 2\theta = \frac{1}{2}$, we need to modify the general solution to take into account the period of the function is now 180°.

So, if the general solution to $\sin\theta = \frac{1}{2}$ is $\theta = 30^\circ + k.360^\circ$ or $\theta = 150^\circ + k.360^\circ, k \in \mathbb{Z}$, then the general solution to $\sin 2\theta = \frac{1}{2}$ is $\theta = 15^\circ + k.180^\circ$ or $\theta = 75^\circ + k.180^\circ, k \in \mathbb{Z}$.

The same is true for cosine and tangent equations.



Example 4.5

Determine the general solution for θ if $5\sin 2\theta = 3$.

Solution

$5 \sin 2\theta = 3$

$\therefore \sin 2\theta = \frac{3}{5}$

Step 1: Use a calculator to determine the reference angle

$\sin 2\theta = \frac{3}{5}$

$\therefore 2\theta = 36.9^\circ$

Note: We keep working with the reference angle as 2θ until we generate the general solution.

Step 2: Use the CAST diagram to determine any other possible solutions

Our equation is $\sin 2\theta = \frac{3}{5}$. $\sin 2\theta > 0$. Sine is positive in the first and second quadrants.

First quadrant: $2\theta = 36.9^\circ$

Second quadrant: $\sin(180^\circ - \theta) = \sin\theta$

$2\theta = 180^\circ - 36.9^\circ = 143.2^\circ$.

Step 3: Generate the general solution

$2\theta = 36.9^\circ + k.360^\circ$ or $2\theta = 143.2^\circ + k.360^\circ, k \in \mathbb{Z}$

$\therefore \theta = 18.45^\circ + k.180^\circ$ or $\theta = 71.6^\circ + k.180^\circ, k \in \mathbb{Z}$

Step 4: Check your general solution

$k = 2$:

$\therefore \theta = 18.45^\circ + 2 \times 180^\circ = 378.45^\circ$ or $\theta = 71.6^\circ + 2 \times 180^\circ = 431.6^\circ$

$\sin(2 \times 378.45^\circ) = 0.6$

$\sin(2 \times 431.6^\circ) = 0.6$



Example 4.6

Solve for θ if $\tan(3\theta - 38^\circ) = -5$ for the interval $[-180^\circ, 180^\circ]$.

Solution

Step 1: Use a calculator to determine the reference angle

$\tan(3\theta - 38^\circ) = 5$ Remember, when finding the reference angle, use the positive ratio

$\therefore 3\theta - 38^\circ = 78.7^\circ$ Keep your reference angle in terms of $3\theta - 38^\circ$

Note: We keep working with the reference angle as $3\theta - 38^\circ$ until we generate the general solution.

Step 2: Use the CAST diagram to determine any other possible solutions

Our equation is $\tan(3\theta - 38^\circ) = -5$. $\tan(3\theta - 38^\circ) < 0$. Tangent is negative in the second and fourth quadrants.

Second quadrant: $\tan(180^\circ - \theta) = -\tan\theta$

$180^\circ - 78.7^\circ = 101.3^\circ$

Step 3: Generate the general solution

$3\theta - 38^\circ = 101.3^\circ + k.180^\circ,\ k \in \mathbb{Z}$ Add 38° to the angle

$\therefore 3\theta = 139.3^\circ + k.180^\circ,\ k \in \mathbb{Z}$ Divide both the angle and the period by 3

$\therefore \theta = 46.37^\circ + k.60^\circ,\ k \in \mathbb{Z}$

Step 4: Generate the solution for the specific range
The interval is $[-180^\circ, 180^\circ]$.
$\theta = 46.37^\circ - 3 \times 60^\circ = -133.63^\circ$ or
$\theta = 46.37^\circ - 2 \times 60^\circ = -73.63^\circ$ or
$\theta = 46.37^\circ - 1 \times 60^\circ = -13.63^\circ$ or
$\theta = 46.37^\circ + 0 \times 60^\circ = 46.37^\circ$ or
$\theta = 46.37^\circ + 1 \times 60^\circ = 106.37^\circ$ or
$\theta = 46.37^\circ + 2 \times 60^\circ = 166.37^\circ$



Exercise 4.3

1. Determine the general solution for $\cos\left(\frac{1}{2}x\right) = 0.54$.
2. Solve for x if $2\sin\left(\frac{3x}{2} + 10^\circ\right) = -1$ for the interval $[-180^\circ, 180^\circ]$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the general solution for trig equations involving sine, cosine and tangent and simple angles.
- How to find the general solution for trig equations involving sine, cosine and tangent and more complex angles.
- How to find the solutions to trig equations involving sine, cosine and tangent for specific intervals.

Unit 4: Assessment

Suggested time to complete: 45 minutes

All the questions below have been adapted from NC(V) examination questions.

1. Determine the general solution for $\sin\theta + 1 = 0.3$.
2. Calculate the value/s of θ if $2\tan\theta = 0.279$ where $\theta \in [0^\circ, 360^\circ]$.
3. Calculate the value/s of θ if $5\cos 2\theta = -2.7$ where $\theta \in [0^\circ, 360^\circ]$.
4. Calculate the value/s of θ if $\tan(3\theta - 48^\circ) = 3.2$ where $\theta \in [0^\circ, 180^\circ]$.
5. Determine the value/s of θ in the following equation without a calculator if $0^\circ \leq \theta \leq 360^\circ$.
   $\sin\theta + \sqrt{2} = -\sin\theta$

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. $2\sin\theta = \sqrt{3}$
   $\therefore \sin\theta = \dfrac{\sqrt{3}}{2}$
   Ref angle: $\theta = 60^\circ$ (you should have recognised this as a special angle ratio)
   $\sin\theta > 0$ in first and second quadrant. Therefore, $\theta = 60^\circ$ or $\theta = 180^\circ - 60^\circ = 120^\circ$.
   General solution: $\theta = 60^\circ + k.360^\circ$ or $\theta = 120^\circ + k.360^\circ, k \in \mathbb{Z}$
2. $5\sin x = -2$
   $\therefore \sin x = -\dfrac{2}{5}$
   Ref angle: $x = 23.6^\circ$
   $\sin x < 0$ in third and fourth quadrants.
   $\theta = 180^\circ + 23.6 = 203.6^\circ$ or $\theta = 360^\circ - 23.6^\circ = 336.4^\circ$.
   General solution: $\theta = 203.6^\circ + k.360^\circ$ or $\theta = 336.4^\circ + k.360^\circ, k \in \mathbb{Z}$
   Specific solution for $0^\circ \leq x \leq 360^\circ$: $\theta = 203.6^\circ$ or $\theta = 336.4^\circ$

Back to Exercise 4.1

Exercise 4.2

1. $5 - 3\tan\theta = 0$
   $\therefore \tan\theta = \dfrac{5}{3}$
   Ref angle: $\theta = 59^\circ$
   $\tan\theta > 0$ in the first and third quadrants.
   $\theta = 59^\circ$ or $\theta = 180^\circ + 59^\circ = 239^\circ$
   General solution: $\theta = 59^\circ + k.180^\circ, k \in \mathbb{Z}$
2. $3\cos x - 1 = -2$
   $\therefore 3\cos x = -1$
   $\therefore \cos x = -\dfrac{1}{3}$
   Ref angle: $x = 70.5^\circ$
   $\cos x < 0$ in the second and third quadrants.
   $x = 180^\circ - 70.5^\circ = 109.5^\circ$ or $x = 180^\circ + 70.5^\circ = 250.5^\circ$
   General solution: $x = 109.5^\circ + k.360^\circ$ or $x = 250.5^\circ + k.360^\circ, k \in \mathbb{Z}$
   Specific solution for $[0^\circ, 360^\circ]$: $x = 109.5^\circ$ or $x = 250.5^\circ$

Back to Exercise 4.2

Exercise 4.3

1.

$\cos\left(\frac{1}{2}x\right) = 0.54$

$\therefore \frac{1}{2}x = 57.3^\circ$

Ref angle: $\frac{1}{2}x = 57.3^\circ$

$\cos\left(\frac{1}{2}x\right) > 0$ in the first and fourth quadrants.

$\frac{1}{2}x = 57.3^\circ$ or $\frac{1}{2}x = 360^\circ - 57.3^\circ = 302.7^\circ$

General solution:

$\frac{1}{2}x = 57.3^\circ + k.360^\circ$ or $\frac{1}{2}x = 302.7^\circ + k.360^\circ, k \in \mathbb{Z}$

$\therefore x = 114.6^\circ + k.720^\circ$ or $x = 605.4^\circ + k.720^\circ, k \in \mathbb{Z}$

2.

$2\sin\left(\frac{3x}{2} + 10^\circ\right) = -1$

$\therefore \sin\left(\frac{3x}{2} + 10^\circ\right) = -\frac{1}{2}$

Ref angle: $\left(\frac{3x}{2} + 10^\circ\right) = 30^\circ$

$\sin\left(\frac{3x}{2} + 10^\circ\right) < 0$ in the third and fourth quadrants.

$\frac{3x}{2} + 10^\circ = 180^\circ + 30^\circ = 210^\circ$ or $\frac{3x}{2} + 10^\circ = 360^\circ - 30^\circ = 330^\circ$

General solution:

$\frac{3x}{2} + 10^\circ = 210^\circ + k.360^\circ$ or $\frac{3x}{2} + 10^\circ = 330^\circ + k.360^\circ, k \in \mathbb{Z}$

$\therefore \frac{3x}{2} = 200^\circ + k.360^\circ$ or $\frac{3x}{2} = 320^\circ + k.360^\circ, k \in \mathbb{Z}$

$\therefore x = 133.33^\circ + k.240^\circ$ or $\frac{3x}{2} = 213.33^\circ + k.240^\circ, k \in \mathbb{Z}$

Specific solution for $[-180^\circ, 180^\circ]$:

$x = 133.33^\circ - 1 \times 240^\circ = -106.67^\circ$ or

$x = 133.33^\circ + 0 \times 240^\circ = 133.33^\circ$ or

$x = 213.33^\circ - 1 \times 240^\circ = -26.67^\circ$

Back to Exercise 4.3

Unit 4: Assessment

1.

$\sin\theta + 1 = 0.3$

$\therefore \sin\theta = -0.7$

Ref angle: $\theta = 44.4^\circ$

$\sin\theta < 0$: $\theta = 180^\circ + 44.4^\circ = 224.4^\circ$ or $\theta = 360^\circ - 44.4^\circ = 315.6^\circ$

General solution: $\theta = 224.4^\circ + k.360^\circ$ or $\theta = 315.6^\circ + k.360^\circ, k \in \mathbb{Z}$

2.

$2\tan\theta = 0.279$

$\therefore \tan\theta = 0.1395$

Ref angle: $\theta = 8.0^\circ$

$\tan\theta > 0 : \theta = 8.0^\circ$ or $\theta = 180^\circ + 8.0^\circ = 188^\circ$

General solution: $\theta = 8.0^\circ + k.180^\circ, k \in \mathbb{Z}$

Specific solution for $\theta \in [0^\circ, 360^\circ]$:

$\theta = 8^\circ + 0 \times 180^\circ = 8^\circ$ or

$\theta = 8.0^\circ + 1 \times 180^\circ = 188.0^\circ$

3. $5\cos 2\theta = -2.7$
   $\therefore \cos 2\theta = -0.54$
   Ref angle: $2\theta = 57.3^\circ$
   $\cos 2\theta < 0$: $2\theta = 180^\circ - 57.3^\circ = 122.7^\circ$ or $2\theta = 180^\circ + 57.3^\circ = 237.3^\circ$
   General solution:
   $2\theta = 122.7^\circ + k.360^\circ$ or $2\theta = 237.3^\circ + k.360^\circ, k \in \mathbb{Z}$
   $\therefore \theta = 61.35^\circ + k.180^\circ$ or $\theta = 118.65^\circ + 180^\circ, k \in \mathbb{Z}$
   Specific solution for $\theta \in [0^\circ, 360^\circ]$:
   $\theta = 61.35^\circ + 0 \times 180^\circ = 61.35^\circ$ or
   $\theta = 61.35^\circ + 1 \times 180^\circ = 241.35^\circ$ or
   $\theta = 118.65^\circ + 0 \times 180^\circ = 118.65^\circ$ or
   $\theta = 118.65^\circ + 1 \times 180^\circ = 298.65^\circ$
4. $\tan(3\theta - 48^\circ) = 3.2$
   Ref angle: $3\theta - 48^\circ = 72.6^\circ$
   $3\theta - 48^\circ = 72.6^\circ$ or $3\theta - 48^\circ = 180^\circ + 72.6^\circ = 252.6^\circ$:
   General solution:
   $3\theta - 48^\circ = 72.6^\circ + k.180^\circ$ or $3\theta - 48^\circ = 252.6^\circ + k.180^\circ, k \in \mathbb{Z}$
   $\therefore 3\theta = 120.6^\circ + k.180^\circ$ or $3\theta = 300.6^\circ + k.180^\circ, k \in \mathbb{Z}$
   $\therefore \theta = 40.2^\circ + k.60^\circ$ or $\theta = 100.2^\circ + k.60^\circ, k \in \mathbb{Z}$
   But $40.2^\circ + 1 \times 60^\circ = 100.2^\circ$
   Therefore, simplest general solution is $\theta = 40.2^\circ + k.60^\circ, k \in \mathbb{Z}$
   Specific solution for $\theta \in [0^\circ, 180^\circ]$:
   $\theta = 40.2^\circ + 0 \times 60^\circ = 40.2^\circ$ or
   $\theta = 40.2^\circ + 1 \times 60^\circ = 100.2^\circ$ or
   $\theta = 40.2^\circ + 2 \times 60^\circ = 160.2^\circ$
5. $\sin\theta + \sqrt{2} = -\sin\theta$
   $\therefore 2\sin\theta = -\sqrt{2}$
   $\therefore \sin\theta = -\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = -\frac{1}{\sqrt{2}}$
   Ref angle: $\theta = 45^\circ$
   $\sin\theta < 0$: $\theta = 180^\circ + 45^\circ = 225^\circ$ or $\theta = 360^\circ - 45^\circ = 315^\circ$
   General solution: $\theta = 225^\circ + k.360^\circ$ or $\theta = 315^\circ + k.360^\circ$
   Specific solution for $0^\circ \leq \theta \leq 360^\circ$:
   $\theta = 225^\circ + 0.360^\circ = 225^\circ$ or
   $\theta = 315^\circ + 0.360^\circ = 315^\circ$

Back to Unit 4: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 5: Apply the area rule

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Apply the area rule in 2-D triangles.

What you should know

Before you start this unit, make sure you can:

- Calculate the area of a triangle using the formula $\text{area} = \frac{1}{2} \times b \times h$.
- Use a calculator to calculate the sine of a given angle. Refer to level 2 subject outcome 3.6 unit 2 if you need help with this.
- Use a calculator to calculate the angle from a given ratio for sine. Refer to level 2 subject outcome 3.6 unit 2 if you need help with this.

Introduction

Trigonometry is not just useful for finding the lengths of unknown sides or the sizes of unknown angles in right-angled triangles. It can also be put to work in non-right-angled triangles and it can be used to find measures other than the lengths of sides and the sizes of angles.

The area rule, for example, is a useful trigonometric identity (remember those from unit 3?) that can be used to find the area of any triangle, even triangles that have no 90° angles. But why do we need another way to calculate the area of a triangle if we already have the well-known formula $\text{Area} = \frac{1}{2} \times b \times h$? Well, sometimes, we might not know the length of the perpendicular height or the base. Instead, we might know the size of one of the angles inside the triangle.

In this unit, we are going to derive the area rule and see how to use it to find the areas of all sorts of different triangles in different contexts.

Derive the area rule

Deriving the area rule is not too difficult. You will be led through this process in Activity 5.1.

Activity 5.1: Derive the area rule

Time required: 15 minutes

What you need:

- a piece of paper
- a pen or pencil
- a ruler

What to do:

1. On your piece of paper, draw any triangle ABC and label the sides in relation to the opposite vertices (see Figure 1). The triangle can be any size and shape.
2. Now drop a perpendicular from B to the opposite side and call this perpendicular h (see Figure 2).
3. Write the expression for the area of triangle ABC based on its base and perpendicular height.
4. Write an expression for $\sin \mathrm{A}$.
5. Write a new expression for the area of triangle ABC that includes the term $\sin A$. Hint: look for a substitution.
6. Now, write an expression for $\sin \mathrm{C}$ and then write a similar expression for the area of the triangle in terms of$\sin \mathrm{C}$.
7. What do you think the area of triangle ABC would be in terms of $\sin \mathrm{B}$?

What did you find?

1. Remember that your triangle ABC can be any size and shape. Figure 1 shows just one possible triangle.

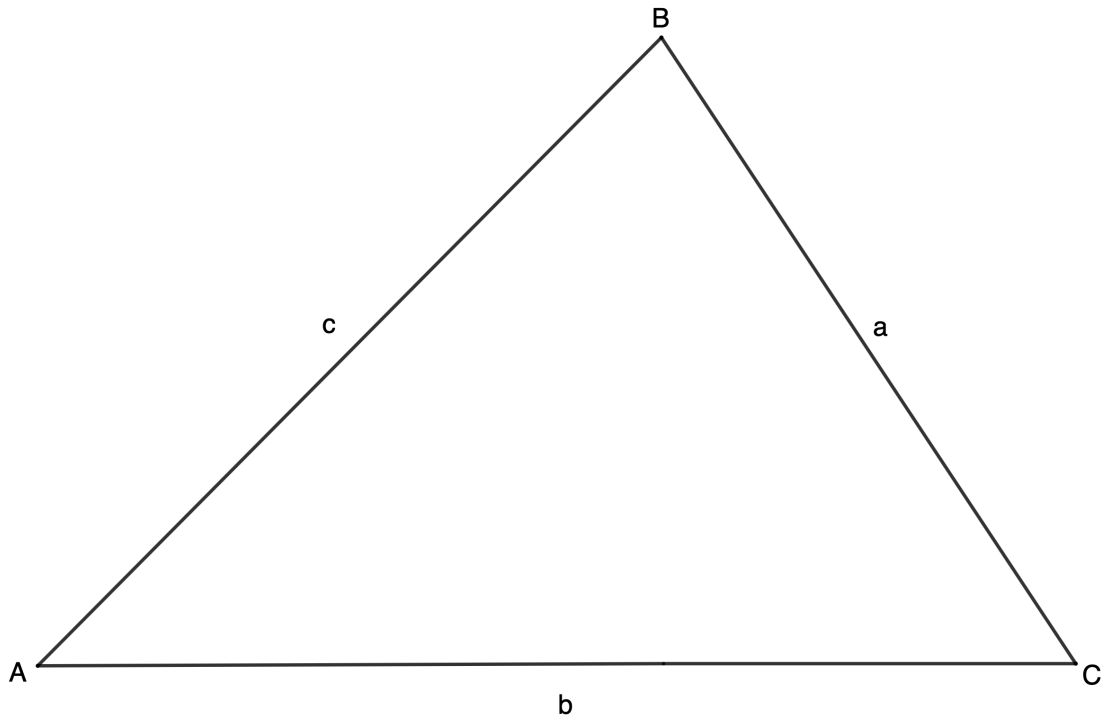


Figure 1: Triangle ABC

2. Remember that dropping a perpendicular means that the new line meets the other line at 90°. Figure 2 shows what your perpendicular line may look like.

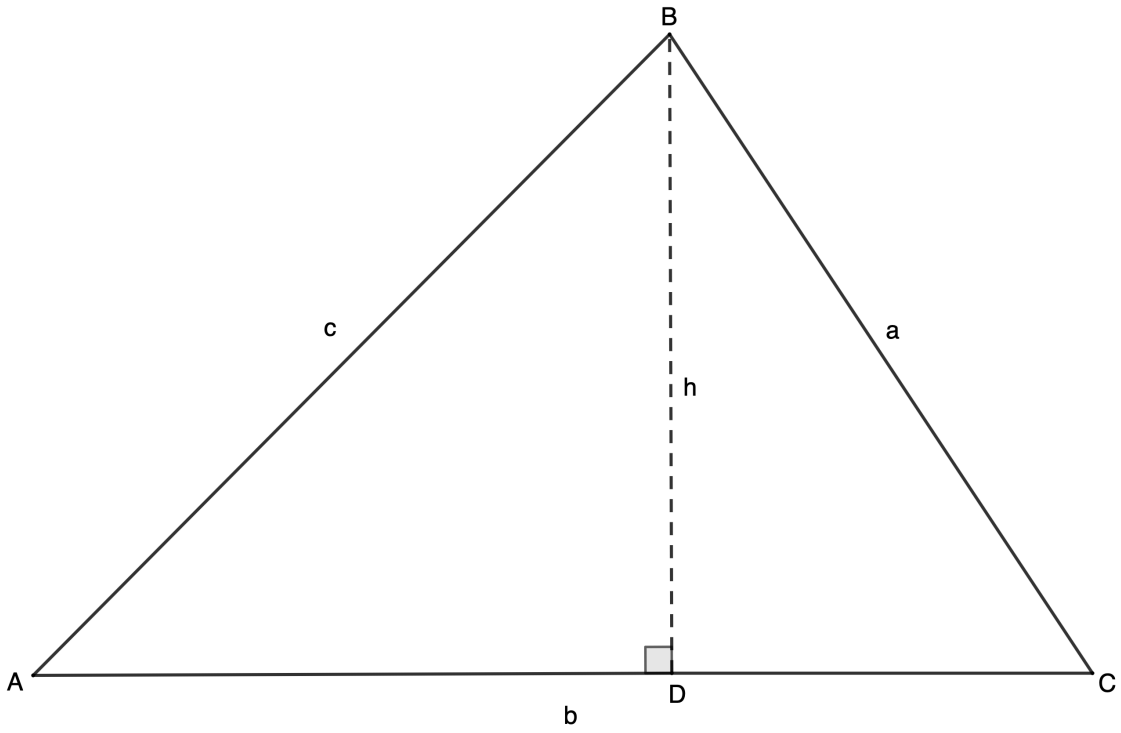
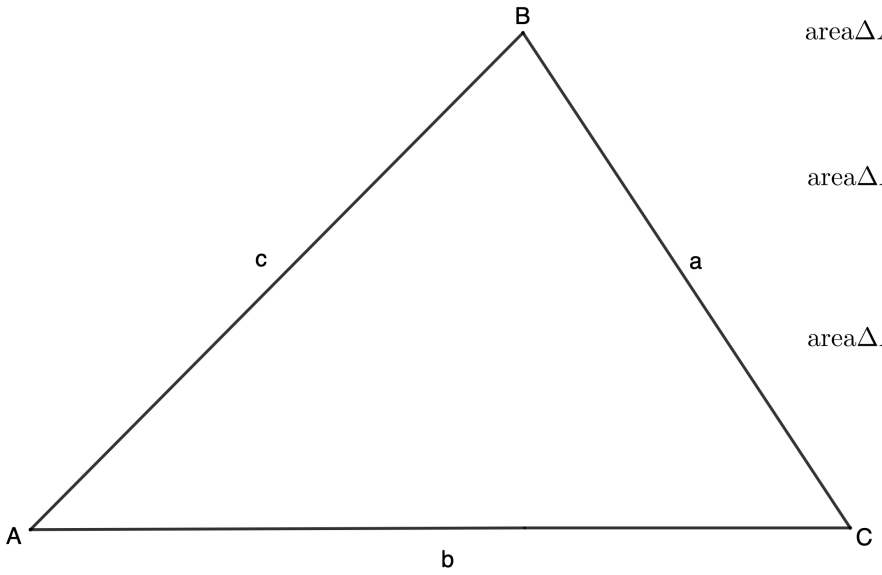


Figure 2: Triangle ABC *with perpendicular* h

3. The area of triangle ABC is given by $A = \frac{1}{2} \times b \times h$ where the base is side b and the perpendicular height is h. Therefore, area of $\Delta\mathrm{ABC}$ is $\frac{1}{2} \times \mathrm{b} \times \mathrm{h}$.
4. $\sin \mathrm{A} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\mathrm{h}}{\mathrm{c}}$.
5. $\sin \mathrm{A} = \frac{\mathrm{h}}{\mathrm{c}} \therefore \mathrm{h} = \mathrm{c} \times \sin \mathrm{A}$. But area $\Delta\mathrm{ABC}$ is $\frac{1}{2} \times \mathrm{b} \times \mathrm{h}$. Therefore, area $\Delta\mathrm{ABC} = \frac{1}{2} \times \mathrm{b} \times \mathrm{c} \times \sin textA = \frac{1}{2}\mathrm{bc} \sin \mathrm{A}$.
6. $\sin \mathrm{C} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\mathrm{h}}{\mathrm{a}}$. Therefore $\mathrm{h} = \mathrm{a} \times \sin \mathrm{C}$ and area $\Delta\mathrm{ABC} = \frac{1}{2} \times \mathrm{b} \times \mathrm{a} \times \sin \mathrm{C} = \frac{1}{2}\mathrm{ab} \sin \mathrm{C}$
7. In each case above, the expression for the area of the triangle is the product of the length of two sides and the sine of the angle between these two sides. Therefore, area $\Delta\mathrm{ABC} = \frac{1}{2}\mathrm{ac} \sin \mathrm{B}$.

The area rule states that in any ΔABC, the area is given by:



$$\text{area}\Delta ABC = \frac{1}{2}bc\sin A$$

$$\text{area}\Delta ABC = \frac{1}{2}ac\sin B$$

$$\text{area}\Delta ABC = \frac{1}{2}ab\sin C$$

Take note!

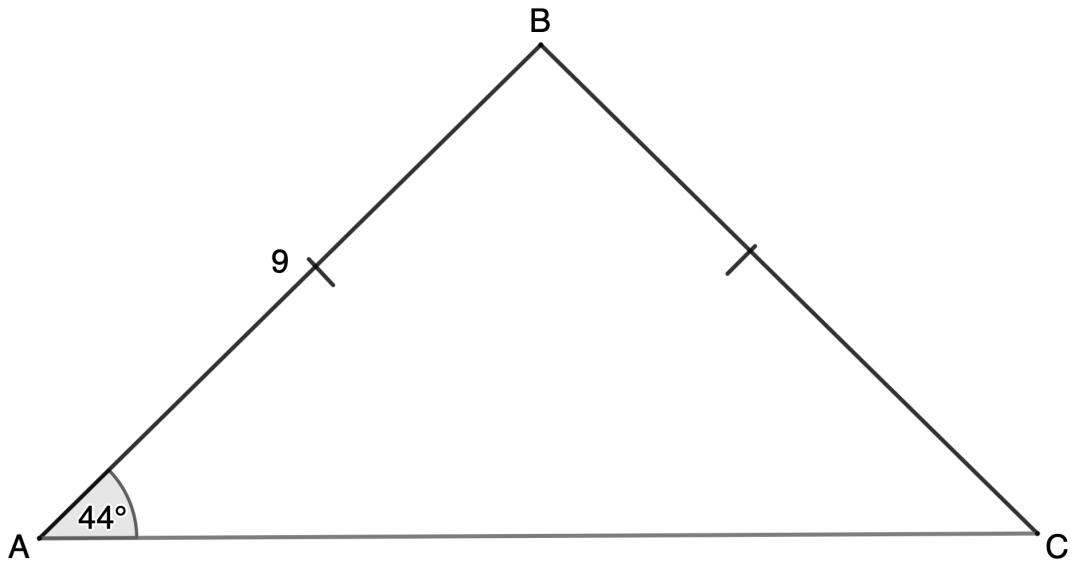
The area rule works for any triangle for which you know the length of **any two sides** and the size of the **included** angle (the angle **between** the two known sides).

Use the area rule

Let's look at some examples of how to apply the area rule.

Example 5.2

Find the area of ΔABC correct to two decimal places.



Solution

We are not given the lengths of two sides and the included angle, so we cannot use the area rule with the information we currently have. We need to find the size of $\hat{B}$. To do this, we can use the fact that $AB = BC$ (given) and therefore, $\hat{A} = \hat{C} = 43^\circ$ (angles opposite equal sides). Then we can use that fact that angles in a triangle are supplementary to find $\hat{B}$.

$AB = BC$ (given)
$\therefore \hat{A} = \hat{C} = 43^\circ$ (angles opposite equal sides)
$\therefore \hat{B} = 180^\circ - \hat{A} - \hat{C}$
$\quad = 180^\circ - 43^\circ - 43^\circ$ (angles in a triangle are supplementary)
$\quad = 94^\circ$

Now we can use the area rule.

$$\begin{aligned}\text{Area } \Delta ABC &= \frac{1}{2}ac\sin B \\ &= \frac{1}{2} \times 8 \times 8 \times \sin 94^\circ \\ &= 31.92 \text{ m}^2\end{aligned}$$

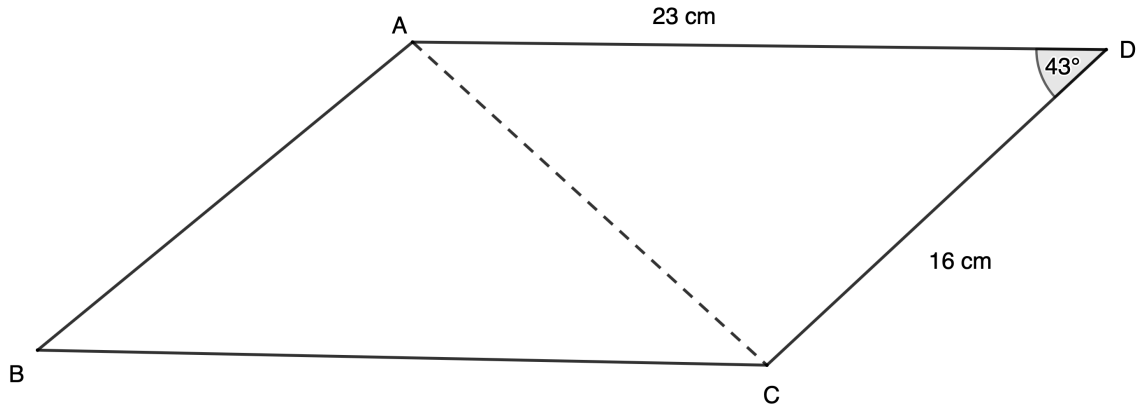
Remember to round off your final answer and include the correct area units.

Example 5.3

A parallelogram has adjacent sides of 16 cm and 23 cm. The angle between them is 43°. Calculate the area of the parallelogram correct to two decimal places.

Solution

If no diagram is given, it is always best to draw your own using the given information. This does not have to be accurate.



We know that $ABCD$ is a parallelogram. Therefore, if we can find the area of ΔACD we can multiply this by 2 to find the area of $ABCD$.

In ΔACD, we know two sides and the included angle so we can use the area rule.

$$\begin{aligned}\text{Area } \Delta ADC &= \frac{1}{2}ac\sin D \\ &= \frac{1}{2} \times 23 \times 16 \times \sin 43^\circ \\ &= 125.49 \text{ cm}^2\end{aligned}$$

The area of parallelogram $ABCD = 2 \times 125.49 \text{ cm}^2 = 250.98 \text{ cm}^2$.

Example 5.4

If ΔEFG has an area of 167.43 cm^2 and $e = 17 \text{ cm}$ and $f = 24.35 \text{ cm}$, what are the two possible sizes of $\hat{G}$, correct to two decimal places.

Solution

This question gives us the area and asks us to calculate the size of $\hat{G}$.

$$\begin{aligned}\text{Area } \Delta EFG &= \frac{1}{2}ef\sin G \\ \therefore 167.43 &= \frac{1}{2} \times 17 \times 24.35 \times \sin G \\ \therefore \sin G &= 53.99^\circ\end{aligned}$$

We have found that $\hat{G} = 53.99^\circ$ but the question mentions two possible solutions. Remember that sine is also positive in the second quadrant, in other words, for angles $90^\circ \leq \theta \leq 180^\circ$. Therefore, $\hat{G} = 180^\circ - 53.99^\circ = 126.01^\circ$ as well. But does this make physical sense? Does such a triangle exist? Figure 3 shows both possible triangles with these dimensions.

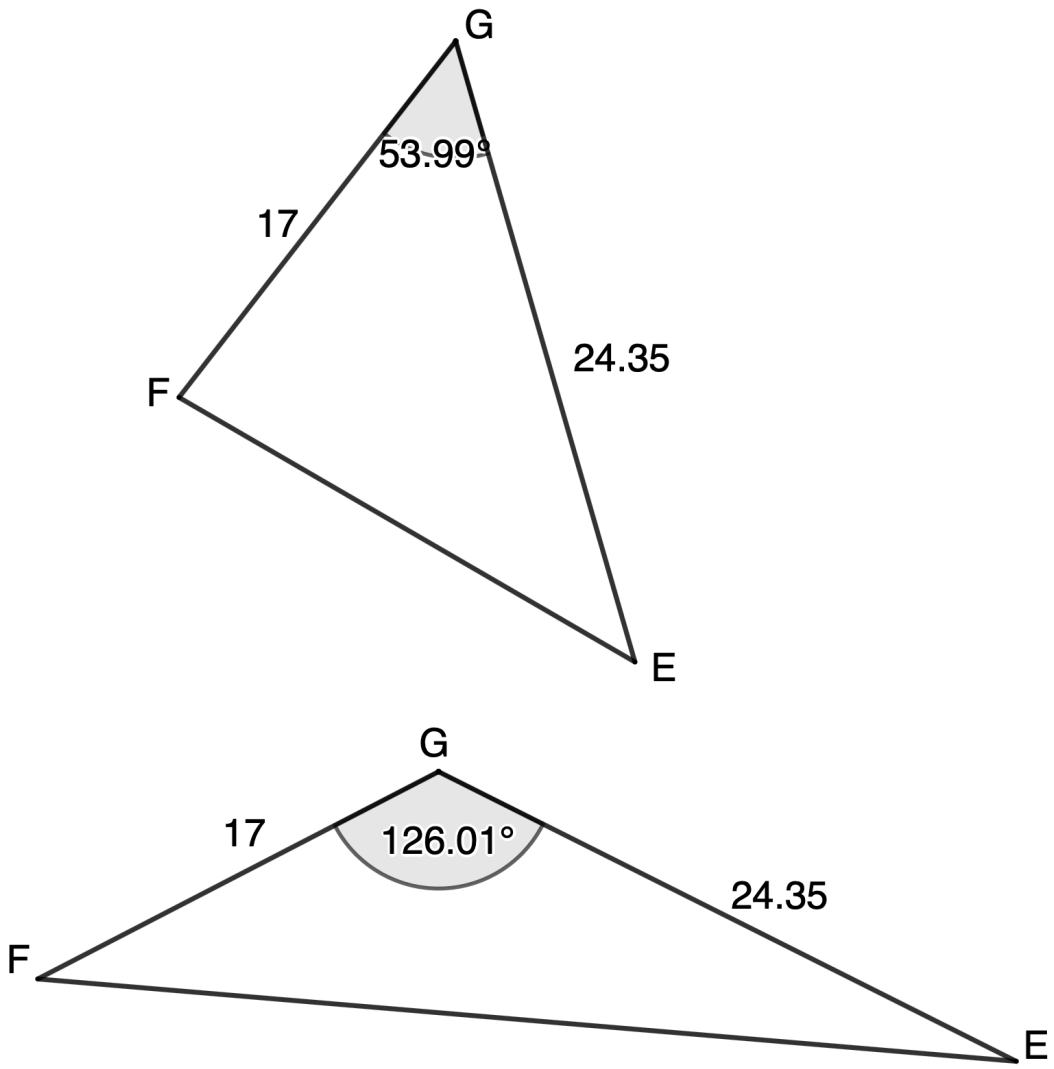
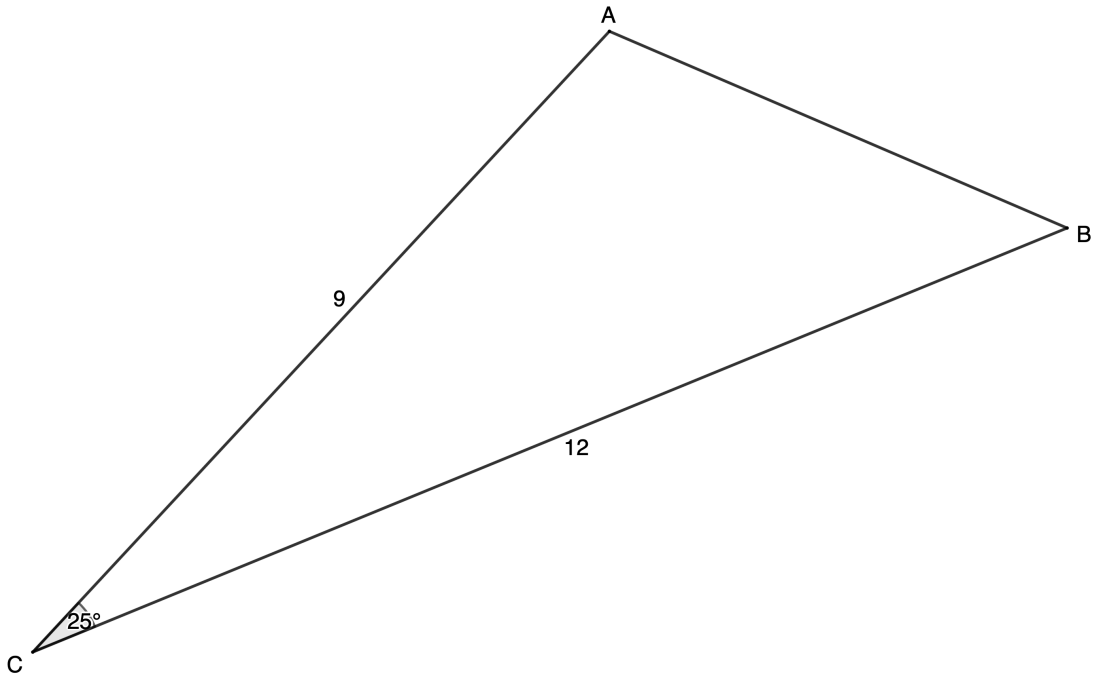


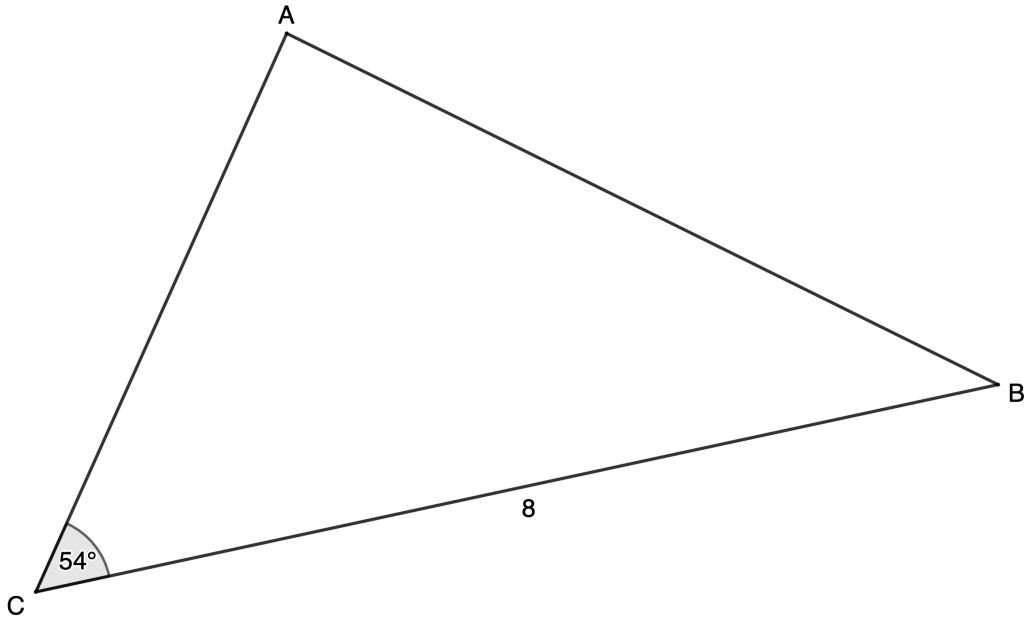
Figure 3: Both possible ΔEFG

Exercise 5.1

1. Calculate the area of ΔABC to two decimal places.



2. Calculate the area of ΔABC given $a = 10\ \text{cm}$, $c = 8\ \text{cm}$ and $\hat{B} = 35°$, to three decimal places.
3. Determine the area of equilateral ΔPQR with $p = 14\ \text{cm}$ to two decimal places.
4. Determine, to two decimal places, the area of a parallelogram in which two adjacent sides are $10\ \text{cm}$ and $13\ \text{cm}$ and the angle between them is $55°$.
5. Determine the length of AC (to one decimal place) if the area of $\Delta ABC = 16.18\ \text{m}^2$.

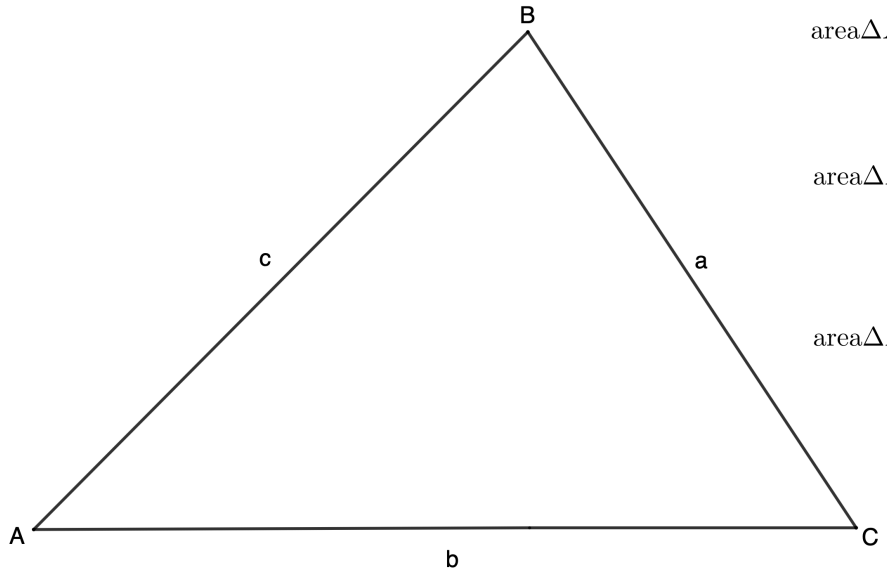


The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- What the area rule is.



$$\text{area}\Delta ABC = \frac{1}{2}bc\sin A$$

$$\text{area}\Delta ABC = \frac{1}{2}ac\sin B$$

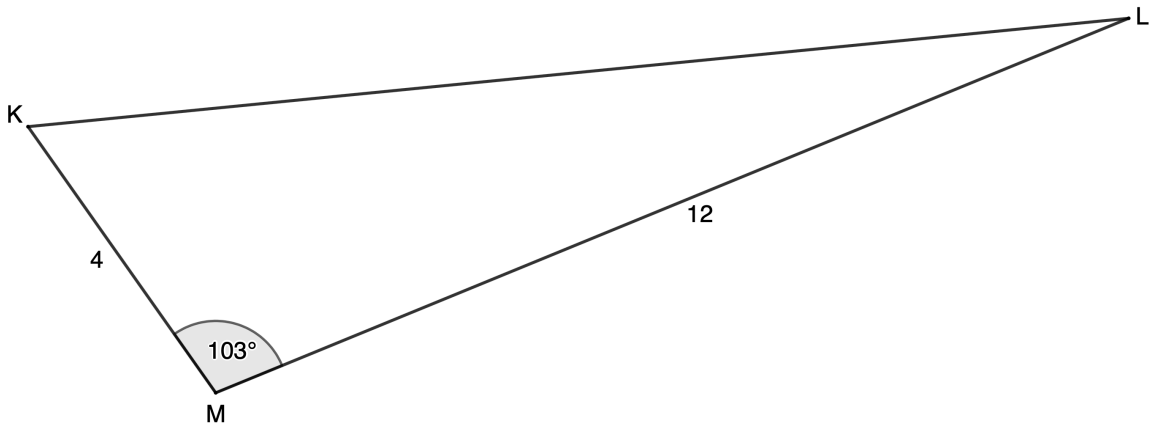
$$\text{area}\Delta ABC = \frac{1}{2}ab\sin C$$

- How to use the area rule to find the area of any triangle where two sides and the included angle are known.

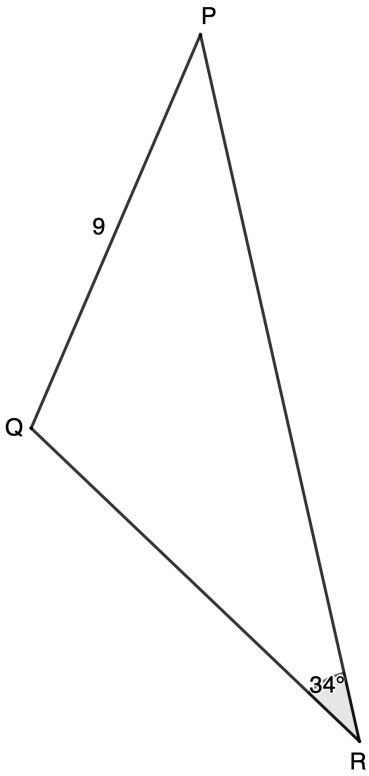
Unit 5: Assessment

Suggested time to complete: 40 minutes

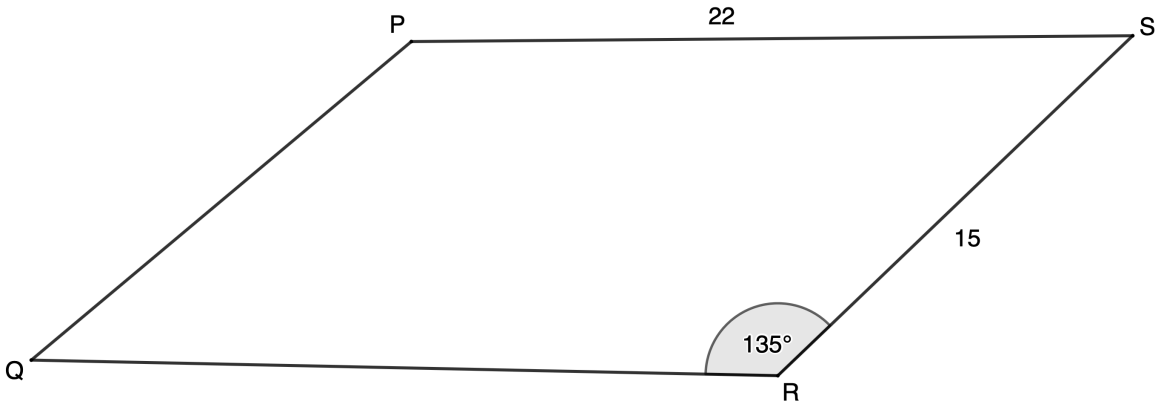
1. Calculate the area of ΔKLM to two decimal places.



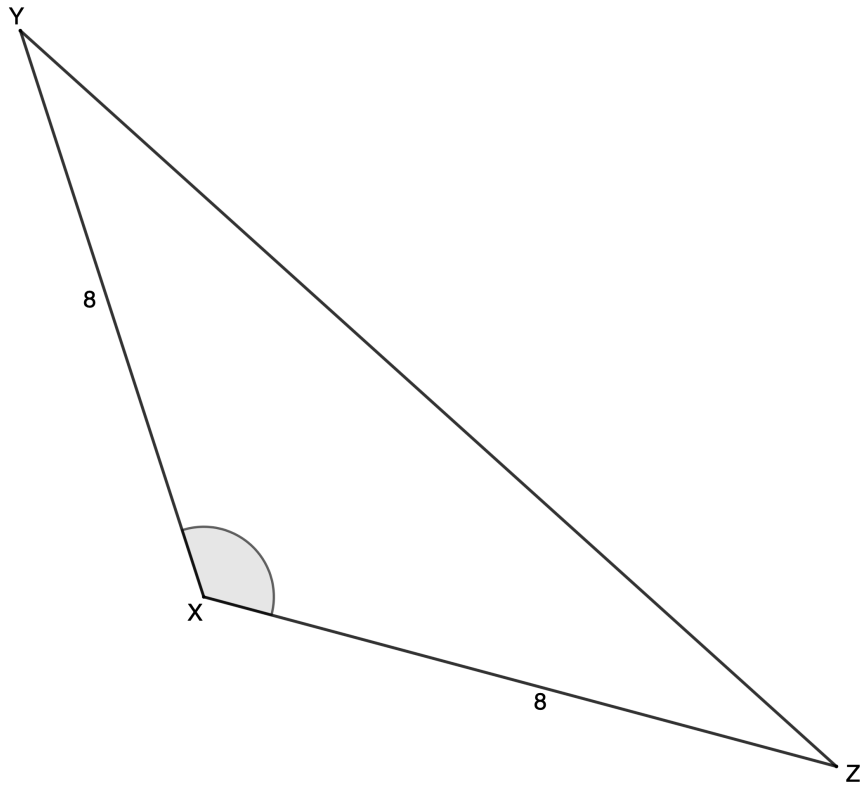
2. Calculate, to two decimal places, the area of ΔABC given $b = 19\ \text{cm}$, $c = 18\ \text{cm}$ and $\hat{A} = 49°$.
3. Determine, to two decimal places, the area of ΔPQR with $p = r = 9\ \text{m}$.



4. Determine the area of a parallelogram $PQRS$ to two decimal places.



5. Determine two possible values for $\hat{X}$ (to one decimal place) if the area of $\Delta XYZ = 26.72\ \text{m}^2$.



The full solutions are at the end of the unit.

Unit 5: Solutions

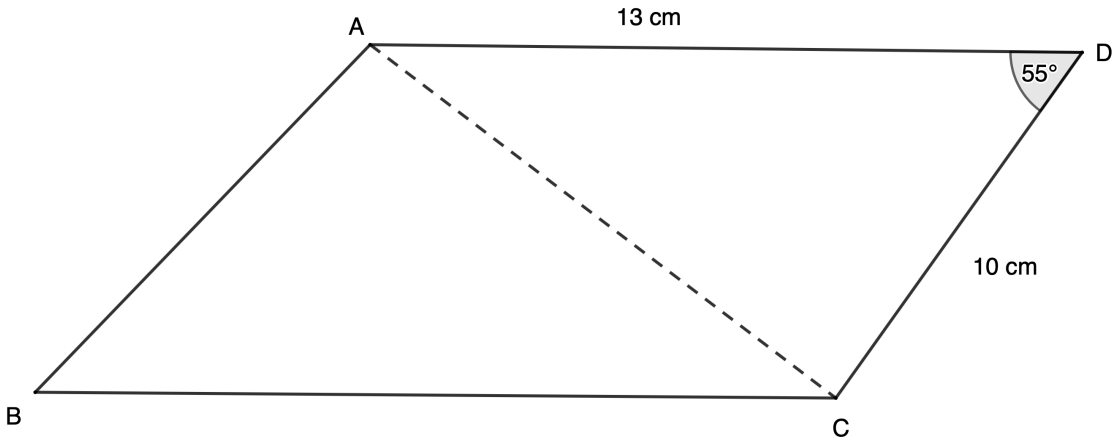
Exercise 5.1

1. $$\begin{aligned}\text{Area } \Delta ABC &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2} \times 12 \times 9 \times \sin 25^\circ \\ &= 22.82 \text{ u}^2\end{aligned}$$
   Note: No units were given so we include u^2 for units squared.

2. $$\begin{aligned}\text{Area } \Delta ABC &= \frac{1}{2}ac\sin B \\ &= \frac{1}{2} \times 10 \times 8 \times \sin 35^\circ \\ &= 22.943 \text{ cm}^2\end{aligned}$$

3. ΔPQR is an equilateral triangle. Therefore, all three sides have length 14 cm and all three angles are 60°. We can choose any combination of sides and angle.
   $$\begin{aligned}\text{Area } \Delta ABC &= \frac{1}{2}pq\sin R \\ &= \frac{1}{2} \times 14 \times 14 \times \sin 60^\circ \\ &= 84.87 \text{ cm}^2\end{aligned}$$

4.



$$\begin{aligned}\text{Area } \Delta ACD &= \frac{1}{2}ac\sin D \\ &= \frac{1}{2} \times 10 \times 13 \times \sin 55^\circ \\ &= 53.24 \text{ cm}^2\end{aligned}$$

Area of parallelogram $ABCD = 2 \times 53.24 \text{ cm}^2 = 106.49 \text{ cm}^2$

5.

$$\begin{aligned}\text{Area } \Delta ABC &= 16.18 \text{ m}^2 \\ \therefore 16.18 &= \frac{1}{2} \times AC \times 8 \times \sin 54^\circ \\ \therefore AC &= 4.045 \sin 54^\circ \\ \therefore AC &= 3.3 \text{ m}\end{aligned}$$

Back to Exercise 5.1

Unit 5: Assessment

1.

$$\begin{aligned}\text{Area } \Delta KLM &= \frac{1}{2}kl\sin M \\ &= \frac{1}{2} \times 12 \times 4 \times \sin 103^\circ \\ &= 23.38 \text{ u}^2\end{aligned}$$

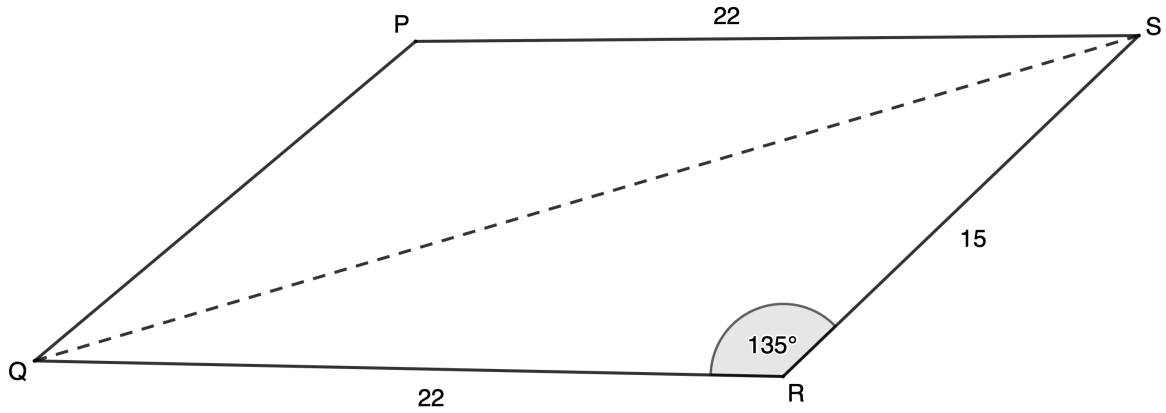
2.

$$\begin{aligned}\text{Area } \Delta ABC &= \frac{1}{2}bc\sin A \\ &= \frac{1}{2} \times 19 \times 18 \times \sin 49^\circ \\ &= 129.06 \text{ cm}^2\end{aligned}$$

3. $PQ = RQ$ (given)
$\therefore \hat{R} = \hat{P} = 34^\circ$ (angles opposite equal sides)
$\therefore \hat{Q} = 180^\circ - R - P$
$= 180^\circ - 34^\circ - 34^\circ$ (angles in a triangle are supplementary)
$= 112^\circ$

$$\begin{aligned}\text{Area } \Delta PQR &= \frac{1}{2}pr\sin Q \\ &= \frac{1}{2} \times 9 \times 9 \times \sin 112^\circ \\ &= 37.55 \text{ m}^2\end{aligned}$$

4. $PQRS$ is a parallelogram (given)
   $\therefore QR = PS = 22$ (opposite sides of parallelogram)



$$\begin{aligned}\text{Area } \Delta QRS &= \frac{1}{2}qs\sin R \\ &= \frac{1}{2} \times 15 \times 22 \times \sin 135^\circ \\ &= 116.67 \text{ u}^2\end{aligned}$$

Area $PQRS = 2 \times 116.67 = 233.35$ u^2 (rounding off in the final step)

5. Area $\Delta XYZ = 26.72$ m^2

$$\begin{aligned}\therefore 26.72 &= \frac{1}{2}yz\sin X \\ &= \frac{1}{2} \times 8 \times 8 \times \sin X \\ \therefore \sin X &= 0.835\end{aligned}$$

$\therefore \hat{X} = 56.6^\circ$ or $\hat{X} = 180^\circ - 56.62^\circ = 123.4^\circ$

Back to Unit 5: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- arearule © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example5.4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise5.1Q1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise5.1Q5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ5 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise5.1A4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 6: Apply the sine rule

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use the sine rule to find an unknown length of a triangle.
- Use the sine rule to find an unknown angle of a triangle.

What you should know

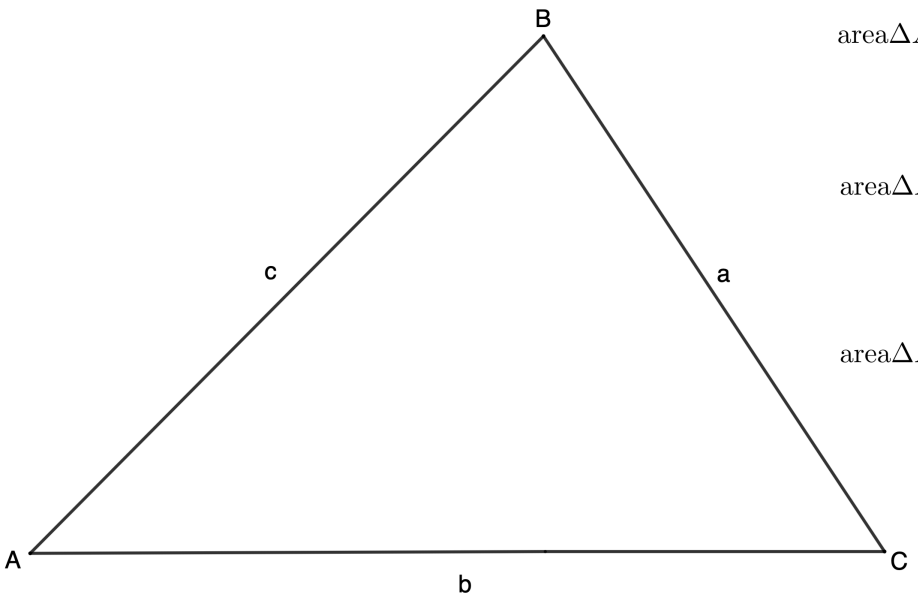
Before you start this unit, make sure you can:

- State and use the area rule. Refer to unit 5 of this subject outcome if you need help with this.
- Use a calculator to calculate the sine of a given angle. Refer to level 2 subject outcome 3.6 unit 2 if you need help with this.
- Use a calculator to calculate the angle from a given ratio for sine. Refer to level 2 subject outcome 3.6 unit 2 if you need help with this.
- Solve simple linear equations.

Introduction

In the previous unit, we derived and used the area rule – a useful trig identity that lets us calculate the area of any triangle if we know the length of any two sides and the size of the included angle.

In ΔABC:



$$\text{area}\Delta ABC = \frac{1}{2}bc\sin A$$

$$\text{area}\Delta ABC = \frac{1}{2}ac\sin B$$

$$\text{area}\Delta ABC = \frac{1}{2}ab\sin C$$

Figure 1: The area ruleEach of these statements for the area of ΔABC must be **equal** – they all give the area of the very same triangle. Therefore, we can write that:

$$\text{Area } \Delta ABC = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

$$\therefore bc\sin A = ac\sin B = ab\sin C$$

Now, if we divide through by abc we get the following:

$$\frac{\cancel{b}\,\cancel{c}\sin A}{a\,\cancel{b}\,\cancel{c}} = \frac{\cancel{a}\,\cancel{c}\sin B}{\cancel{a}\,b\,\cancel{c}} = \frac{\cancel{a}\,\cancel{b}\sin C}{\cancel{a}\,\cancel{b}\,c}$$

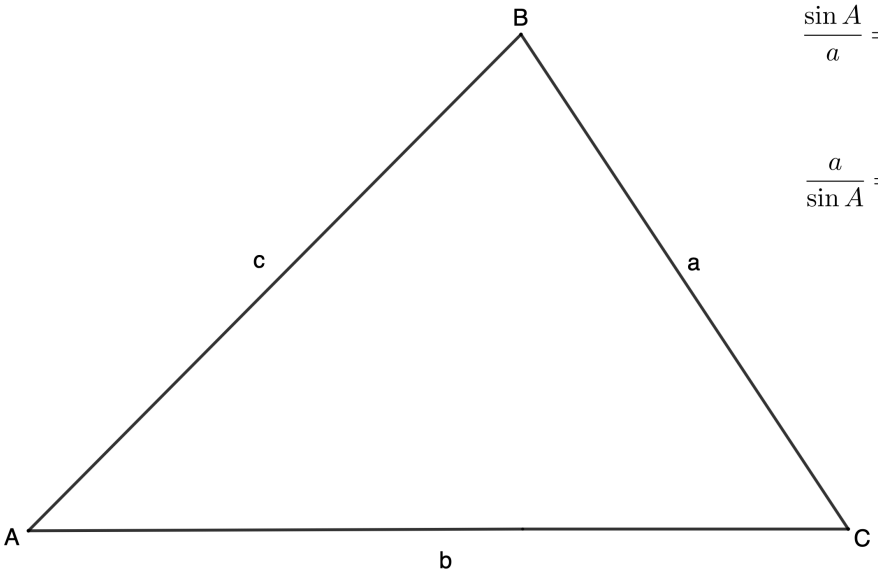
$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This trig identity is called the **sine rule**. It is very useful for finding missing sides and angles of triangles, as we shall see.

The sine rule

Let's now look at the sine rule in more detail.

The sine rule states that in any ΔABC:



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Take note!

You must make sure that you get the naming of vertices and sides in your triangle correct. The side opposite vertex A is a. The side opposite vertex B is b. The side opposite vertex C is c. The sine rule will only work if you label your triangle like this.

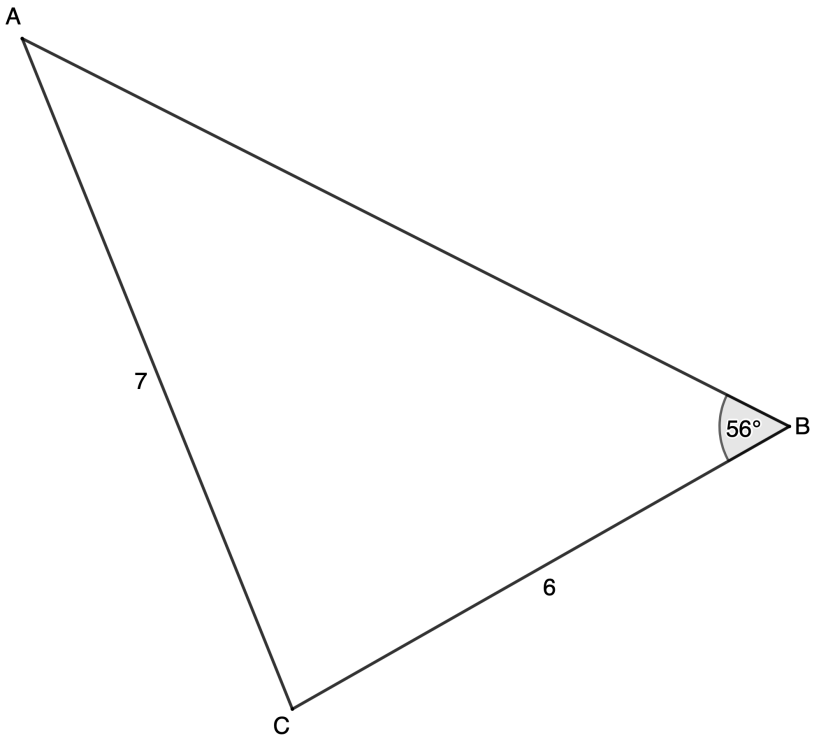
Take note!

The sine rule can also be expressed in reciprocal form.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 6.1

Find all the missing sides and angles.



Solution

We have been given $\hat{B} = 56^\circ$, $b = 7$ and $a = 6$. We need to find the length of c and the size of angles $\hat{A}$ and $\hat{C}$. To do this, we can use the sine rule.

To use the sine rule we need to know three of the four unknowns. Therefore, we can use the information we have been given to find $\hat{A}$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\therefore \frac{\sin A}{6} = \frac{\sin 56^\circ}{7}$$
$$\therefore \sin A = \frac{6 \times \sin 56^\circ}{7}$$
$$\therefore \hat{A} = 45.28^\circ$$

Now that we have $\hat{A}$, we can find $\hat{C}$.

$$\hat{C} = 180^\circ - \hat{A} - \hat{B} \qquad (\text{angles in a triangle are supplementary})$$
$$= 180^\circ - 45.28^\circ - 56^\circ$$
$$= 78.72^\circ$$

Now we can use either $\hat{A}/a$ or $\hat{B}/b$ to find the length of c. Because, we are finding a side, it is easier to

use the reciprocal of the sine rule.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{7}{\sin 56^\circ} = \frac{c}{\sin 78.72^\circ}$$

$$\therefore c = \frac{7 \times \sin 78.72^\circ}{\sin 56^\circ}$$

$$\therefore c = 8.28$$

The ambiguous case

In ΔABC, $\hat{B} = 32^\circ$, $b = 6$ and $c = 9$. Make a quick sketch of this triangle using the given information. Don't read on until you have made your sketch. It does not need to be very accurate. Just a rough sketch will do.

Now look at Figures 2 and 3. Which one of these two triangles did yours look like? Which one is correct?

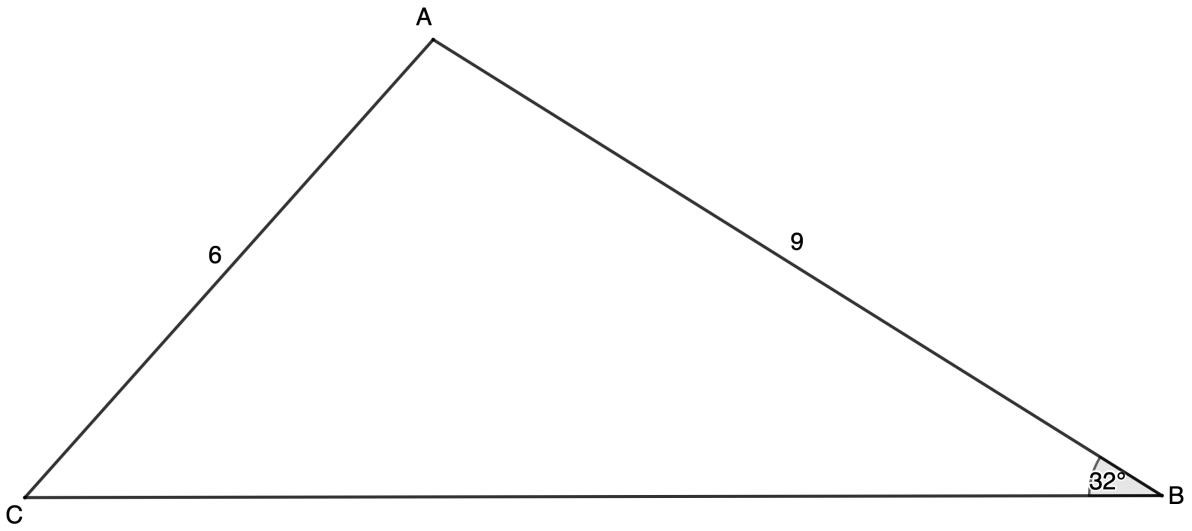


Figure 2: Case 1

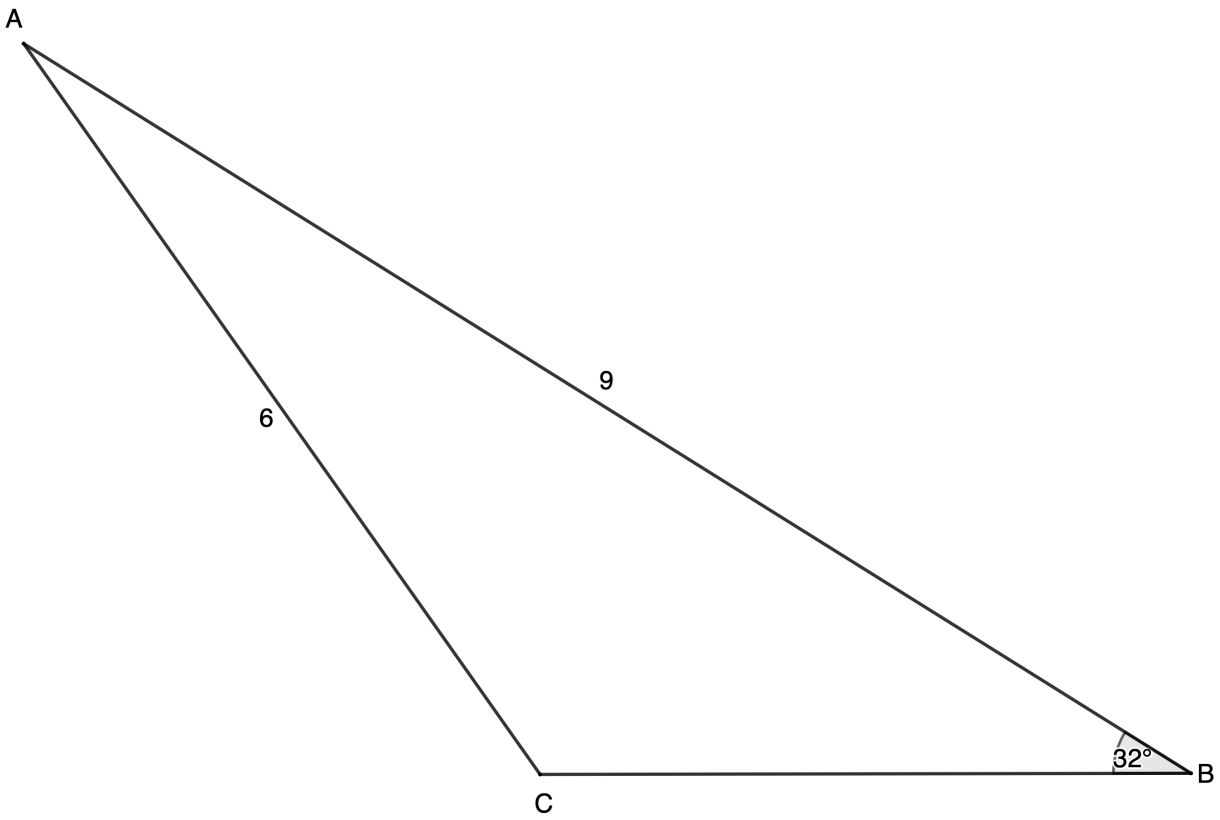


Figure 3: Case 2

Can you see that both of them are correct interpretations of the information that was given? The differences are that the sizes of $\hat{A}$ and $\hat{C}$ and the length of a in each case are different. This is an example of the **ambiguous case** and is a result of the fact that $\sin\theta = \sin(180^\circ - \theta)$. In Figure 3 $90^\circ \leq \hat{C} \leq 180^\circ$ and in Figure 2 $0^\circ \leq \hat{C} \leq 90^\circ$.

The ambiguous case happens when you are given two sides and an angle but the side **opposite the given angle is the shorter of the two sides**.

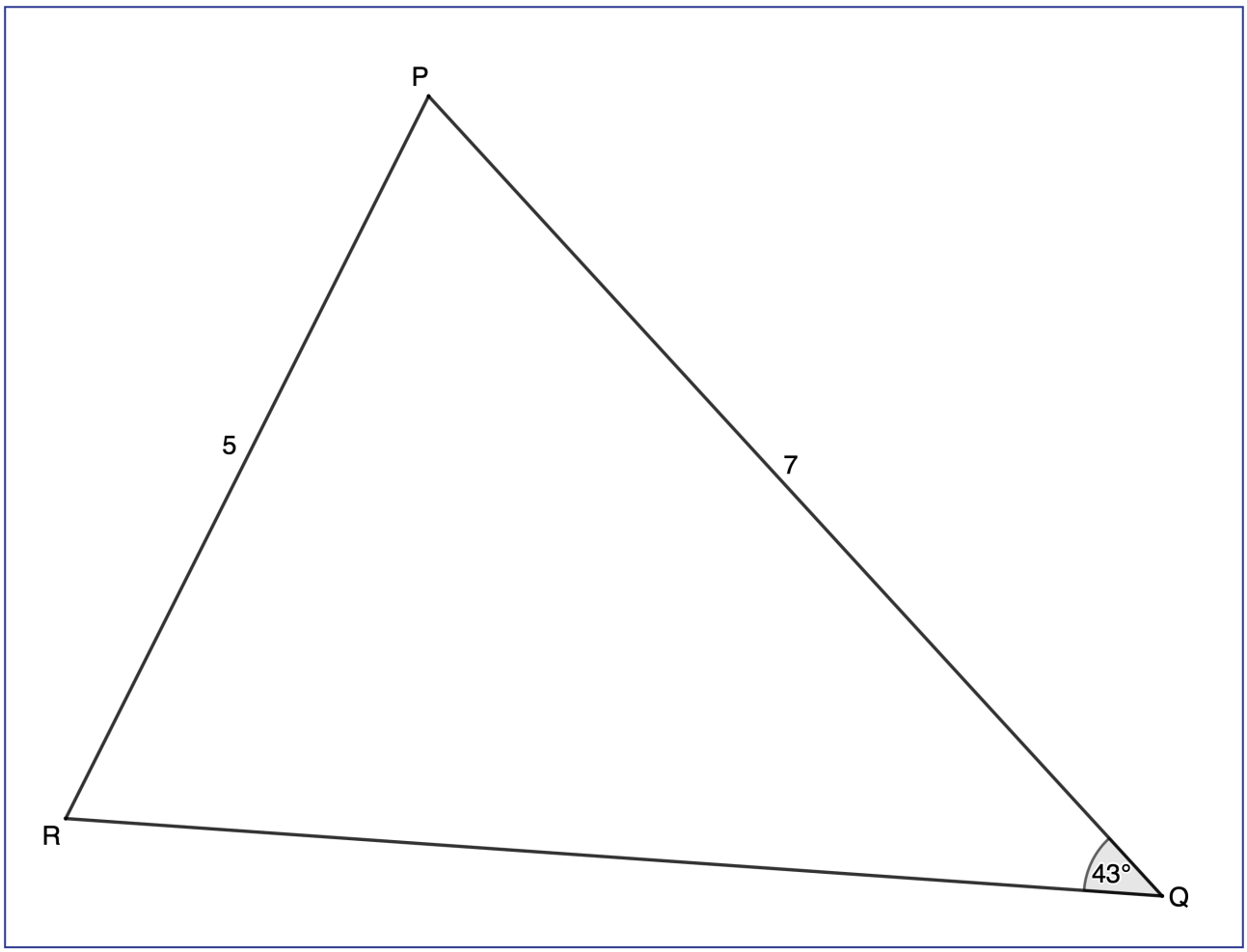


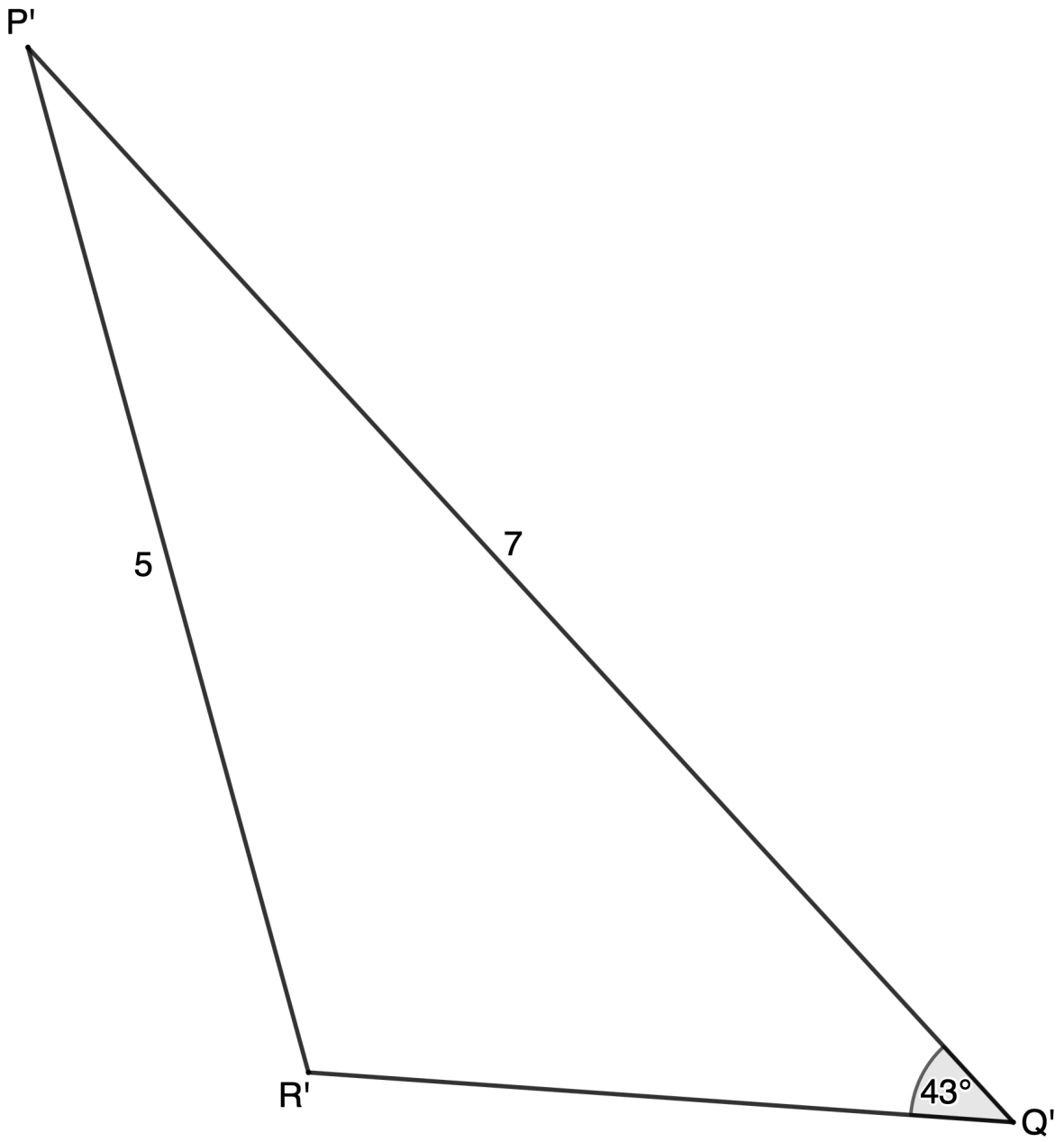
Example 6.2

In ΔPQR, $\hat{Q} = 43^\circ$, $r = 7$ and $q = 5$. Find the sizes of $\hat{R}$ and $\hat{P}$ and the length of p.

Solution

This is an example of the ambiguous case although you might not be able to recognise it immediately. It always helps to make a quick sketch of the given information. If you do, you will notice that there are two possible triangles based on the information given.


P
5
7
R
43°
Q



Remember, if the side opposite the given angle is the shorter side, then you have the ambiguous case. Here we can see that it is $\hat{R}$ that is either acute or obtuse.

$$\frac{\sin R}{r} = \frac{\sin Q}{q}$$
$$\therefore \frac{\sin R}{7} = \frac{\sin 43^\circ}{5}$$
$$\therefore \sin R = \frac{7 \times \sin 43^\circ}{5} \text{ equation } (1)$$

There are two solutions to this equation: since sine is positive in the first and second quadrants, so R can be acute or obtuse.

Firstly, in $\Delta PQR\hat{R}$ is acute:

$$\hat{R} = 72.71^\circ \qquad (\text{angles in a triangle are supplementary})$$
$$\therefore P = 180^\circ - 72.71^\circ - 43^\circ$$
$$= 64.29^\circ$$

$$\frac{p}{\sin P} = \frac{q}{\sin Q}$$
$$\therefore \frac{p}{\sin 64.29^\circ} = \frac{5}{\sin 43^\circ}$$
$$\therefore p = \frac{5 \times \sin 64.29^\circ}{\sin 43^\circ}$$
$$\therefore p = 6.61$$

Now we can start working in the other possible triangle $\Delta P'Q'R'$ (where $\hat{R}$ is obtuse).
In $\Delta P'Q'R'$, and from equation (1) above:

$$\hat{R}' = 180^\circ - 72.71^\circ$$
$$= 107.29^\circ$$
$$\therefore P' = 180^\circ - 107.29^\circ - 43^\circ \qquad \text{(angles in a triangle are supplementary)}$$
$$= 29.71^\circ$$
$$\frac{p'}{\sin P'} = \frac{q'}{\sin Q'}$$
$$\therefore \frac{p'}{\sin 29.71^\circ} = \frac{5}{\sin 43^\circ}$$
$$\therefore p' = \frac{5 \times \sin 29.71^\circ}{\sin 43^\circ}$$
$$\therefore p' = 3.63$$

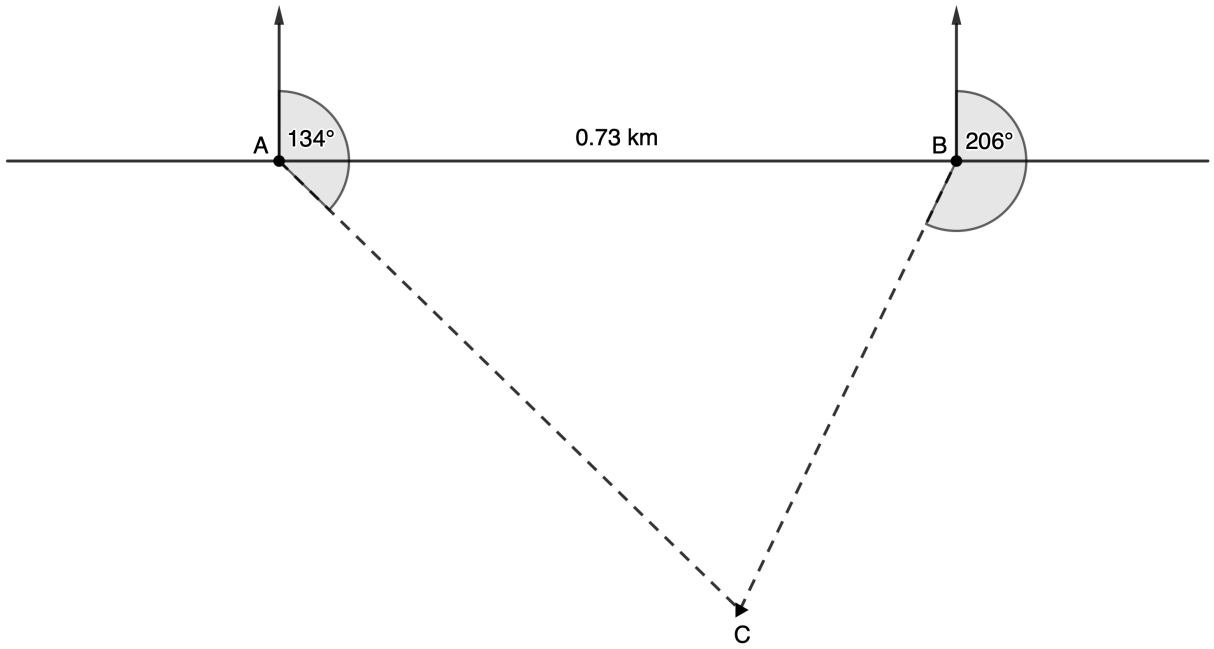
Take note!

The ambiguous case occurs when you are given two sides and an angle but the side **opposite the given angle is the shorter of the two sides**.

Example 6.3

This question is adapted from Everything Maths Grade 11 Worked example 27

There is a coastline with two lighthouses, A and B, that are 0.73 km apart. B is exactly due east of A. The lighthouses can tell how close a boat (C) is to the shore by taking bearings on the boat. These bearings are shown on the diagram below. How far is the boat from each lighthouse?



Solution

We need to calculate the distances AC and BC. First, we need to calculate the size of $\hat{C}$.
$B\hat{A}C = 134^\circ - 90^\circ = 44^\circ$ (angles on a straight line)
$A\hat{B}C = 360^\circ - 90^\circ - 206^\circ = 64^\circ$ (angles around a point)
$\therefore A\hat{C}B = 180^\circ - 64^\circ - 44^\circ = 72^\circ$ (angles in a triangle are supplementary)

Side AC is opposite $A\hat{B}C$, therefore we will refer to it as b.
Side BC is opposite $B\hat{A}C$, therefore we will refer to it as a.
Side AB is opposite $A\hat{C}B$, therefore we will refer to it as c.

$$\frac{a}{\sin B\hat{A}C} = \frac{c}{\sin A\hat{C}B}$$
$$\therefore \frac{a}{\sin 44^\circ} = \frac{0.73}{\sin 72^\circ}$$
$$\therefore a = \frac{0.73 \sin 44^\circ}{\sin 72^\circ}$$
$$= 0.53$$

$$\frac{b}{\sin A\hat{B}C} = \frac{c}{\sin A\hat{C}B}$$
$$\therefore \frac{b}{\sin 64^\circ} = \frac{0.73}{\sin 72^\circ}$$
$$\therefore b = \frac{0.73 \sin 64^\circ}{\sin 72^\circ}$$
$$= 0.69$$

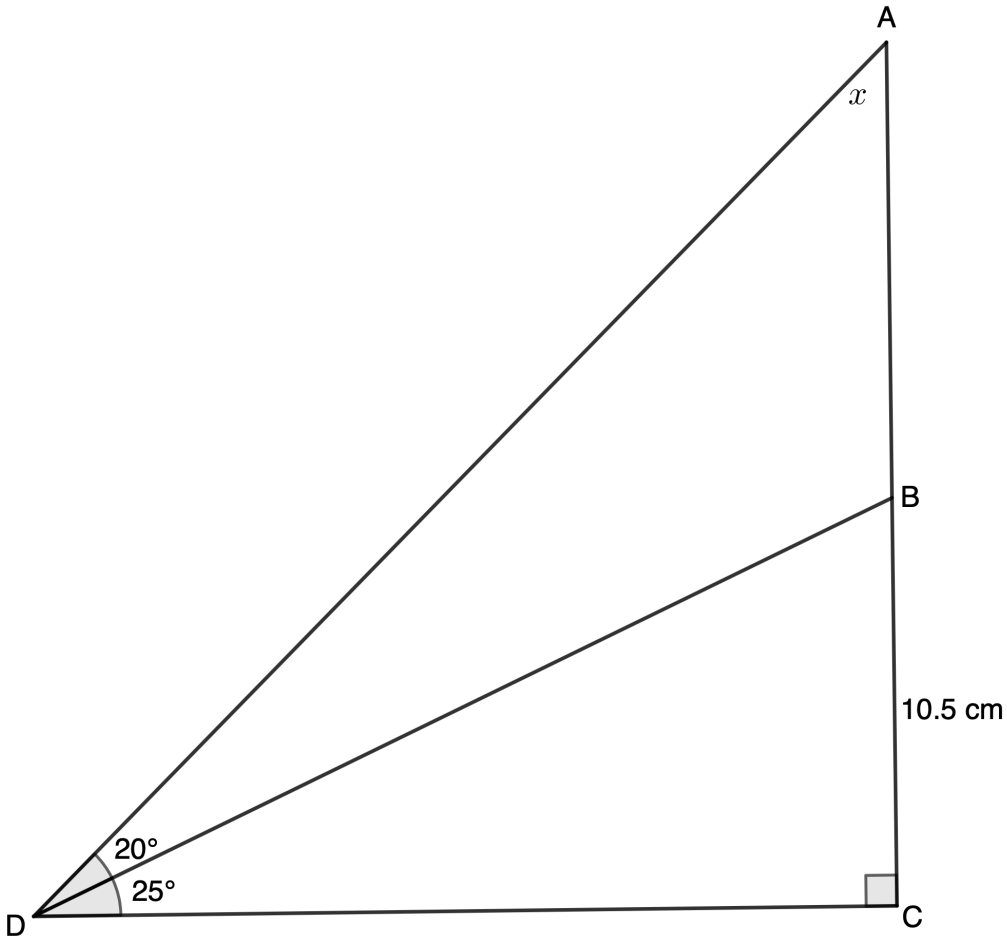
The boat is $0.53 \text{ km} = 530 \text{ m}$ away from lighthouse B and $0.69 \text{ km} = 690 \text{ m}$ away from lighthouse A.

Exercise 6.1

1. Find all the unknown sides and angles in the following triangles. Round all answers to two decimal places:
   a. ΔABC where $\hat{A} = 68^\circ$, $\hat{B} = 48^\circ$ and $b = 7$
   b. ΔPQR where $\hat{P} = 33^\circ$, $\hat{R} = 58^\circ$ and $p = 6$
   c. ΔSTU where $\hat{U} = 34^\circ$, $u = 9$ and $s = 15$
   d. ΔLMN where $\widehat{L} = 87^\circ$, $l = 13$ and $n = 10$

Question 2 is adapted from NC(V) Mathematics Level 3 Paper 2 November 2011 question 1.7

2. In the figure, $A\hat{C}D = 90^\circ$, $A\hat{D}B = 20^\circ$, $B\hat{D}C = 25^\circ$ and $BC = 10.5$ cm. Let $D\hat{A}B = x$.



   a. Calculate the value of x.
   b. Determine the length of DB.
   c. Determine the length of AB.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

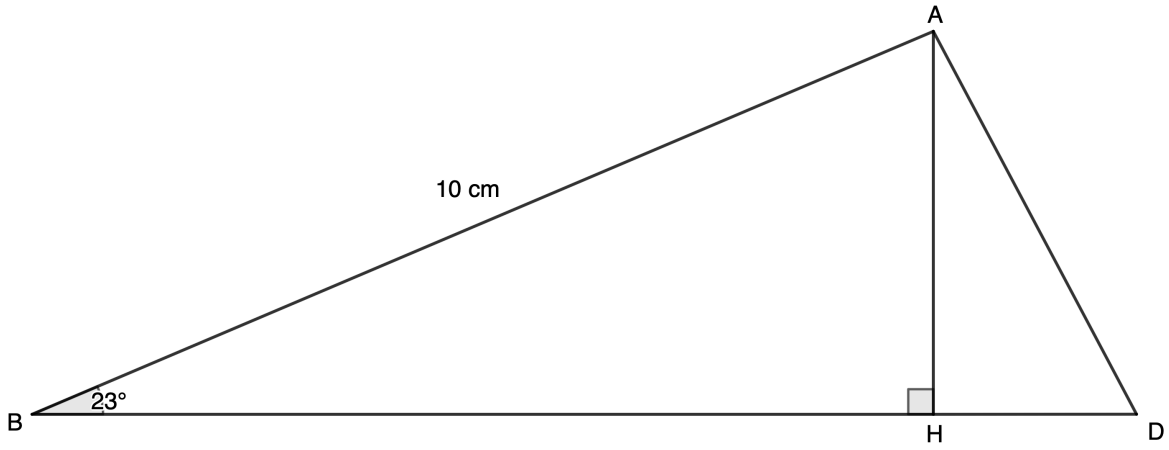
- How to find an unknown angle in a triangle using the sine rule.
- How to find an unknown side of a triangle using the sine rule.
- How to recognise and deal with the ambiguous case when solving triangles using the sine rule.

Unit 6: Assessment

Suggested time to complete: 45 minutes

Question 1 is adapted from NC(V) Mathematics Level 3 Paper 2 March 2014 question 2.6

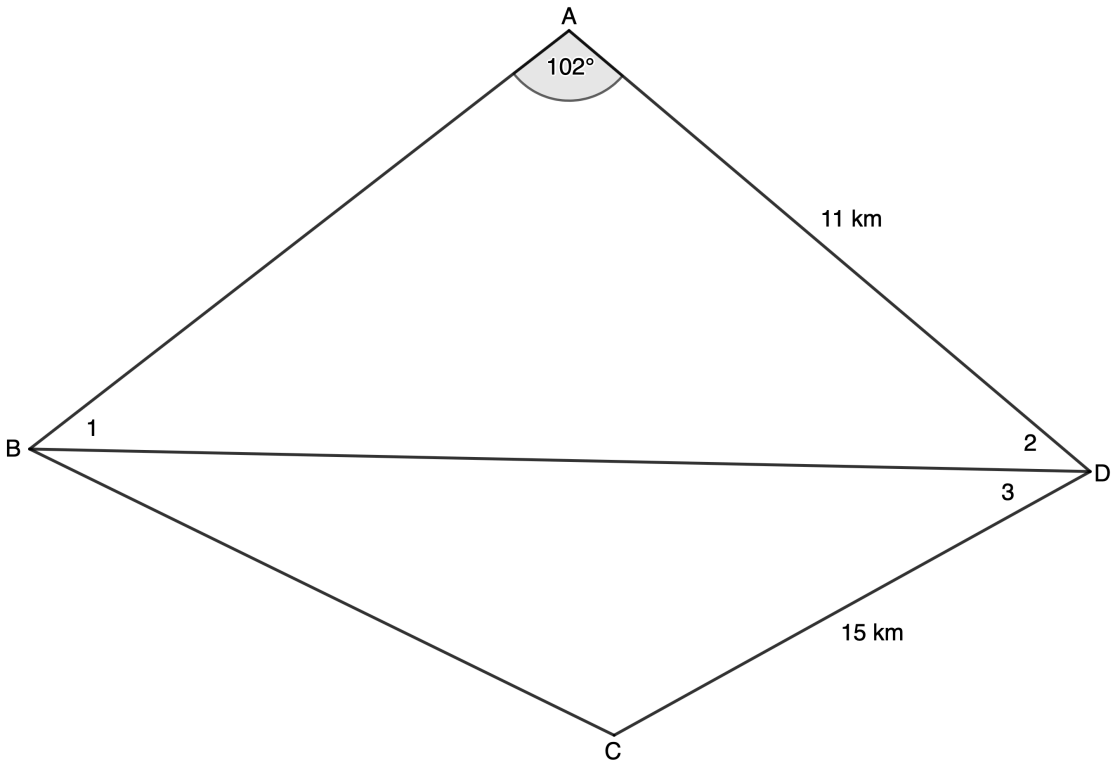
1. In ΔABD, $DA \perp AB$ and $AH \perp BD$. $AB = 10\ \text{cm}$ and $\hat{B} = 23^\circ$.



   a. Calculate the length of AH.
   b. Calculate the length of HD.

Question 2 is adapted from NC(V) Mathematics Level 3 Paper 2 March 2014 question 2.7

2. Below is a diagram of a farm in the shape of a quadrilateral. A fence from B to D divides the farm into two sections. $AD = 11\ \text{km}$ and $DC = 15\ \text{km}$. $D\hat{A}B = 102^\circ$, $\hat{D}_2 = 41^\circ$ and $A\hat{D}C = 75^\circ$



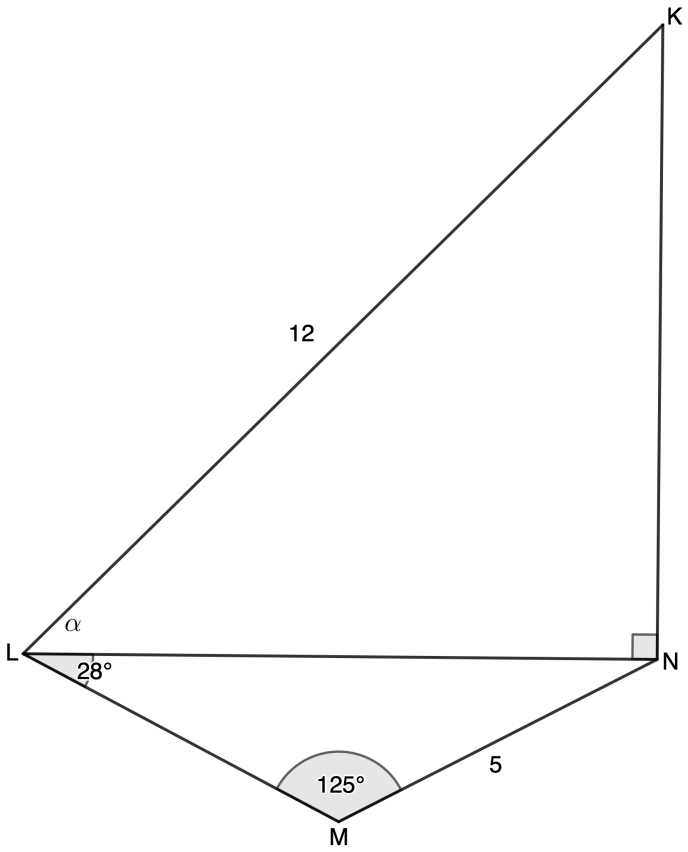
a. Calculate $\hat{B}_1$.

b. Calculate the length of the fence.

c. If $C = 130°$, calculate the length of BC.

Question 3 is adapted from NC(V) Mathematics Level 3 Paper 2 March 2011 question 1.6

3. In the figure below, $KLMN$ is a quadrilateral with $\hat{M} = 125°$, $N\hat{L}M = 28°$, $K\hat{N}L = 90°$, $MN = 5\ \text{cm}$ and $KL = 12\ \text{cm}$.



a. Prove that $LN = 8.7 \text{ cm}$ (to one decimal place).
b. Determine the size of α.
c. Calculate the area of ΔLMN.

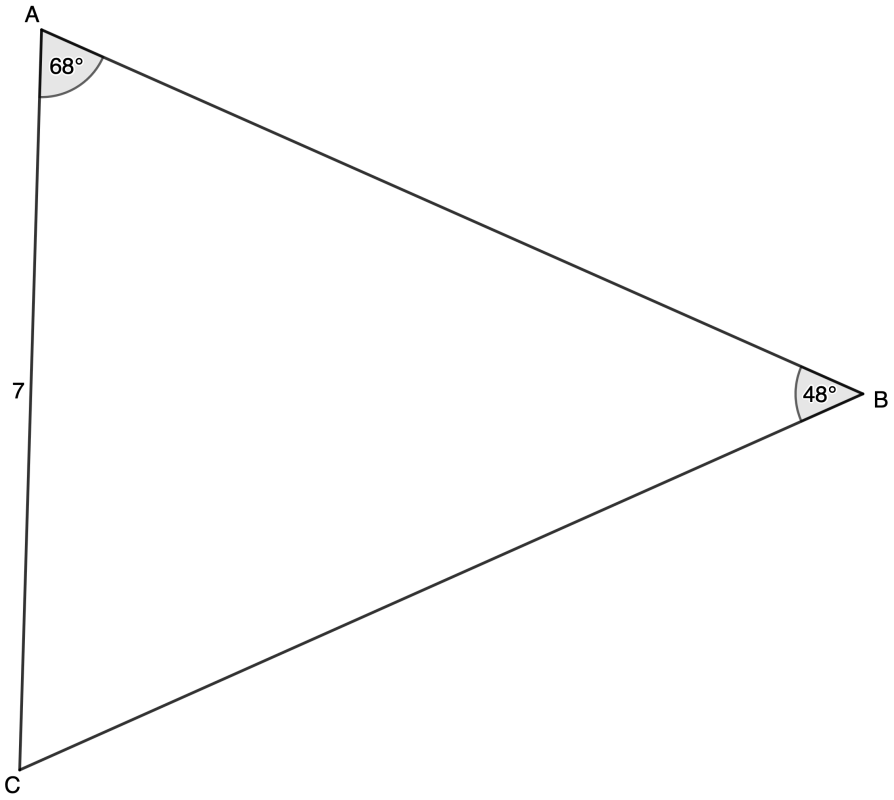
The full solutions are at the end of the unit.

Unit 6: Solutions

Exercise 6.1

1.

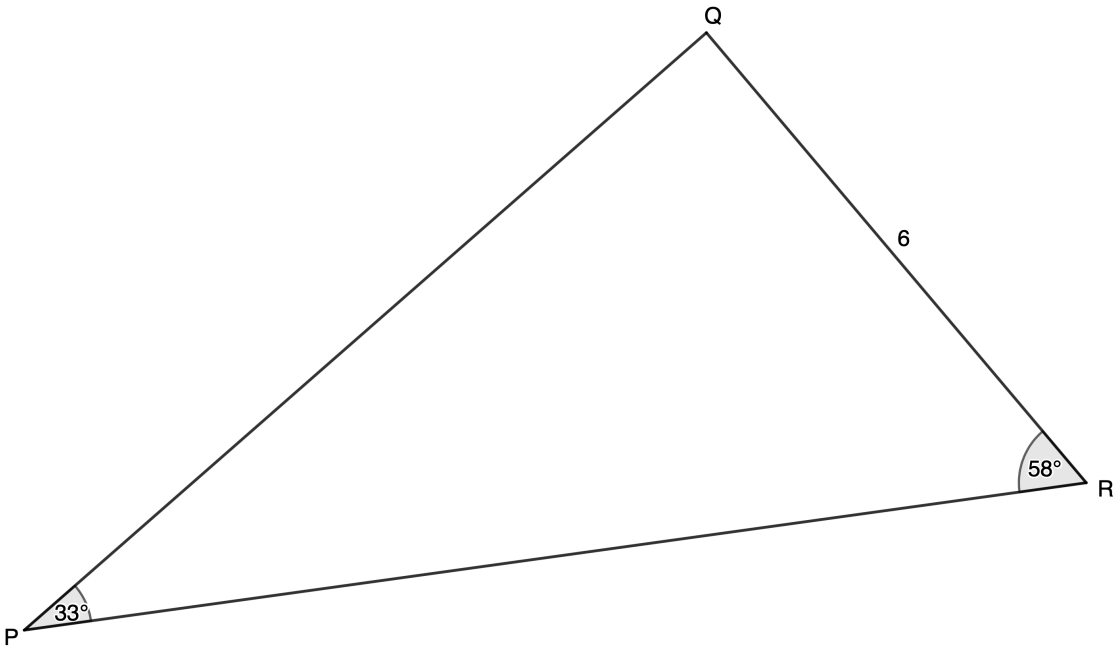
a. First draw a sketch of the given information. Because we are given two angles and a side, we do not need to worry about the ambiguous case.



$\hat{C} = 180^\circ - 68^\circ - 48^\circ = 64^\circ$ (angles in a triangle are supplementary)

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\therefore \frac{a}{\sin 68^\circ} = \frac{7}{\sin 48^\circ}$$
$$\therefore a = \frac{7 \sin 68^\circ}{\sin 48^\circ}$$
$$= 8.73$$
$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\therefore \frac{c}{\sin 64^\circ} = \frac{7}{\sin 48^\circ}$$
$$\therefore c = \frac{7 \sin 64^\circ}{\sin 48^\circ}$$
$$= 8.47$$

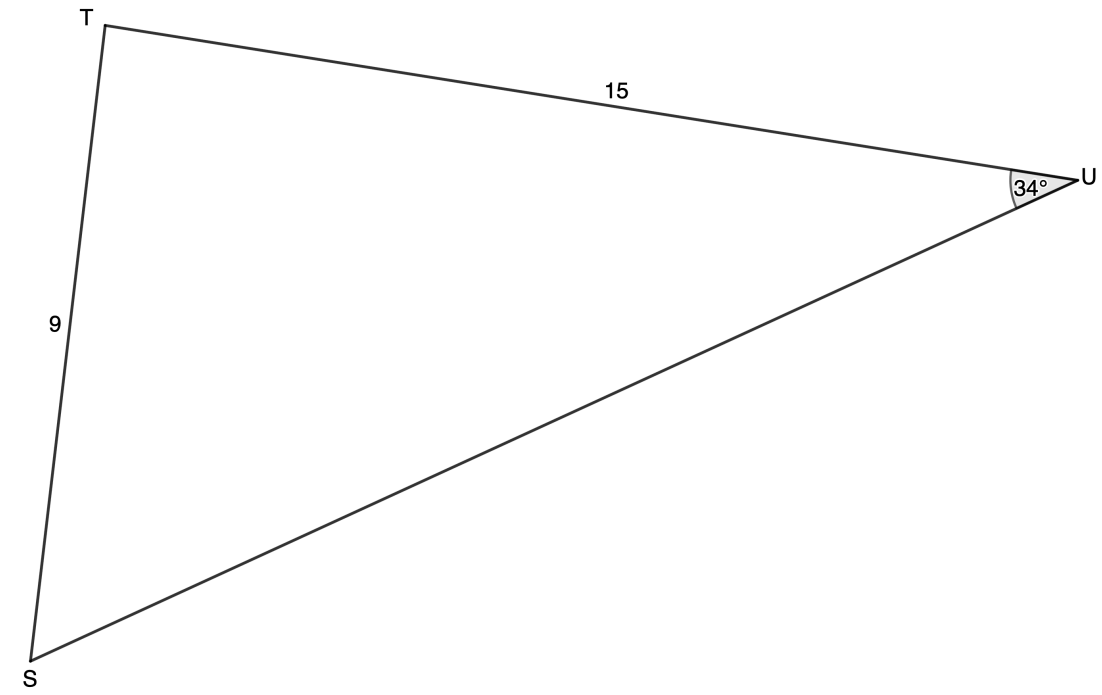
b. First draw a sketch of the given information. Because we are given two angles and a side, we do not need to worry about the ambiguous case.

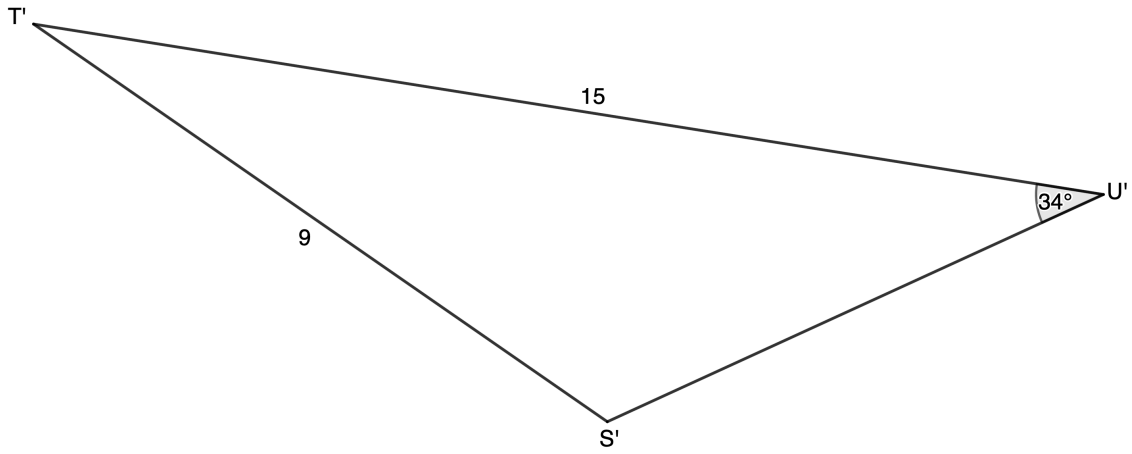


$\hat{Q} = 180^\circ - 58^\circ - 33^\circ = 89^\circ$ (angles in a triangle are supplementary)

$$\frac{q}{\sin Q} = \frac{p}{\sin P}$$
$$\therefore \frac{q}{\sin 89^\circ} = \frac{6}{\sin 33^\circ}$$
$$\therefore q = \frac{6 \sin 89^\circ}{\sin 33^\circ}$$
$$= 11.01$$
$$\frac{r}{\sin R} = \frac{p}{\sin P}$$
$$\therefore \frac{r}{\sin 58^\circ} = \frac{6}{\sin 33^\circ}$$
$$\therefore r = \frac{6 \sin 58^\circ}{\sin 33^\circ}$$
$$= 9.34$$

c. First draw a sketch of the given information. Because we are given two sides and an angle, we need to check for the ambiguous case. Because the side opposite the given angle is the shorter, we have the ambiguous case. Sketch both cases.





In ΔSTU:

$$\frac{\sin S}{s} = \frac{\sin U}{u}$$

$$\therefore \frac{\sin S}{15} = \frac{\sin 34^\circ}{9}$$

$$\therefore \sin S = \frac{15 \sin 34^\circ}{9}$$

$$\therefore \hat{S} = 68.75^\circ$$

$\hat{T} = 180^\circ - 68.75^\circ - 34^\circ = 77.25^\circ$ (angles in a triangle are supplementary)

$$\frac{t}{\sin T} = \frac{u}{\sin U}$$

$$\therefore \frac{t}{\sin 77.25^\circ} = \frac{9}{\sin 34^\circ}$$

$$\therefore t = \frac{9 \sin 77.25^\circ}{\sin 34^\circ}$$

$$= 15.70$$

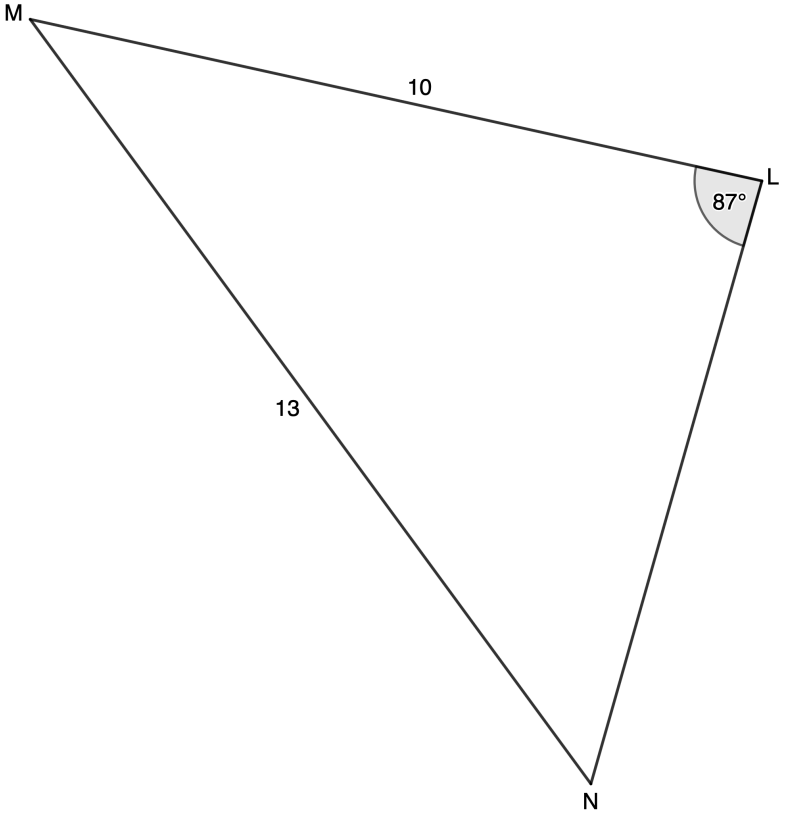
In $\Delta S'T'U'$:

$\hat{S} = 180^\circ - 68.75^\circ = 111.25^\circ$

$\hat{T} = 180^\circ - 111.25^\circ - 34^\circ = 34.75^\circ$ (angles in a triangle are supplementary)

$$\frac{t}{\sin T} = \frac{u}{\sin U}$$
$$\therefore \frac{t}{\sin 34.75^\circ} = \frac{9}{\sin 34^\circ}$$
$$\therefore t = \frac{9 \sin 34.75^\circ}{\sin 34^\circ}$$
$$= 9.17$$

d. First draw a sketch of the given information. Because we are given two sides and an angle, we need to check for the ambiguous case. Because the side opposite the given angle is the longer, we do not have the ambiguous case.



$$\frac{\sin N}{n} = \frac{\sin L}{l}$$
$$\therefore \frac{\sin N}{10} = \frac{\sin 87^\circ}{13}$$
$$\therefore \sin N = \frac{10 \sin 87^\circ}{13}$$
$$\therefore \hat{N} = 50.19^\circ$$
$\hat{M} = 180^\circ - 87^\circ - 50.19^\circ = 42.81^\circ$ (angles in a triangle are supplementary)
$$\frac{m}{\sin M} = \frac{l}{\sin L}$$
$$\therefore \frac{m}{\sin 42.81^\circ} = \frac{13}{\sin 87^\circ}$$
$$\therefore m = \frac{13 \sin 42.81^\circ}{\sin 87^\circ}$$
$$= 8.85$$

2.

a. In ΔACD:
$x = 180^\circ - 90^\circ - (20^\circ + 25^\circ) = 45^\circ$ (angles in a triangle are supplementary)

b. In ΔBCD:

$$\frac{DB}{\sin C} = \frac{BC}{\sin B\hat{D}C}$$
$$\therefore \frac{DB}{\sin 90^\circ} = \frac{10.5}{\sin 25^\circ}$$
$$\therefore DB = \frac{10.5 \sin 90^\circ}{\sin 25^\circ}$$
$$= 24.85 \text{ cm}$$

c. In ΔABD:
$$\frac{AB}{\sin A\hat{D}B} = \frac{DB}{\sin x}$$
$$\therefore \frac{AB}{\sin 20^\circ} = \frac{24.85}{\sin 45^\circ}$$
$$\therefore AB = \frac{24.85 \sin 20^\circ}{\sin 45^\circ}$$
$$= 12.02 \text{ cm}$$

Back to Exercise 6.1

Unit 6: Assessment

1.

a.
$$\frac{AH}{\sin B} = \frac{AB}{\sin A\hat{H}B}$$
$$\therefore \frac{AH}{\sin 23^\circ} = \frac{10}{\sin 90^\circ}$$
$$\therefore AH = \frac{10 \sin 23^\circ}{\sin 90^\circ}$$
$$= 3.91 \text{ cm}$$

b. $B\hat{A}D = 90^\circ$ (given)
$\hat{D} = 180^\circ - \hat{B} - B\hat{A}D = 180^\circ - 23^\circ - 90^\circ = 67^\circ$ (angles in a triangle are supplementary)
$D\hat{A}H = 180^\circ - A\hat{H}D - \hat{D} = 180^\circ - 90^\circ - 67^\circ = 23^\circ$ (angles in a triangle are supplementary)
$$\frac{HD}{\sin D\hat{A}H} = \frac{AH}{\sin \hat{D}}$$
$$\therefore \frac{HD}{\sin 23^\circ} = \frac{3.91}{\sin 67^\circ}$$
$$\therefore HD = \frac{3.91 \sin 23^\circ}{\sin 67^\circ}$$
$$= 1.66 \text{ cm}$$

2.

a. $\hat{B}_1 = 180^\circ - \hat{A} - \hat{D}_2 = 180^\circ - 102^\circ - 41^\circ = 37^\circ$ (angles in a triangle are supplementary)

b.
$$\frac{BD}{\sin A} = \frac{AD}{\sin B_1}$$
$$\therefore \frac{BD}{\sin 102^\circ} = \frac{11}{\sin 37^\circ}$$
$$\therefore BD = \frac{11 \sin 102^\circ}{\sin 37^\circ}$$
$$= 17.88 \text{ km}$$

c. $A\hat{D}C = 75^\circ = \hat{D}_2 + \hat{D}_3$ (given)
$\therefore D_3 = 75^\circ - 41^\circ = 34^\circ$
$\hat{C} = 130^\circ$ (given)

$$\frac{BC}{\sin \hat{D}_3} = \frac{BD}{\sin \hat{C}}$$
$$\therefore \frac{BC}{\sin 34^\circ} = \frac{17.88}{\sin 130^\circ}$$
$$\therefore BC = \frac{17.88 \sin 34^\circ}{\sin 130^\circ}$$
$$= 13.05 \text{ km}$$

3.

a.
$$\frac{LN}{\sin M} = \frac{MN}{\sin L}$$
$$\therefore \frac{LN}{\sin 125^\circ} = \frac{5}{\sin 28^\circ}$$
$$\therefore LN = \frac{5 \sin 125^\circ}{\sin 28^\circ}$$
$$= 8.7$$

b.
$$\frac{\sin K}{LN} = \frac{\sin K\hat{N}L}{KL}$$
$$\therefore \frac{\sin K}{8.7} = \frac{\sin 90^\circ}{12}$$
$$\therefore \sin K = \frac{8.7 \sin 90^\circ}{12}$$
$$\therefore K = 46.47^\circ$$
$\alpha = 180^\circ - 90^\circ - 46.47^\circ = 43.53^\circ$ (angles in triangle are supplementary)

c. $L\hat{N}M = 180^\circ - 28^\circ - 125^\circ = 27^\circ$ (angles in triangle are supplementary)
$$\text{Area } \Delta LMN = \frac{1}{2} \times LN \times MN \times \sin L\hat{N}M$$
$$= \frac{1}{2} \times 8.7 \times 5 \times \sin 27^\circ$$
$$= 9.87 \text{ u}^2$$

Back to Unit 6: Assessment

Media Attributions

- arearule © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- sinerule © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example6.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example6.2.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example6.2.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example6.3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise6.1Q2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise6.1A1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise6.1A1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise6.1A1c1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise6.1A1c2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise6.1A1d © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 7: Apply the cosine rule

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use the cosine rule to find the length of an unknown side of a triangle.
- Use the cosine rule to find an unknown angle of a triangle.

What you should know

Before you start this unit, make sure you can:

- State and use the area rule. Refer to unit 5 of this subject outcome if you need help with this.
- State and use the sine rule to find unknown sides and angles of triangles. Refer to unit 6 of this subject outcome if you need help with this.
- Use a calculator to calculate the sine or cosine of a given angle. Refer to level 2 subject outcome 3.6 unit 2 if you need help with this.
- Use a calculator to calculate the angle from a given ratio for sine or cosine. Refer to level 2 subject outcome 3.6 unit 2 if you need help with this.
- Solve simple equations.

Introduction

The sine rule that we learnt about in unit 6 is very powerful. With it, we can solve most triangles (find all the unknown sides and angles). However, there are some triangles that the sine rule cannot solve. Have a look at Figure 1. Can you use the sine rule to solve this triangle?

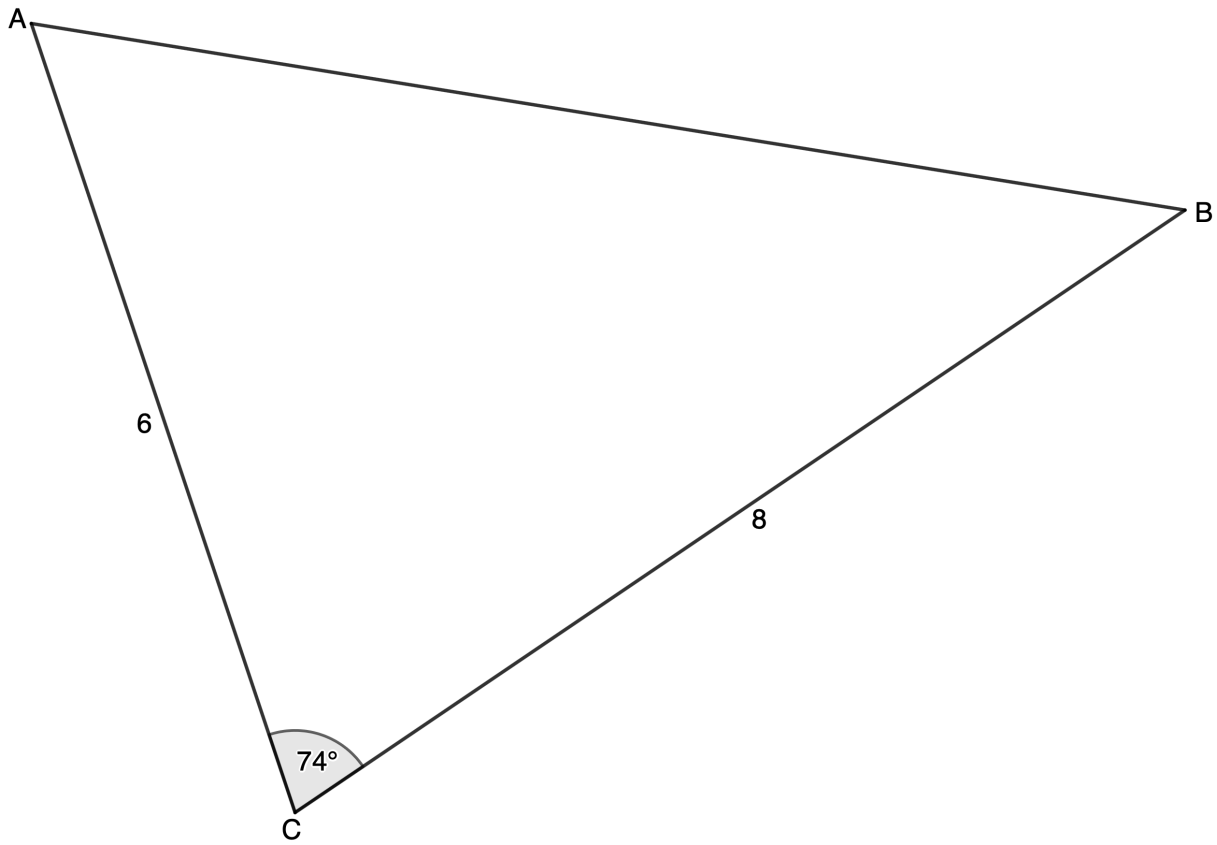


Figure 1: ΔABC

To use the sine rule we need an angle and opposite side pair. In other words, we would either need to know the length of AB (side c) or we would need to know the size of angle $\hat{A}$ or $\hat{B}$.

Thankfully, there is a way to solve triangles like this with another useful trig identity called the **cosine rule**.

The cosine rule

The cosine rule says that in any ΔABC:

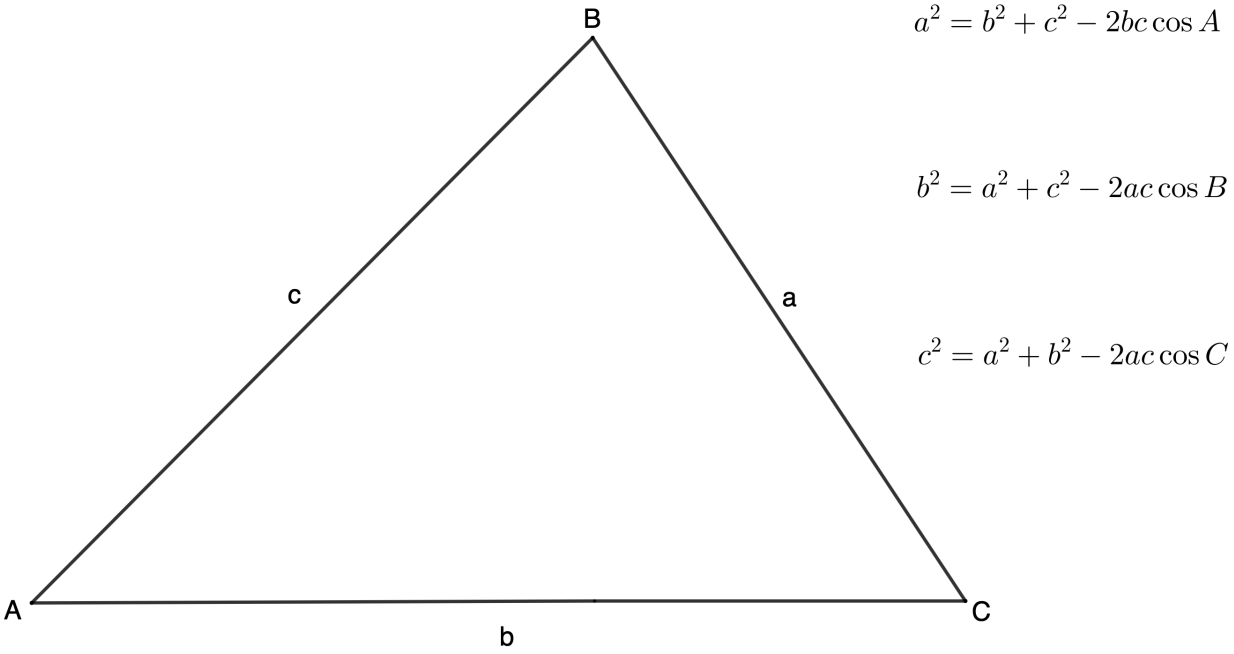


Figure 2: The cosine rule

Notice two important things about the cosine rule:

- The cosine rule works when we know the lengths of any two sides and the size of the **included** angle.
- The side on the LHS is always the side **opposite** the angle of which we take the cosine.

We can use the cosine rule to work out what c is in the triangle in Figure 3 because we know the lengths of two sides and the size of the **included** angle.

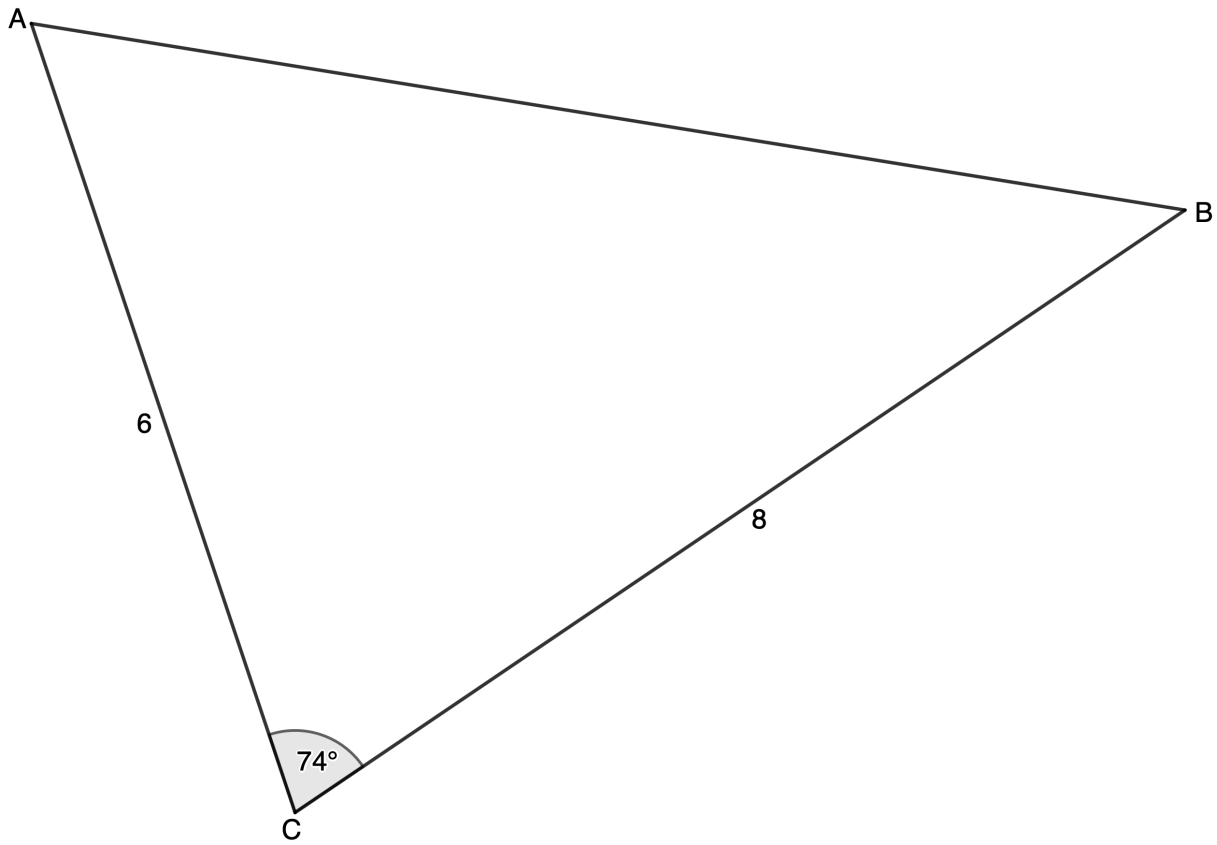


Figure 1: ΔABC

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
&= 8^2 + 6^2 - 2 \cdot 8 \cdot 6 \cdot \cos 74^\circ \\
\therefore c &= \sqrt{8^2 + 6^2 - 2 \cdot 8 \cdot 6 \cdot \cos 74^\circ} \\
&= 8.58
\end{aligned}
$$

Because we are dealing with a physical length, we do not need to consider the negative square root.

When calculating with the cosine rule, it is important to round off only your final answer. In the calculation above, this is the sequence of keys you need to press on your Casio calculator to compute c.

$\sqrt{}$ 8 x^2 + 6 x^2 − 2 × 8 × 6 × cos 74) =

Take note!

You must make sure that you get the naming of vertices and sides in your triangle correct. The side opposite vertex A is a. The side opposite vertex B is b. The side opposite vertex C is c. The cosine rule will only work if you label your triangle like this.

Deriving the cosine rule is a little harder than deriving either the area rule or sine rule. If you would like to know how to derive it, and you have an internet connection, watch the video “Derivation of the cosine rule”.

Derivation of the cosine rule (Duration: 4.12)

Because the sine rule is simpler to work with, it is advisable to use it if possible. Only if you cannot use the sine rule or you are explicitly told to use the cosine rule should you use the cosine rule.

Note

If you have an internet connection, watch the video called “Sine or Cosine Rule?” to see when you should use each rule.

Sine or Cosine Rule? (Duration: 2.51)

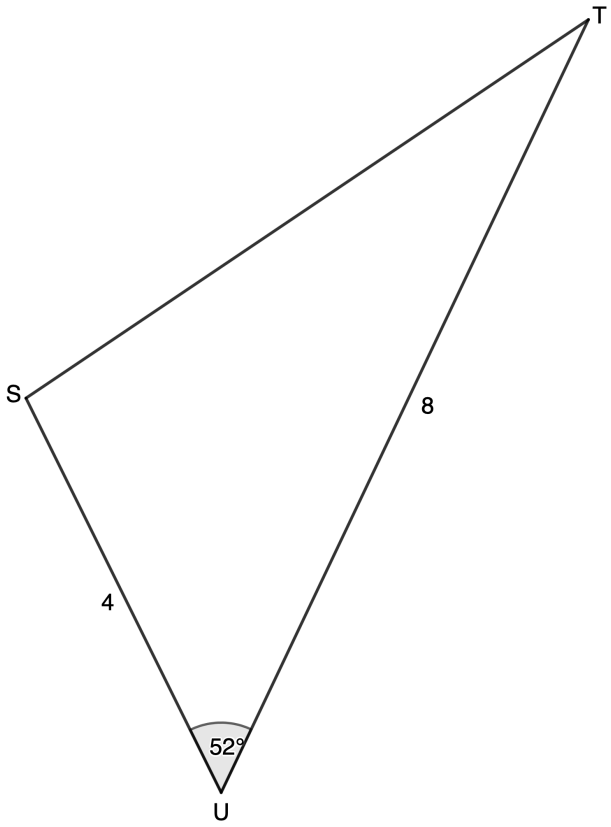
Use the cosine rule to find an unknown side

Let’s look at an example of when you would use the cosine rule.



Example 7.1

Calculate the length of ST.



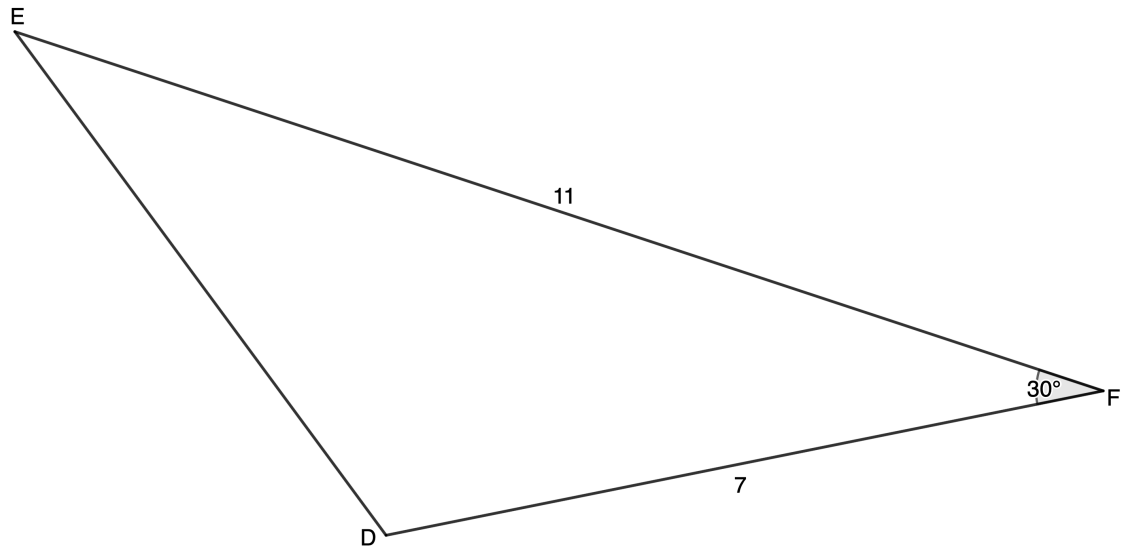
Solution

Check that we have two sides and the included angle. We know that $ST = u$ because it is opposite $\hat{U}$, and we know that $t = 4$ and $s = 8$.

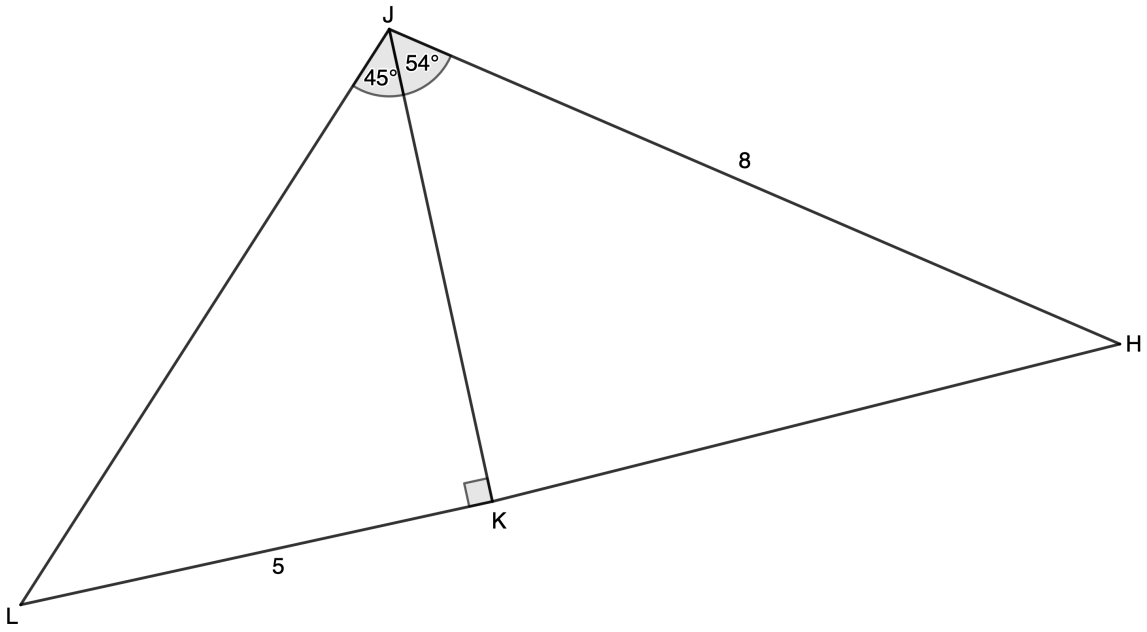
$$u^2 = s^2 + t^2 - 2st\cos U$$
$$\therefore u^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cdot \cos 52^\circ$$
$$\therefore u = \sqrt{8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cdot \cos 52^\circ}$$
$$= 6.37$$

Exercise 7.1

1. Find the length of DE in ΔDEF.



2. Find the length of HK.



The full solutions are at the end of the unit.

Use the cosine rule to find an unknown angle

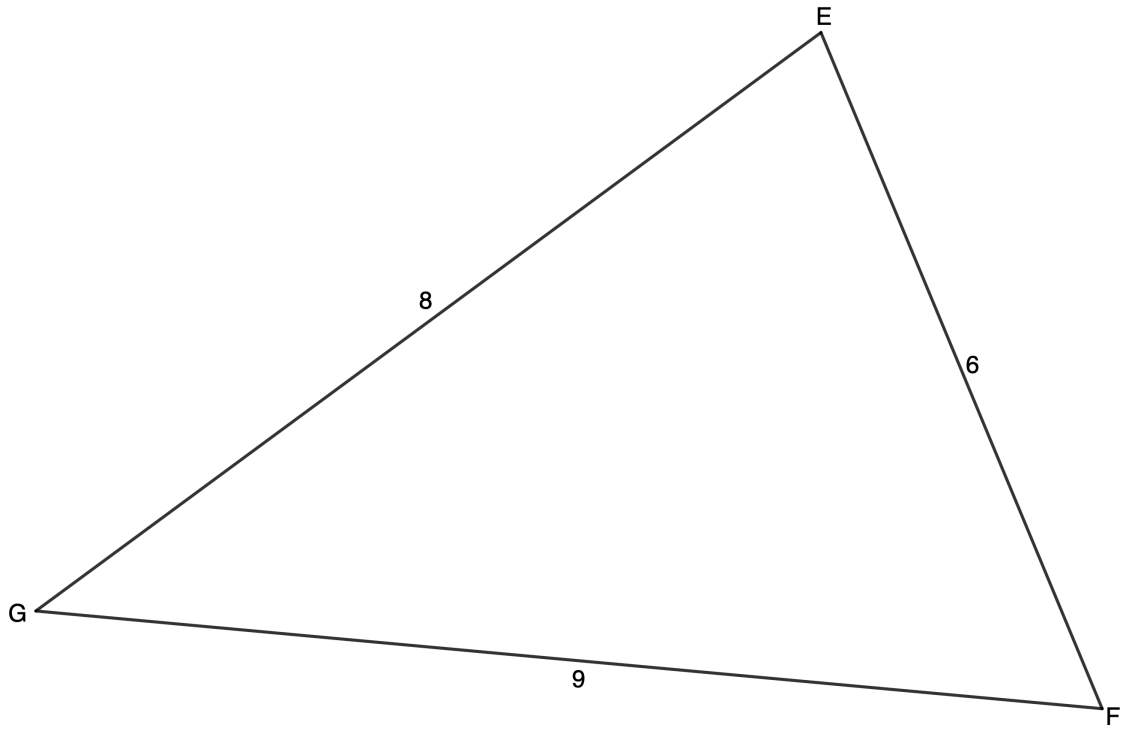
If we know the length of all three sides of a triangle, we can rearrange the cosine rule to solve for any of the unknown angles.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\therefore a^2 - b^2 - c^2 = -2bc\cos A$$
$$\therefore \frac{a^2 - b^2 - c^2}{-2bc} = \cos A$$
$$\therefore \cos A = \frac{-(-a^2 + b^2 + c^2)}{-2bc}$$
$$= \frac{b^2 + c^2 - a^2}{2bc}$$

Just remember the side a is opposite angle $\hat{A}$, etc.

Example 7.2

Solve the following triangle:



Solution

Because we have been given three sides, we know we need to use the cosine rule. We can solve for any of the unknown angles but must make sure that we arrange the three sides in the equation correctly.

We will solve for $\hat{E}$ first.

$$e^2 = f^2 + g^2 - 2fg\cos E$$
$$\therefore \cos E = \frac{f^2 + g^2 - e^2}{2fg}$$
$$= \frac{8^2 + 6^2 - 9^2}{2 \cdot 8 \cdot 6}$$
$$= 0.1979$$
$$\therefore \hat{E} = 78.58^\circ$$

Note: Make sure that you round off only your final answer.

Now that we have a side/angle pair, we can complete solving, using the sine rule.

$$\frac{\sin F}{f} = \frac{\sin E}{e}$$
$$\therefore \frac{\sin F}{8} = \frac{\sin 78.58^\circ}{9}$$
$$\therefore \sin F = \frac{8 \sin 78.58^\circ}{9}$$
$$\therefore F = 60.61^\circ$$
$$\frac{\sin G}{g} = \frac{\sin E}{e}$$
$$\therefore \frac{\sin G}{6} = \frac{\sin 78.58^\circ}{9}$$
$$\therefore \sin G = \frac{6 \sin 78.58^\circ}{9}$$
$$\therefore G = 40.80^\circ$$



Exercise 7.2

1. Solve the following triangles:
   a. ΔPQR where $\hat{P} = 58^\circ$, $q = 9$ cm and $r = 7$ cm.
   b. ΔLMN where $l = 8.1$, $m = 3.8$ and $n = 10.3$.
2. Determine the largest angle in ΔJHK where $HJ = 6$, $JK = 5$ and $KH = 9$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

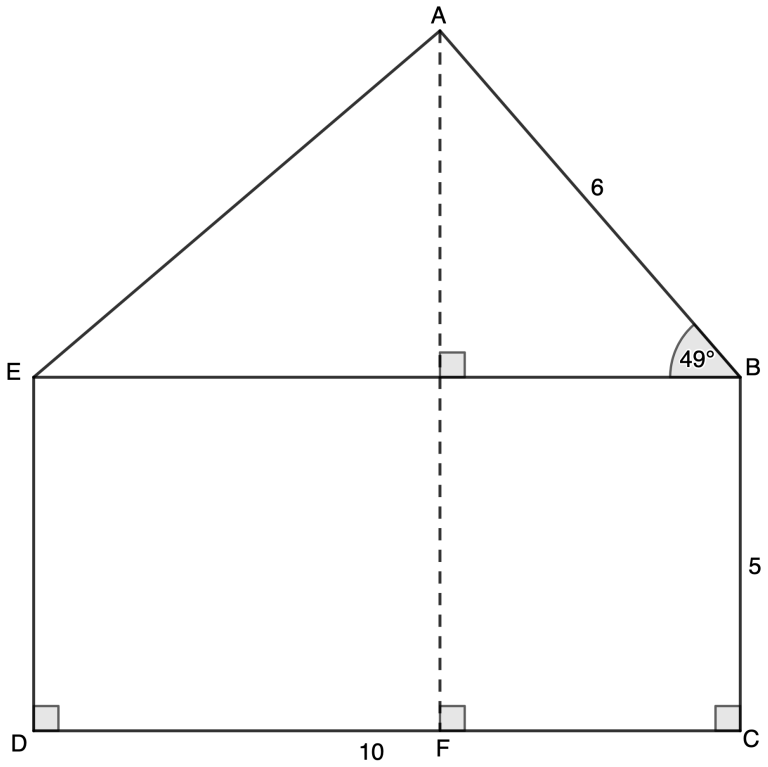
- How to use the cosine rule to find the length of a side of a triangle.
- How to use the cosine rule to find the size of an angle of a triangle.

Unit 7: Assessment

Suggested time to complete: 30 minutes

Question 1 is adapted from NC(V) Mathematics Paper 2 March 2013 question 2.4

1. The sketch below shows a side view of a house. $AB = 6$ m, $BC = 5$ m, $CD = 10$ m and $A\hat{B}E = 49^\circ$.



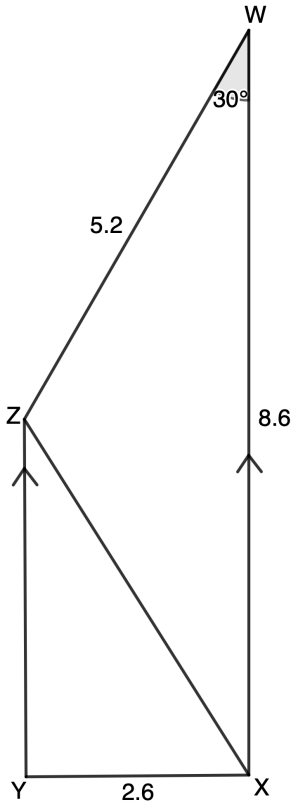
a. Determine the length of AE.

b. Calculate the magnitude of $A\hat{E}B$.

c. Calculate the area of ΔABE.

d. Determine the length of AF.

2. $WXYZ$ is a trapezium, $WX \parallel YZ$ with $WX = 8.6$ m, $XY = 2.6$ m, $ZW = 5.2$ m and $\hat{W} = 30°$.



a. Determine XZ.

b. If $W\hat{Z}X = 117.7°$, determine $\hat{Y}$.

c. Determine area of $WXYZ$.

The full solutions are at the end of the unit.

Unit 7: Solutions

Exercise 7.1

1. We have the length of two sides and the included angle. Therefore, we can use the cosine rule.
   $DE = f, FD = e = 7, EF = d = 11$
   $$f^2 = d^2 + e^2 - 2de\cos F$$
   $$\therefore f^2 = 7^2 + 11^2 - 2 \cdot 7 \cdot 11 \cdot \cos 30°$$
   $$\therefore f = \sqrt{7^2 + 11^2 - 2 \cdot 7 \cdot 11 \cdot \cos 30°}$$
   $$= 6.05$$
2. We do not yet have enough information to use the sine or cosine rules in ΔHJK.
   In ΔKJL:
   $$\tan 45° = \frac{LK}{KJ}$$
   $$\therefore KJ = \frac{LK}{\tan 45°} = \frac{5}{1} = 5$$

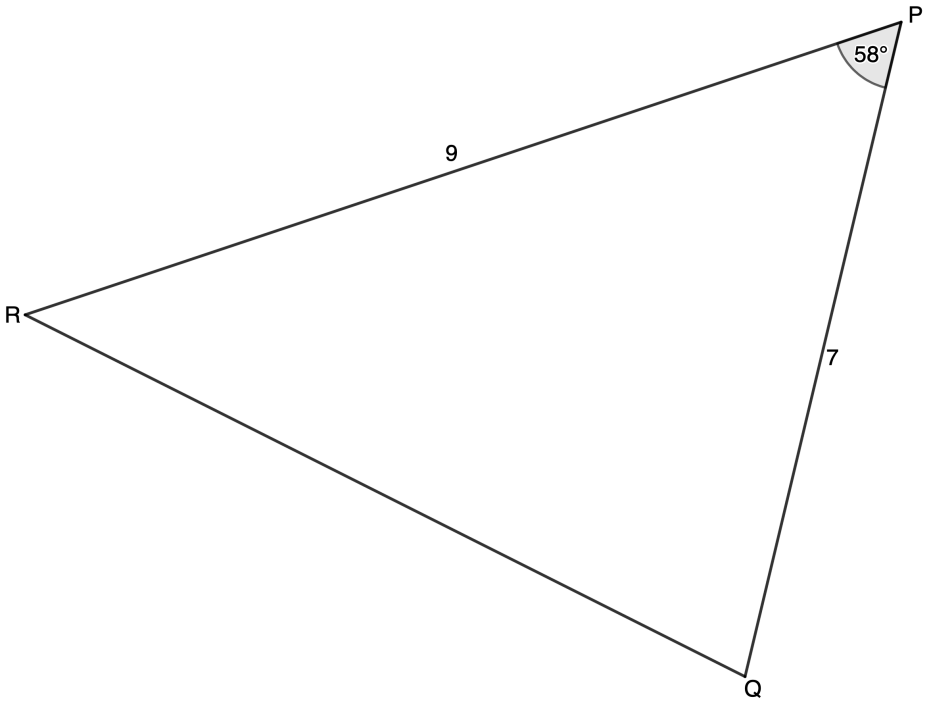
   In ΔHJK:
   We know the length of two sides (HJ and JK) and the included angle. Therefore, we can use the cosine rule to find HK.
   $$HK^2 = HJ^2 + JK^2 - 2HJ \cdot JK \cdot \cos H\hat{J}K$$
   $$\therefore HK^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5 \cdot \cos 54°$$
   $$\therefore HK = \sqrt{8^2 + 5^2 - 2 \cdot 8 \cdot 5 \cdot \cos 54°}$$
   $$= 6.48$$

Back to Exercise 7.1

Exercise 7.2

1.

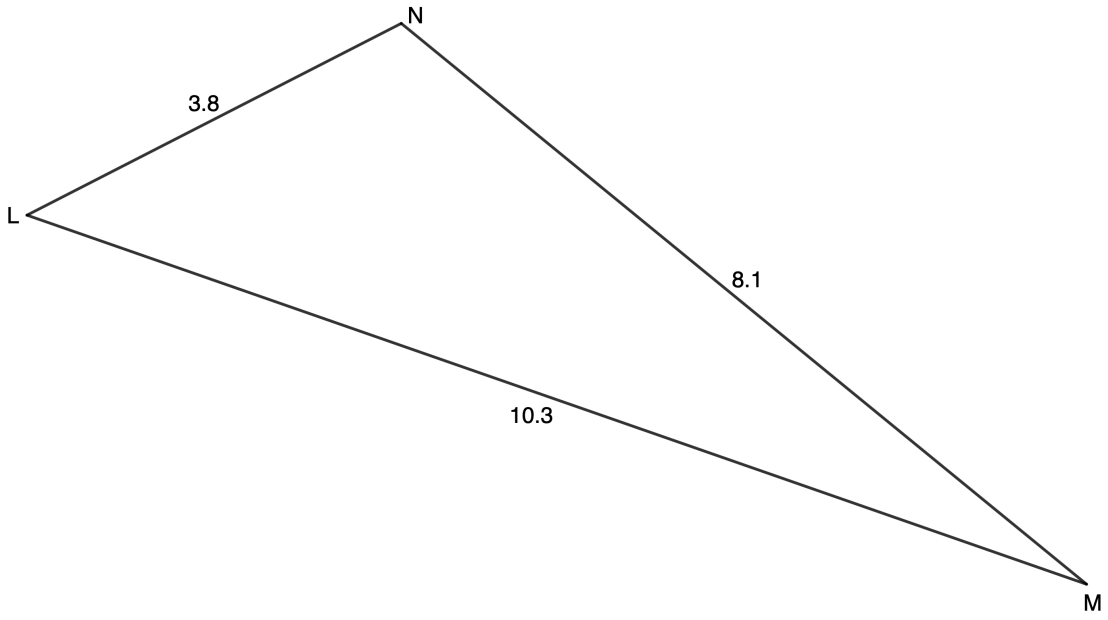
a. Make a sketch of the triangle with the given information.



We can use the cosine rule to find QR (side p).

$$QR^2 = PQ^2 + RP^2 - 2 \cdot PQ \cdot RP \cdot \cos P$$
$$\therefore QR^2 = 7^2 + 9^2 - 2 \cdot 7 \cdot 9 \cdot \cos 58^\circ$$
$$\therefore QR = \sqrt{7^2 + 9^2 - 2 \cdot 7 \cdot 9 \cdot \cos 58^\circ}$$
$$= 7.95$$
$$\frac{\sin R}{PQ} = \frac{\sin P}{QR}$$
$$\therefore \frac{\sin R}{7} = \frac{\sin 58^\circ}{7.95}$$
$$\therefore \sin R = \frac{7 \sin 58^\circ}{7.95}$$
$$\therefore R = 48.31^\circ$$
$$\hat{Q} = 180^\circ - \hat{P} - \hat{R} \qquad \text{(angles in a triangle are supplementary)}$$
$$= 180^\circ - 58^\circ - 48.31^\circ$$
$$= 73.69^\circ$$

b. Make a sketch of the triangle with the given information.



Given three sides, we use the cosine rule to find any of the angles.

$$MN^2 = LM^2 + NL^2 - 2 \cdot LM \cdot NL \cdot \cos L$$

$$\therefore \cos L = \frac{LM^2 + NL^2 - MN^2}{2 \cdot LM \cdot NL}$$

$$\therefore \cos L = \frac{10.3^2 + 3.8^2 - 8.1^2}{2 \cdot (10.3) \cdot (3.8)}$$

$$\therefore \hat{L} = 45.45^\circ$$

$$\frac{\sin M}{NL} = \frac{\sin L}{MN}$$

$$\therefore \frac{\sin M}{3.8} = \frac{\sin 45.45^\circ}{8.1}$$

$$\therefore \sin M = \frac{3.8 \sin 45.45^\circ}{8.1}$$

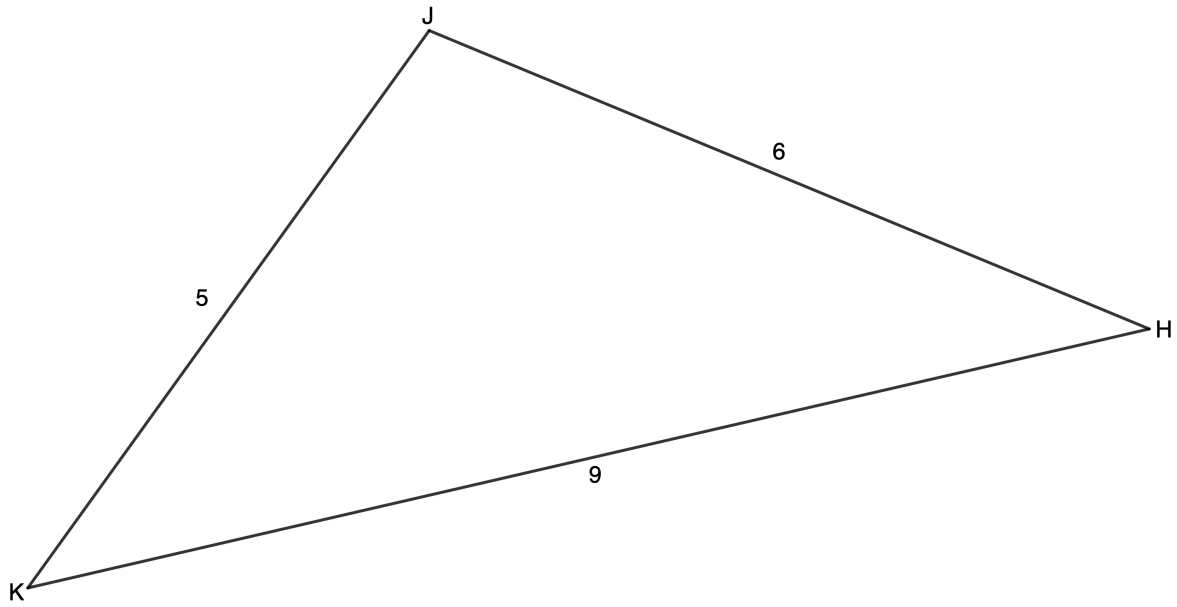
$$\therefore \hat{M} = 19.53^\circ$$

$$\hat{N} = 180^\circ - \hat{L} - \hat{M} \qquad \text{(angles in a triangle are supplementary)}$$

$$= 180^\circ - 45.45^\circ - 19.53^\circ$$

$$= 115.02^\circ$$

2. Make a sketch of the triangle with the given information.



The largest angle is the angle opposite the longest side. Therefore, $\hat{J}$ is the largest angle.

$$KH^2 = HJ^2 + JK^2 - 2 \cdot HJ \cdot JK \cdot \cos J$$
$$\therefore \cos J = \frac{HJ^2 + JK^2 - KH^2}{2 \cdot HJ \cdot JK}$$
$$\therefore \cos J = \frac{6^2 + 5^2 - 9^2}{2 \cdot 6 \cdot 5}$$
$$\therefore \hat{J} = 109.47^\circ$$

Back to Exercise 7.2

Unit 7: Assessment

1.

a. $CD \parallel EB$ (corresponding angles are equal)
$BC \parallel ED$ (co-interior angles are supplementary)
$\therefore BCDE$ is a rectangle (both pairs of opposite sides are parallel and interior angles equal to 90°)
$\therefore CD = EB = 10 \text{ m}$ (opposite sides of a rectangle are equal)

$$AE^2 = AB^2 + BE^2 - 2 \cdot AB \cdot BE \cdot \cos A\hat{B}E$$
$$\therefore AE^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 49^\circ$$
$$\therefore AE = \sqrt{6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 49^\circ}$$
$$= 7.57 \text{ m}$$

b.

$$\frac{\sin A\hat{E}B}{AB} = \frac{\sin A\hat{B}E}{AE}$$
$$\therefore \frac{\sin A\hat{E}B}{6} = \frac{\sin 49^\circ}{7.57}$$
$$\therefore \sin A\hat{E}B = \frac{6 \sin 49^\circ}{7.57}$$
$$\therefore A\hat{E}B = 36.74^\circ$$

c.

$$\begin{aligned}\text{Area } \Delta ABE &= \frac{1}{2} \cdot AB \cdot EB \cdot \sin A\hat{B}E \\ &= \frac{1}{2} \cdot 6 \cdot 10 \cdot \sin 49^\circ \\ &= 22.64 \text{ m}^2\end{aligned}$$

d. Let AF intersect BE at G. Then $AF = AG + GF$.
$GF = BC = 5$ m (opposites sides of a rectangle are equal)
AG is the perpendicular height of ΔABE and area $\Delta ABE = 22.64$ m^2.

$$\begin{aligned}\text{Area } \Delta ABE &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times EB \times AG \\ \therefore AG &= \frac{2 \times \text{Area } \Delta ABE}{EB} \\ &= \frac{2 \times 22.64}{10} \\ &= 4.53 \text{ m}\end{aligned}$$

Therefore $AF = 4.53 + 5 = 9.53$ m

2.

a.

$$\begin{aligned}XZ^2 &= WZ^2 + WX^2 - 2 \cdot WZ \cdot WX \cdot \cos W \\ \therefore XZ^2 &= 5.2^2 + 8.6^2 - 2 \cdot (5.2) \cdot (8.6) \cdot \cos 30^\circ \\ \therefore XZ &= \sqrt{5.2^2 + 8.6^2 - 2 \cdot (5.2) \cdot (8.6) \cdot \cos 30^\circ} \\ &= 4.85\end{aligned}$$

b. $W\hat{Z}X = 117.7^\circ$ (given)
$W\hat{X}Z = 180^\circ - 117.7^\circ - 30^\circ = 32.3^\circ$ (angles in a triangle are supplementary)
$WX \parallel YZ$ (given)
$W\hat{X}Z = X\hat{Z}Y = 32.3^\circ$ (alternate angles are equal, $WX \parallel YZ$)

$$\begin{aligned}\frac{\sin Y}{XZ} &= \frac{\sin X\hat{Z}Y}{XY} \\ \therefore \frac{\sin Y}{4.85} &= \frac{\sin 32.3^\circ}{2.6} \\ \therefore \sin Y &= \frac{4.85 \times \sin 32.3^\circ}{2.6} \\ \therefore \hat{Y} &= 85.40^\circ\end{aligned}$$

c. $\text{area } WXYZ = \text{area } \Delta WXZ + \text{area } \Delta XYZ$

$$\begin{aligned}\text{area } \Delta WXZ &= \frac{1}{2} \times WZ \times WX \times \sin W \\ &= \frac{1}{2} \times 5.2 \times 8.6 \times \sin 30^\circ \\ &= 11.18 \text{ u}^2\end{aligned}$$

In ΔXYZ:

$$\begin{aligned}Z\widehat{X}Y &= 180^\circ - \hat{Y} - X\hat{Z}Y \\ &= 180^\circ - 85.40^\circ - 32.3^\circ \\ &= 62.3^\circ\end{aligned}$$

$$\begin{aligned}\text{area } \Delta XYZ &= \frac{1}{2} \times XZ \times XY \times \sin Z\hat{X}Y \\ &= \frac{1}{2} \times 4.85 \times 2.6 \times \sin 62.3^\circ \\ &= 5.58 \text{ u}^2\end{aligned}$$

Therefore $\text{area } WXYZ = 11.18 + 5.58 = 16.76$ u^2

Back to Unit 7: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example7.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.1Q1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.1Q2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example7.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.2A1a © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.2A1b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise7.2A2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

Unit 8: Solve 2-D trig problems

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve real problems by applying the area, sine and cosine rules.

What you should know

- Before you start this unit, make sure you can:
- Use the area rule. Refer to unit 5 of this subject outcome if you need help with this.
- Use the sine rule to find unknown sides and angles of triangles. Refer to unit 6 of this subject outcome if you need help with this.
- Use the cosine rule to find unknown sides and angles of triangles. Refer to unit 7 of this subject outcome if you need help with this.

Introduction

This is the final unit in this subject outcome. In this unit, we are going to bring much of what we have already learnt together to solve some problems in two dimensions. We will particularly be using the area, sine and cosine rules to work out unknown areas, sides and angles of triangles. Read the following note carefully about how to determine which of these rules to use, and when.

Take note!

Use the area rule if:

- no perpendicular height is given.

Use the sine rule if:

- no right angle is given
- two sides and an angle are given (not the included angle)
- two angles and a side are given.

Use the cosine rule if:

- no right angle is given
- two sides and the included angle are given

- three sides are given.

Bearings and angles of elevation and depression

Before we get into solving 2-D problems, it is important that you understand what is meant by a bearing, and by angles of elevation and depression.

We can measure direction using **bearings**. Bearings are measured as the angle from the **north** rotated in a **clockwise** direction. For example, Figure 1 shows a point A at a bearing of 215°.

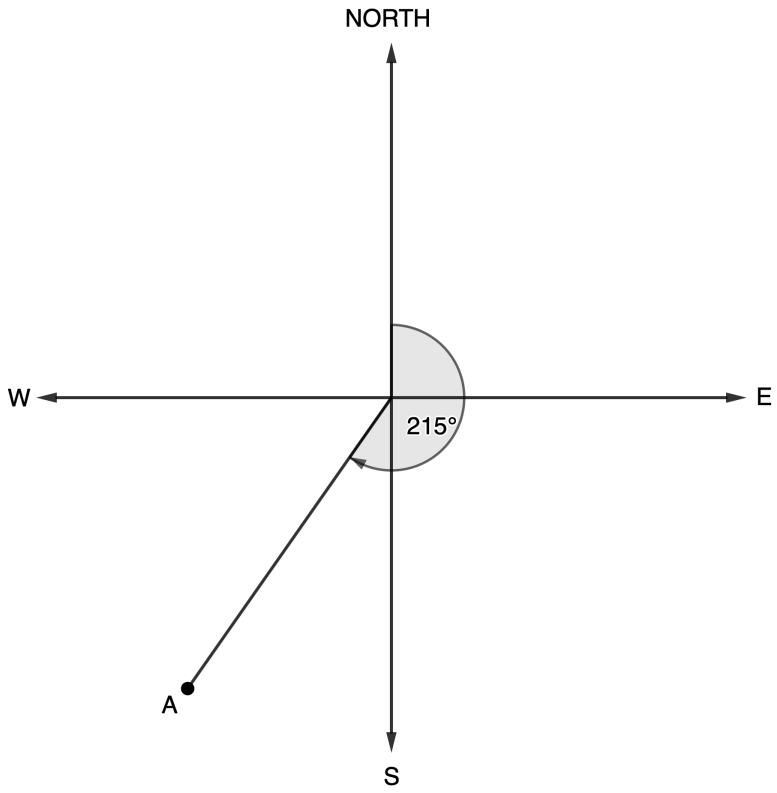


Figure 1: Point A at a bearing of 215°

We can also measure direction as the number of degrees from any of the cardinal directions. For example, in Figure 2, point A is $\mathrm{W}16^\circ\mathrm{S}$ or 16° south of due west.

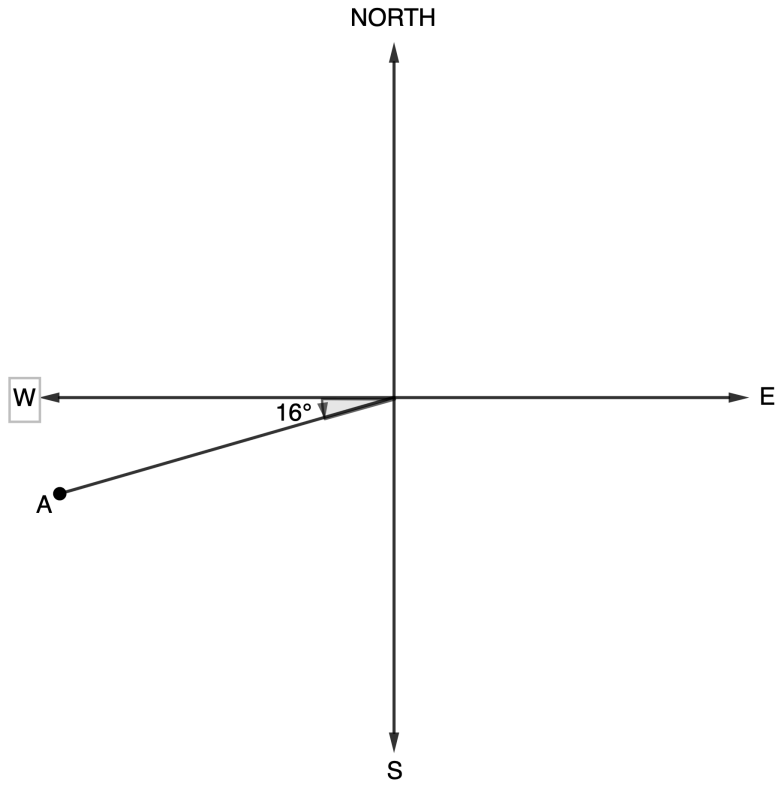


Figure 2: Point A is $\mathrm{W}16^\circ\mathrm{S}$ *or* 16° *south of due west*

We measure how far an object is above a horizontal line using an **angle of elevation**. Figure 3 shows point A at an angle of elevation of 35°.

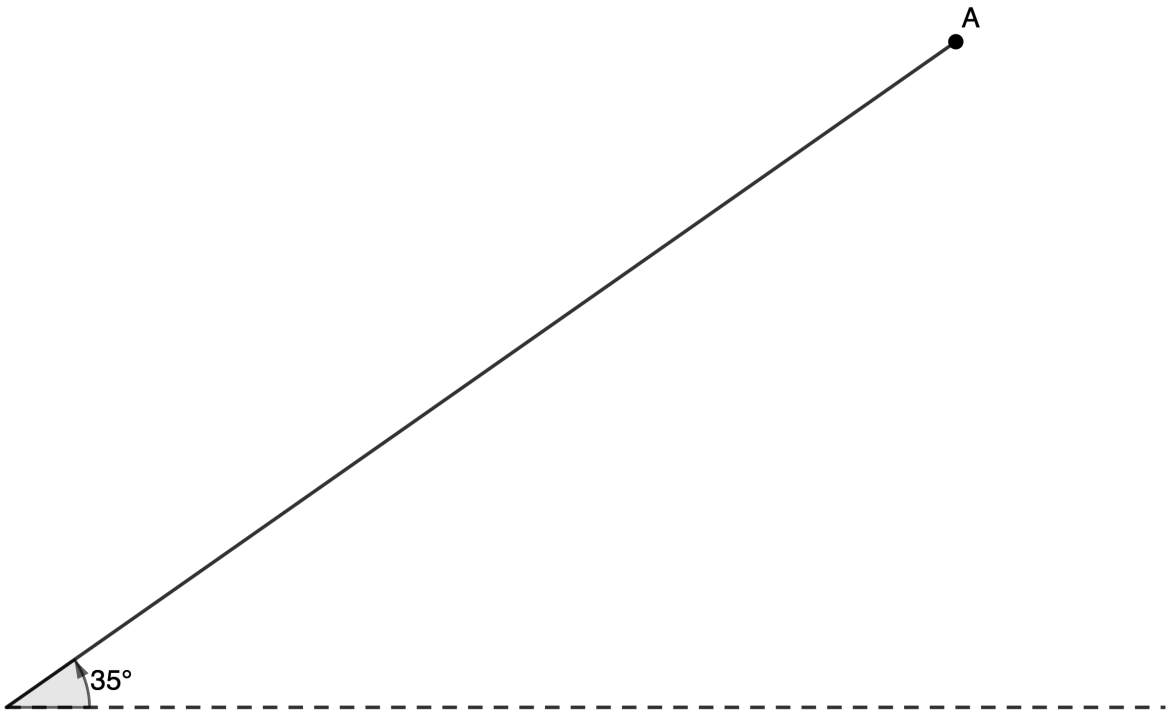


Figure 3: Point A at an angle of elevation of 35°

Similarly, we measure how far below the horizontal something is using an angle of depression. Figure 4 shows point A at an angle of depression of 47°.

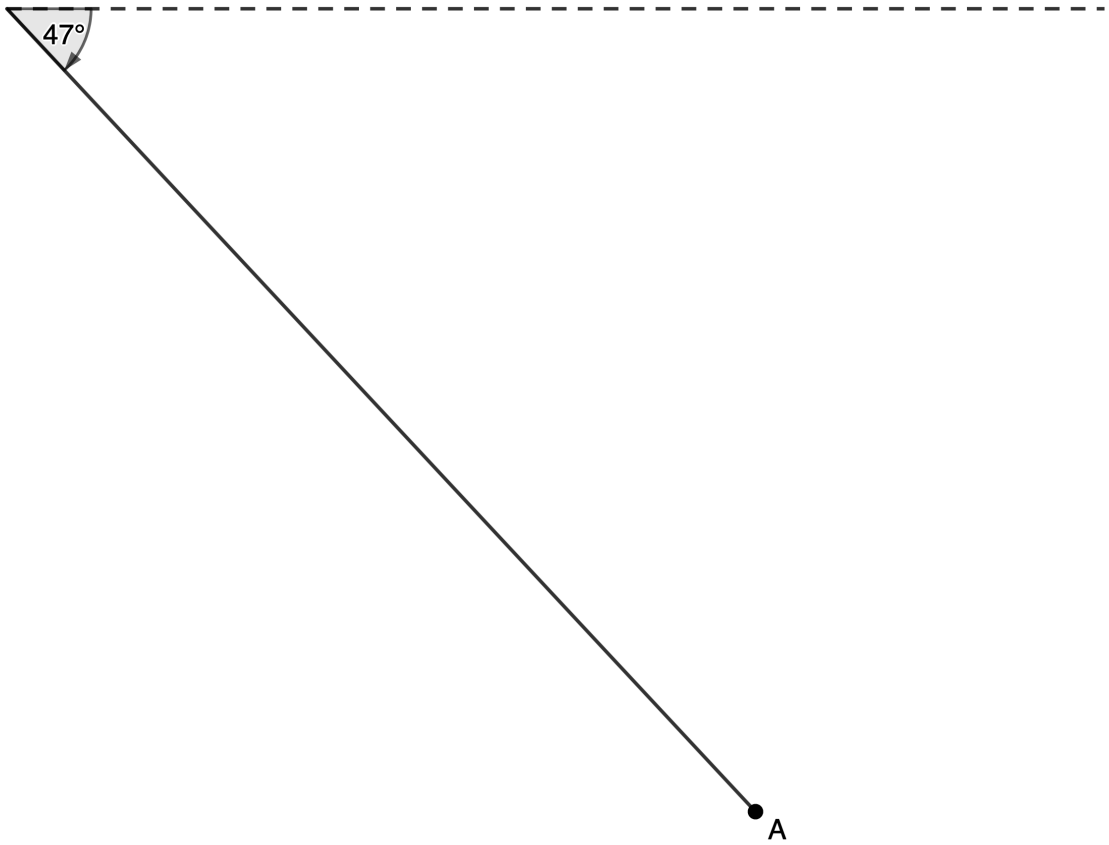


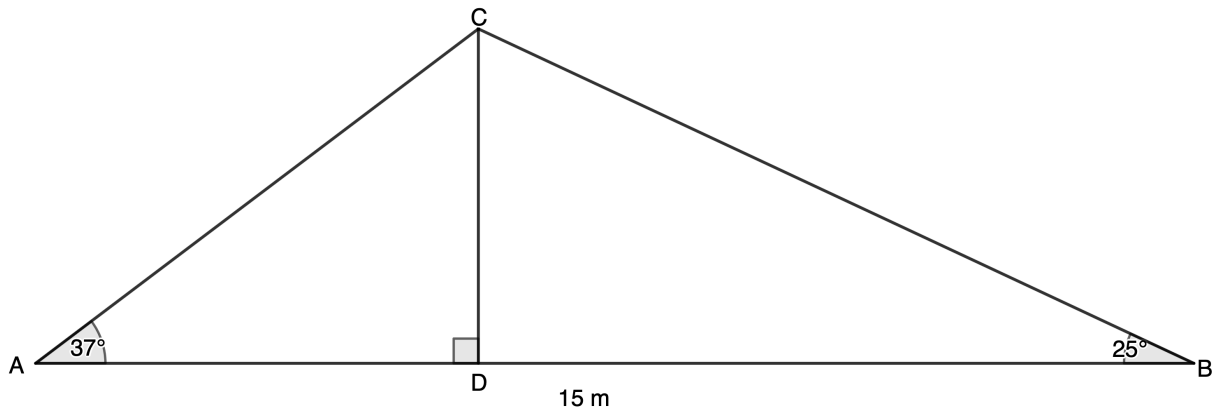
Figure 4: Point A at an angle of depression of $47°$

Solving problems in 2-D

Let's now look at how we solve different types of two-dimensional problems.

Example 8.2

Two observers at A and B respectively, are watching workmen fixing power lines at C. The angle of elevation from A to C is $37°$ and the angle of elevation from B to C is $25°$. If the observers are standing $15\ \text{m}$ apart, calculate the height of the pole CD, correct to one decimal place.



Solution

To calculate the length of CD, we first need to find the length of AC or BC. Then we can work in either of these smaller triangles to calculate CD. To get AC or BC we need $\hat{C}$ which we can easily calculate.

$\hat{C} = 180° - \hat{A} - \hat{B} = 180° - 37° - 25° = 118°$ (angles in a triangle are supplementary)

$$\frac{AC}{\sin B} = \frac{AB}{\sin C}$$
$$\therefore \frac{AC}{\sin 25°} = \frac{15}{\sin 118°}$$
$$\therefore AC = \frac{15 \sin 25°}{\sin 118°}$$
$$= 7.18 \text{ m}$$

Now that we have the length of AC we can work in ΔACD to determine CD.

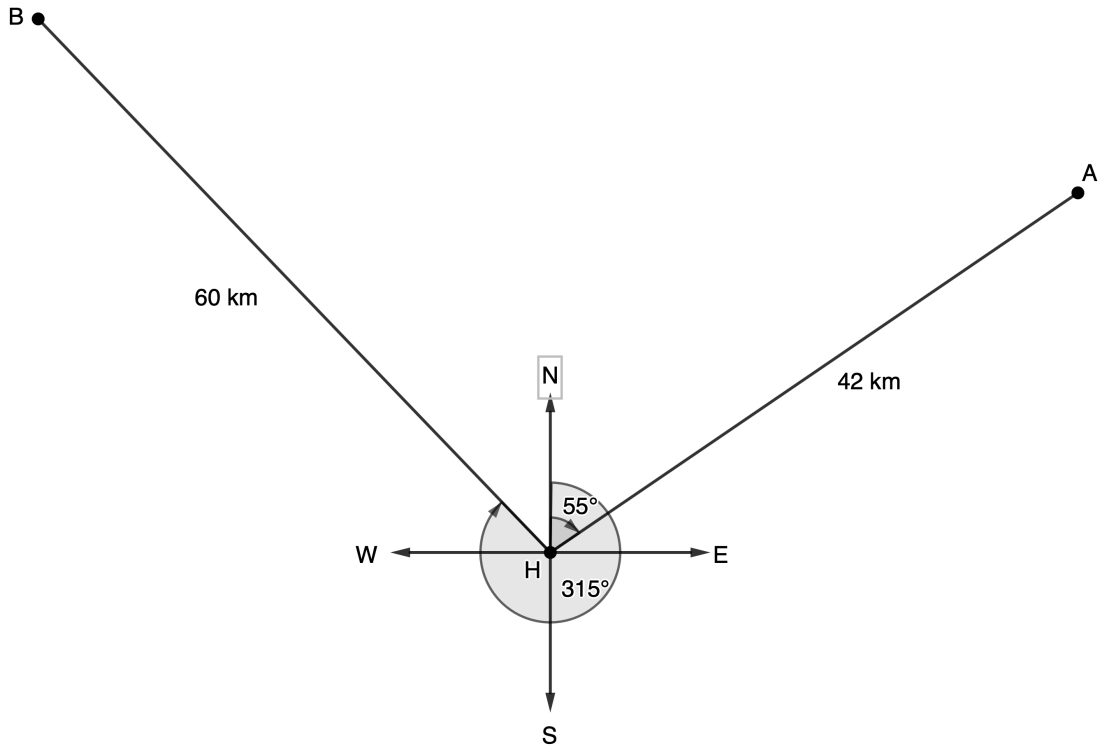
$$\frac{CD}{\sin A} = \frac{AC}{\sin C\hat{D}A}$$
$$\therefore \frac{CD}{\sin 37°} = \frac{7.18}{\sin 90°}$$
$$\therefore CD = \frac{7.18 \sin 37°}{\sin 90°}$$
$$= 4.32 \text{ m}$$

Example 8.2

Two ships leave a harbour, H, at the same time. The first ship sails 42 km to point A in a direction of $\text{N}55°\text{E}$. The second ship sails for 60 km to point B on a bearing of $315°$. How far apart are the points A and B?

Solution

We are not given a diagram of the situation so the first thing we need to do is to sketch our own. It does not need to be perfectly accurate or to scale. Always make sure you have a diagram representing the given information before trying to calculate anything.



We need to calculate the distance AB. We will be working in ΔABH.
The angle between BH and north is 45° (angles around a point). Therefore, $A\hat{H}B = 45^\circ + 55^\circ = 100^\circ$.

We can use the cosine rule to determine AB.

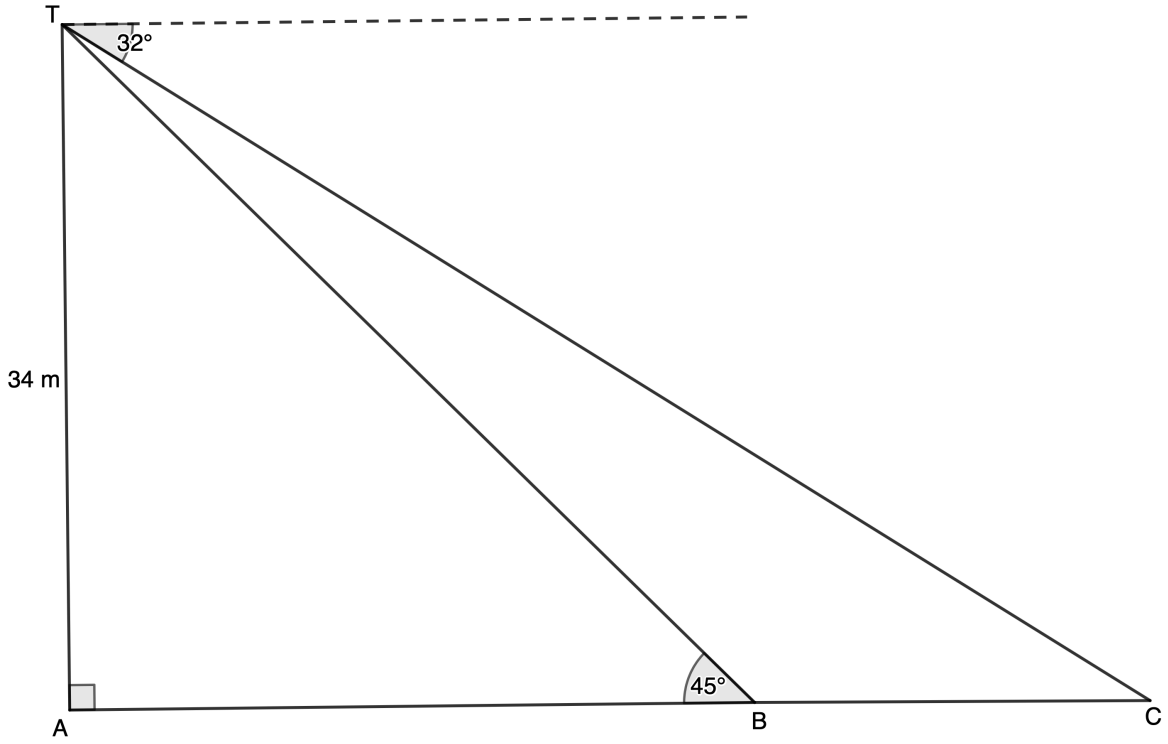
$$\begin{aligned} AB^2 &= AH^2 + BH^2 - 2 \cdot AH \cdot BH \cdot \cos A\hat{H}B \\ \therefore AB^2 &= 42^2 + 60^2 - 2 \cdot 42 \cdot 60 \cdot \cos 100^\circ \\ \therefore AB &= \sqrt{42^2 + 60^2 - 2 \cdot 42 \cdot 60 \cdot \cos 100^\circ} \\ &= 78.99 \text{ km} \end{aligned}$$

NOTE: You might have noticed that the partial calculation for $2 \cdot 42 \cdot 60 \cdot \cos 100^\circ$ gave a negative number. Remember that cosine is negative in the second quadrant.

Exercise 8.1

1. CP is a radio mast. CA and CB are cables used to support the mast. B and A are on the same level as P. $CA = 11$ m, $CB = 8$ m and $BA = 10$ m.
   a. Determine $C\hat{A}B$.
   b. Hence, determine CP.
   c. Determine the area of ΔCAB.
2. The diagram below shows a 34 m vertical observation tower AT, and two cars, B and C, on a road. The angle of depression from T to car C is 32°. The angle of elevation from car B to the top of the tower is 45°. A, B and C lie in a straight line and lie on the same horizontal plane as the base of the

tower.

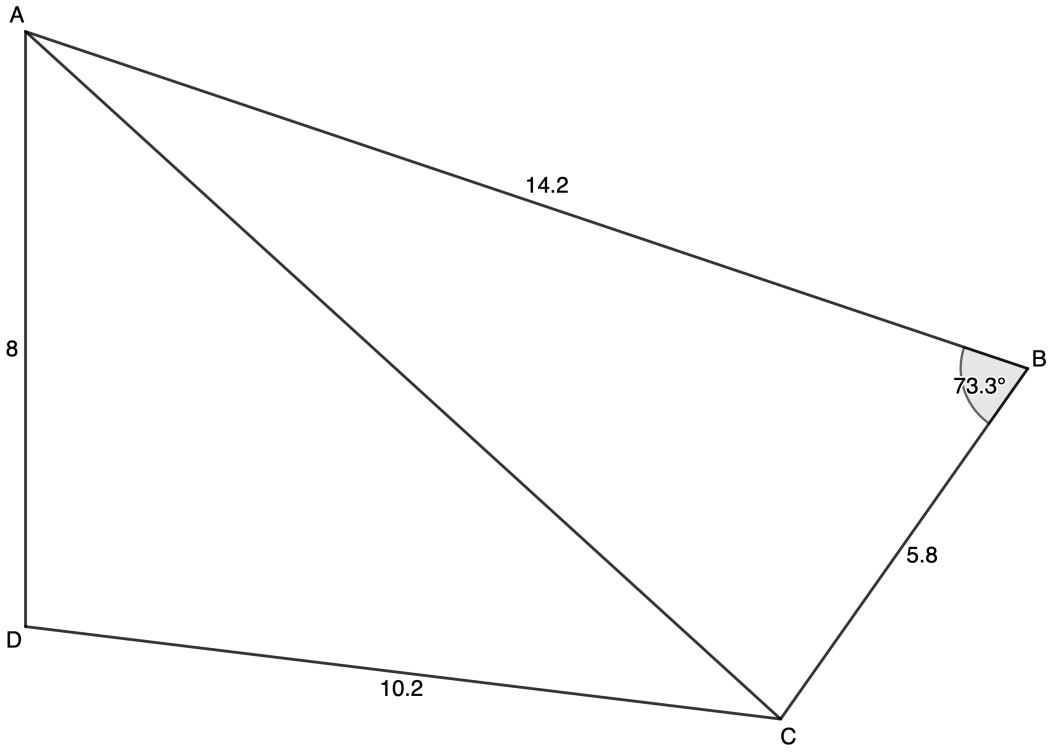


a. Calculate the size of $\hat{C}$.

b. Determine the length of BT.

c. Hence, calculate the distance between the two cars.

3. A piece of land has the form of a quadrilateral $ABCD$ as shown below. All distances are in kilometres.



The owner decides to divide the land into two plots by erecting a fence from A to C.

a. Calculate the length of the fence AC correct to one decimal place.

b. Calculate the size of $B\hat{A}C$ correct to the nearest degree.

c. Calculate the size of $\hat{D}$, correct to the nearest degree.

d. Calculate the area of the entire piece of land $ABCD$, correct to one decimal place.

The full solutions are at the end of the unit.

Summary

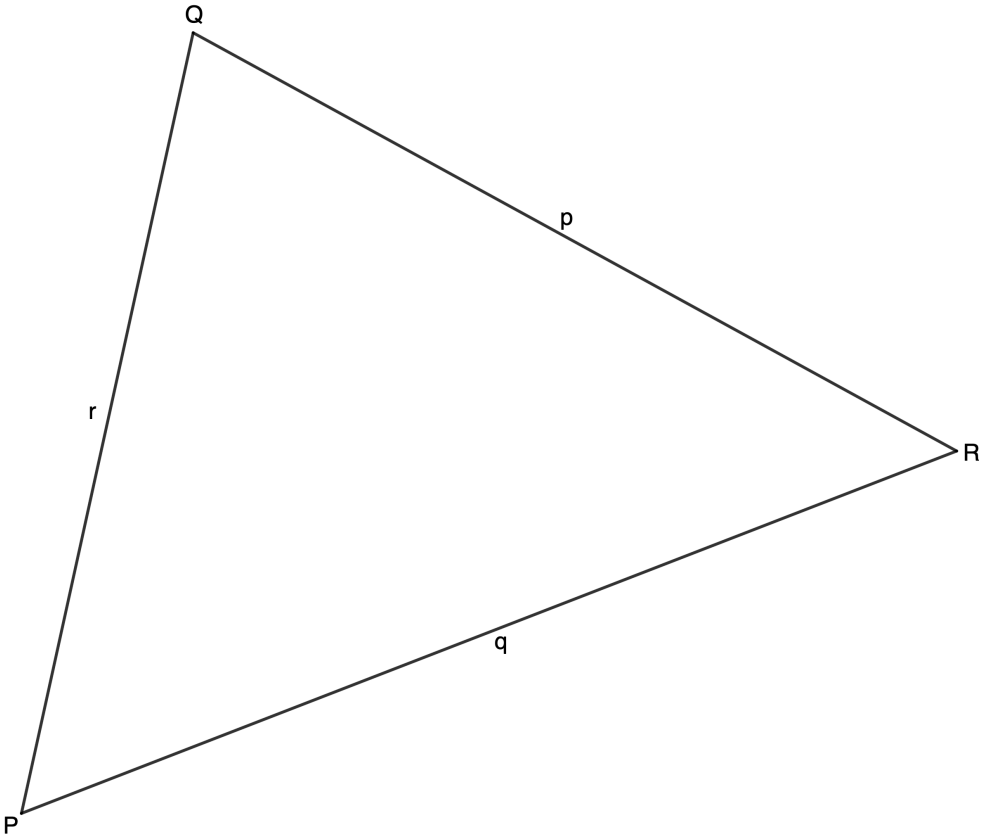
In this unit you have learnt the following:

- How to solve 2-D trigonometry problems using a combination of the area rule, sine rule and cosine rule.

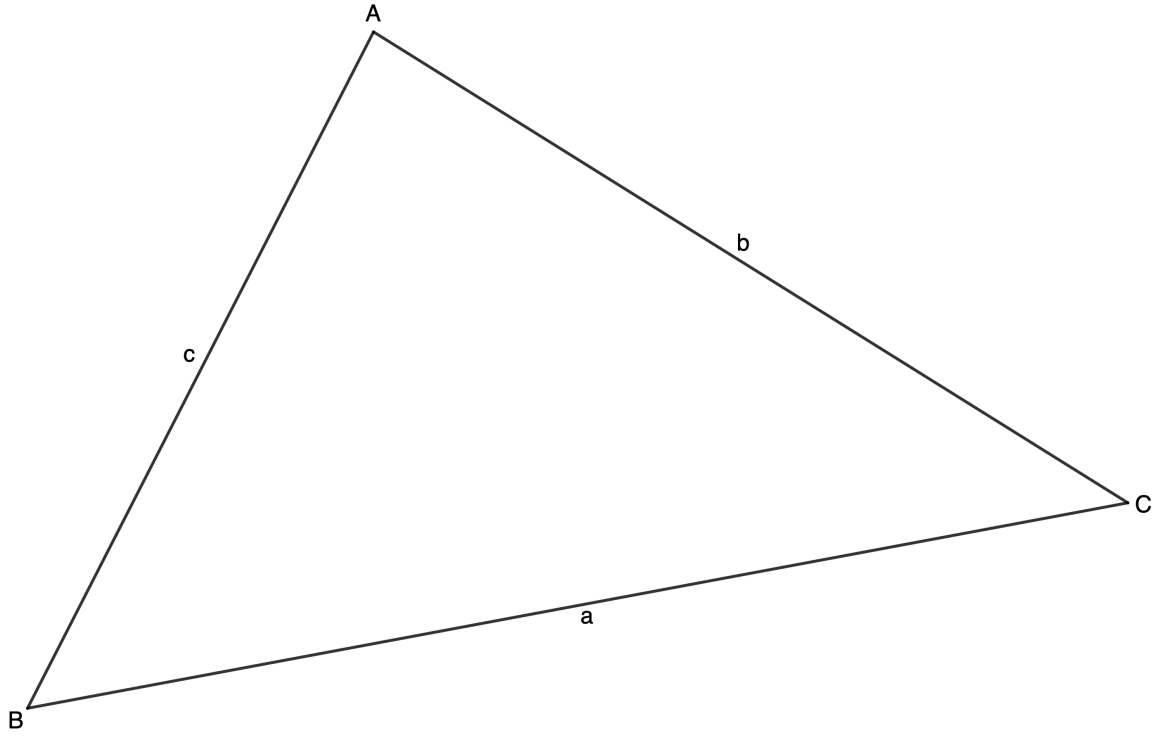
Unit 8: Assessment

Suggested time to complete: 45 minutes

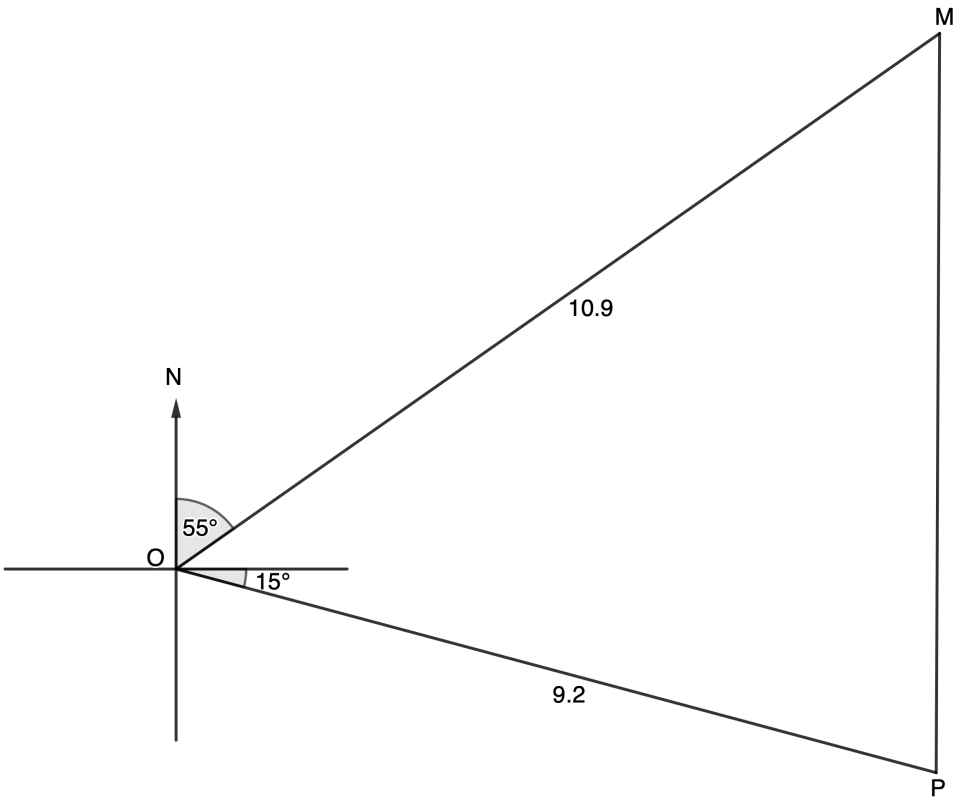
1. ΔPQR is an isosceles triangle with $PQ = QR$. By using the cosine rule, prove that $\cos Q = 1 - \frac{q^2}{2p^2}$.



2. If $b = c$ and $a^2 = 7b^2$, show why it is impossible to construct ΔABC.



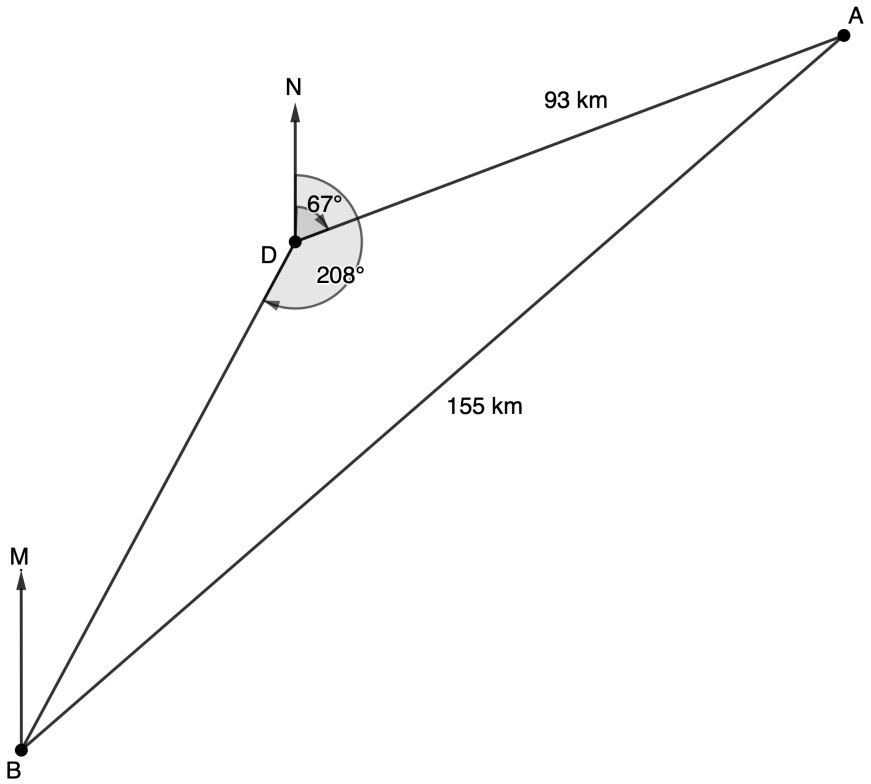
3. A game ranger M is $10.9\ \text{km}$ from control centre O at a bearing of $55°$, when he receives a call that there is an elephant at P, injured and needing attention as a result of a failed poaching attempt. The elephant is located $9.2\ \text{km}$ at $\text{E}15°\text{S}$ of the control centre. The diagram below is a representation of the situation.



a. Determine how far the game ranger is from the injured elephant.

b. If it is believed that the poachers are in the area formed by ΔMOP, calculate the area that needs to be searched.

4. Two ships, A and B, are $155\ \text{km}$ apart. Ship A is at a bearing of $67°$ from D and $93\ \text{km}$ away from D. DN points due north. Ship B is at a bearing of $208°$ from D.



a. Determine the bearing of Ship A from Ship B, when $BM \parallel DN$.

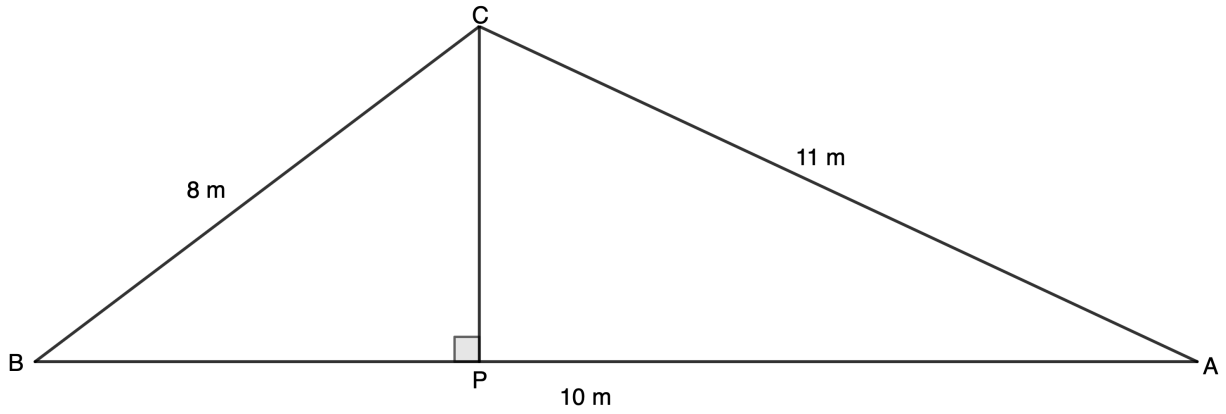
b. If Ship B travels due north, and Ship A travels due south, then at some instant in time, Ship A is due east of Ship B. Calculate the distance between the two ships at that instant.

The full solutions are at the end of the unit.

Unit 8: Solutions

Exercise 8.1

1. Start by making a sketch of the given information.



a. $C\hat{A}B = \hat{A}$

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos A$$

$$\therefore \cos A = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$$

$$= \frac{10^2 + 11^2 - 8^2}{2 \times 10 \times 11}$$

$$\therefore A = 44.47^\circ$$

b.

$$\frac{CP}{\sin A} = \frac{AC}{\sin A\hat{P}C}$$

$$\therefore \frac{CP}{\sin 44.47^\circ} = \frac{11}{\sin 90^\circ}$$

$$\therefore CP = \frac{11 \sin 44.47^\circ}{\sin 90^\circ}$$

$$= 7.71 \text{ m}$$

c.

$$\text{area } \Delta ABC = \frac{1}{2} \times AC \times AB \times \sin A$$

$$= \frac{1}{2} \times 11 \times 10 \times \sin 44.47^\circ$$

$$= 38.53 \text{ m}^2$$

2.

a. AC is parallel to the horizontal line from which the angle of depression to car C is measured. Therefore, $\hat{C} = 32^\circ$ (alternate angles are equal)

b. In ΔABT:

$$\frac{BT}{\sin A} = \frac{AT}{\sin A\hat{B}T}$$
$$\therefore \frac{BT}{\sin 90^\circ} = \frac{34}{\sin 45^\circ}$$
$$\therefore BT = \frac{34 \sin 90^\circ}{\sin 45^\circ}$$
$$= 48.08 \text{ m}$$

c. $A\hat{T}B = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ (angles in a triangle are supplementary)
$B\hat{T}C = 90^\circ - A\hat{T}B - 32^\circ = 90^\circ - 45^\circ - 32^\circ = 13^\circ$
In ΔBCT:
$$\frac{BC}{\sin B\hat{T}C} = \frac{BT}{\sin C}$$
$$\therefore \frac{BC}{\sin 13^\circ} = \frac{48.08}{\sin 32^\circ}$$
$$\therefore BC = \frac{48.08 \sin 13^\circ}{\sin 32^\circ}$$
$$= 20.41 \text{ m}$$

3.

a.
$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos B$$
$$\therefore AC^2 = 14.2^2 + 5.8^2 - 2 \cdot (14.2) \cdot (5.8) \cdot \cos 73.3^\circ$$
$$\therefore AC = \sqrt{14.2^2 + 5.8^2 - 2 \cdot (14.2) \cdot (5.8) \cdot \cos 73.3^\circ}$$
$$= 13.7 \text{ km}$$

b.
$$\frac{\sin B\hat{A}C}{BC} = \frac{\sin B}{AC}$$
$$\therefore \frac{\sin B\hat{A}C}{5.8} = \frac{\sin 73.3^\circ}{13.71}$$
$$\therefore \sin B\hat{A}C = \frac{5.8 \sin 73.3^\circ}{13.71}$$
$$\therefore B\hat{A}C = 24^\circ$$

c.
$$AC^2 = AD^2 + CD^2 - 2 \cdot AD \cdot CD \cdot \cos D$$
$$\therefore \cos D = \frac{AD^2 + CD^2 - AC^2}{2 \cdot AD \cdot CD}$$
$$\therefore \cos D = \frac{8^2 + 10.2^2 - 13.7^2}{2 \cdot 8 \cdot (10.2)}$$
$$\therefore \hat{D} = 97^\circ$$

d. $\text{area } \Delta ABCD = \text{area } \Delta ABC + \text{area } \Delta ACD$
$$\text{area } \Delta ABC = \frac{1}{2} \cdot AB \cdot BC \cdot \sin B$$
$$= \frac{1}{2} \cdot (14.2) \cdot (5.8) \cdot \sin 73.3^\circ$$
$$= 39.4 \text{ km}^2$$
$$\text{area } \Delta ACD = \frac{1}{2} \cdot AD \cdot CD \cdot \sin D$$
$$= \frac{1}{2} \cdot 8 \cdot (10.2) \cdot \sin 97^\circ$$
$$= 40.5 \text{ km}^2$$
$$\text{area } ABCD = 39.4 + 40.5 = 79.9 \text{ km}^2$$

Back to Exercise 8.1

Unit 8: Assessment

1. To prove: $\cos Q = 1 - \dfrac{q^2}{2p^2}$ using the cosine rule.

$$\begin{aligned} q^2 &= p^2 + r^2 - 2pr\cos Q \\ \therefore \cos Q &= \frac{p^2 + r^2 - q^2}{2pr} \\ \text{But: } PQ &= QR \therefore p = r \\ \therefore \cos Q &= \frac{p^2 + p^2 - q^2}{2p \cdot p} \\ &= \frac{2p^2 - q^2}{2p^2} \\ &= \frac{2p^2}{2p^2} - \frac{q^2}{2p^2} \\ &= 1 - \frac{q^2}{2p^2} \end{aligned}$$

2. $b = c$ and $a^2 = 7b^2$, and assume one can construct ΔABC. Then:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\begin{aligned} \text{But: } a^2 = 7b^2 \text{ and } b = c& \\ \therefore 7b^2 &= b^2 + b^2 - 2b^2\cos A \\ \therefore 7b^2 &= 2b^2 - 2b^2\cos A \\ \therefore 7b^2 &= 2b^2(1 - \cos A) \\ \therefore \frac{7\cancel{b^2}}{2\cancel{b^2}} &= 1 - \cos A \\ \therefore \cos A &= 1 - \frac{7}{2} = -\frac{5}{2} \end{aligned}$$

But $\cos A$ cannot be less than -1. Therefore, the construction is not possible.

3.

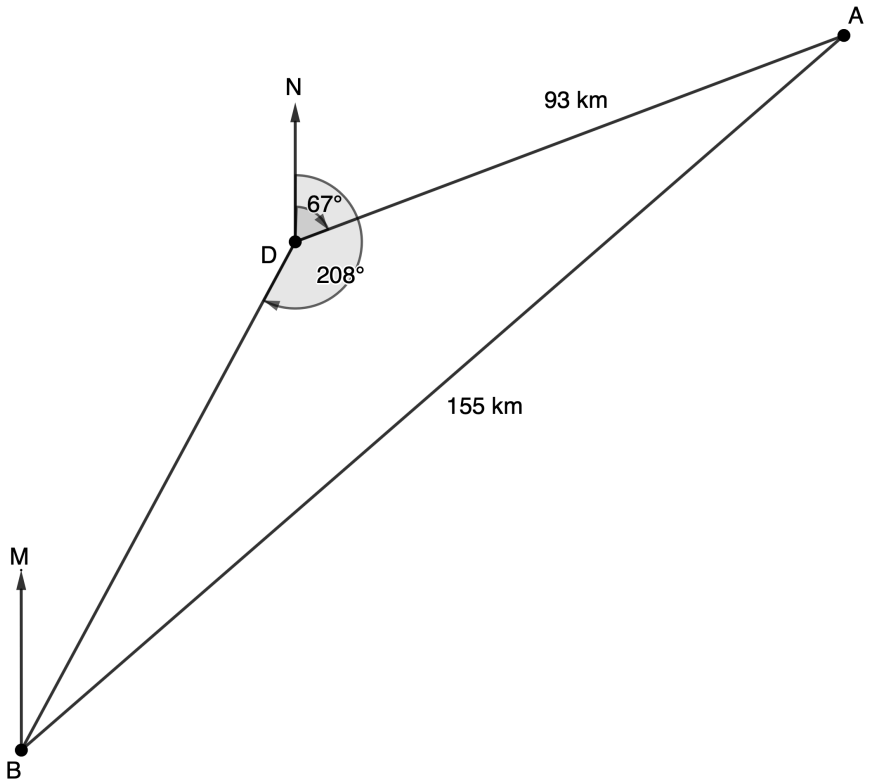
a. $M\hat{O}P = (90^\circ - 55^\circ) + 15^\circ = 50^\circ$

$$\begin{aligned} MP^2 &= MO^2 + PO^2 - 2 \cdot MO \cdot PO \cdot \cos M\hat{O}N \\ \therefore MP^2 &= 10.9^2 + 9.2^2 - 2 \cdot (10.9) \cdot (9.2) \cdot \cos 50^\circ \\ \therefore MP &= \sqrt{10.9^2 + 9.2^2 - 2 \cdot (10.9) \cdot (9.2) \cdot \cos 50^\circ} \\ &= 8.63 \text{ km} \end{aligned}$$

b.

$$\begin{aligned} \text{area } \Delta MOP &= \frac{1}{2} \cdot MO \cdot PO \cdot \sin M\hat{O}N \\ &= \frac{1}{2} \cdot (10.9) \cdot (9.2) \cdot \sin 50^\circ \\ &= 38.41 \text{ km}^2 \end{aligned}$$

4. A sketch of the information was given.



a. The bearing of Ship A from Ship B is $M\hat{B}A$.

$M\hat{B}A = D\hat{B}M + A\hat{B}D$

$N\hat{D}B = 360^\circ - 208^\circ = 152^\circ$ (angles at a point)

$BM \parallel DN$ (given). Therefore, $D\hat{B}M = 180^\circ - 152^\circ = 28^\circ$ (co-interior angles are supplementary, $BM \parallel DN$)

In ΔADB:

$A\hat{D}B = 208^\circ - 67^\circ = 141^\circ$

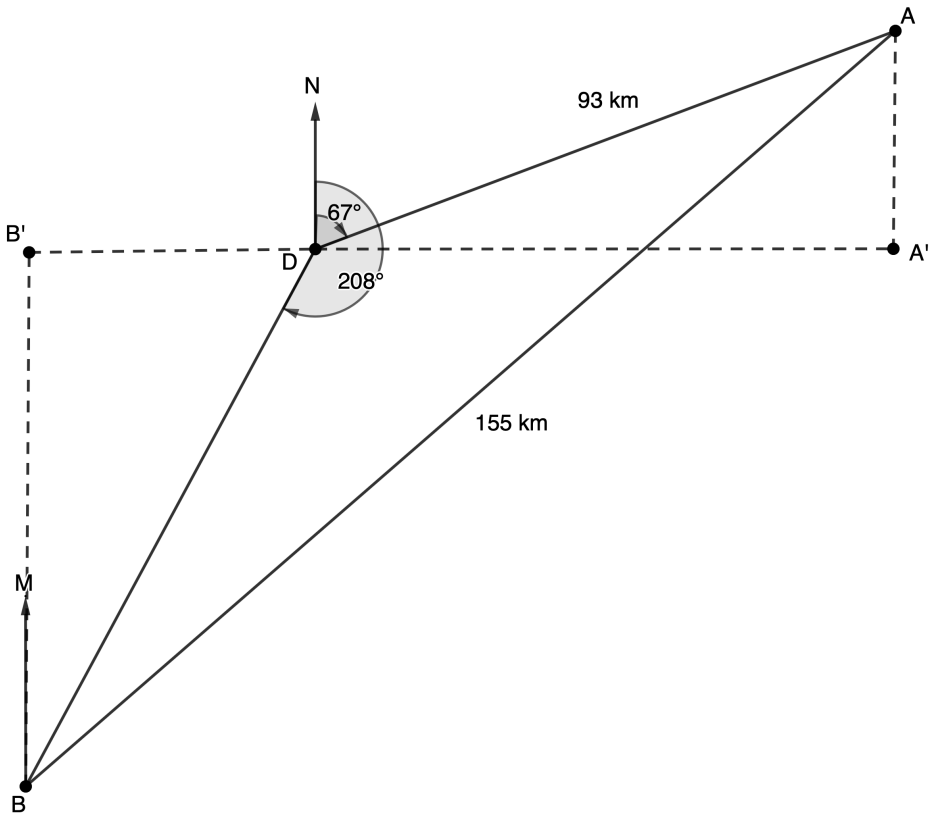
$$\frac{\sin A\hat{B}D}{AD} = \frac{\sin A\hat{D}B}{AB}$$

$$\therefore \frac{\sin A\hat{B}D}{93} = \frac{\sin 141^\circ}{155} \qquad M\hat{B}A = 28^\circ + 22.18^\circ = 50.18^\circ$$

$$\therefore \sin A\hat{B}D = \frac{93 \sin 141^\circ}{155}$$

$$\therefore A\hat{B}D = 22.18^\circ$$

b. It helps to makes another sketch to see what is happening.



The sketch shows when Ship A is due east of Ship B. We need to determine that the length of $A'B'$ which is equal to $A'D + B'D$. Because we are told that the ships sail due north and due south respectively, we know that we are dealing with right-angled triangles.

In $\Delta ADA'$:

$A\hat{D}A' = 90^\circ - 67^\circ = 23^\circ$

$\cos 23^\circ = \dfrac{DA'}{93}$

$\therefore DA' = 93 \cos 23^\circ$

$= 85.61 \text{ km}$

To determine $B'D$, we need the length of BD. To determine BD, we need to determine $D\hat{A}B$.

In ΔABD:

$A\hat{B}D = 22.18^\circ$ (from a.)

$\therefore D\hat{A}B = 180^\circ - A\hat{D}B - A\hat{B}D$ (angles in a triangle are supplementary)

$= 180^\circ - 141^\circ - 22.18^\circ$

$= 16.82^\circ$

$\dfrac{BD}{\sin D\hat{A}B} = \dfrac{AB}{\sin A\hat{D}B}$

$\therefore \dfrac{BD}{\sin 16.82^\circ} = \dfrac{155}{\sin 141^\circ}$

$\therefore BD = \dfrac{155 \sin 16.82^\circ}{\sin 141^\circ}$

$= 71.27 \text{ km}$

In $\Delta BDB'$:

$B\hat{D}B' = 270^\circ - 208^\circ = 62^\circ$

$\cos 62^\circ = \dfrac{DB'}{BD}$

$\therefore DB' = 71.27 \cos 62^\circ$

$= 33.46 \text{ km}$

Therefore:

$$\begin{aligned} A'B' &= 85.61 + 33.46 \\ &= 119.07 \text{ km} \end{aligned}$$

Back to Unit 8: Assessment

Media Attributions

- figure1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- figure4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example8.1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- example8.2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.1Q2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.1Q3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ2 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ3 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentQ4 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- exercise8.1A1 © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license
- assessmentA4b © Geogebra is licensed under a CC BY-SA (Attribution ShareAlike) license

SUBJECT OUTCOME XI

DATA HANDLING: CALCULATE, REPRESENT AND INTERPRET MEASURES OF CENTRAL TENDENCY AND DISPERSION IN UNIVARIATE NUMERICAL UNGROUPED DATA

Subject outcome

Subject outcome 4.1: Calculate, represent and interpret measures of central tendency and dispersion in univariate numerical ungrouped data.

Learning outcomes

- Work out the five-number summary by:
  - Calculating the maximum, minimum and quartiles
  - Determining the fences
  - Constructing the box and whisker diagram
  - Indicating any outliers.
- Interpret the meaning of the representation of the box and whisker diagram with its outliers.

Unit 1 outcomes

By the end of this unit you will be able to:

- Find the mean for ungrouped data.
- Find the median for ungrouped data.
- Find the mode for ungrouped data.
- Find the range and interquartile range.

Unit 2 outcomes

By the end of this unit you will be able to:

- Combine measures of central tendency and dispersion to work out the five-number summary.
- Construct the box-and-whisker plot.
- Interpret the box-and-whisker plot.

Unit 1: Revise measures of central tendency and dispersion for ungrouped data

NATASHIA BEARAM-EDMUNDS



Unit outcomes

By the end of this unit you will be able to:

- Find the mean for ungrouped data.
- Find the median for ungrouped data.
- Find the mode for ungrouped data.
- Find the range and interquartile range.

What you should know

Before you start this unit, make sure you can:

- Calculate central tendencies and dispersion of data and represent data effectively. Revise the following level 2, topic 4 units on data handling:
  - Subject outcome 4.1, unit 1: data collecting.
  - Subject outcome 4.1, unit 2: measures of central tendency of ungrouped data.
  - Subject outcome 4.1, unit 3: measures of dispersion of ungrouped data.
  - Subject outcome 4.2, unit 1: data representation.
  - Subject outcome 4.2, unit 2: bar graphs and histograms.
  - Subject outcome 4.2, unit 3: frequency polygon and line graphs.

Introduction

You will come across many formulas to calculate statistics. The purpose of statistics is not to perform numerous calculations using formulas, but to gain an understanding of your data. The calculations are often done using a calculator or a computer but basic calculations can be done by hand too. The understanding must come from you. If you can thoroughly understand the basics of statistics, you can be more confident in the decisions you make in life.

In this unit, we revise how to calculate statistics that give us information about the central values and spread of data.

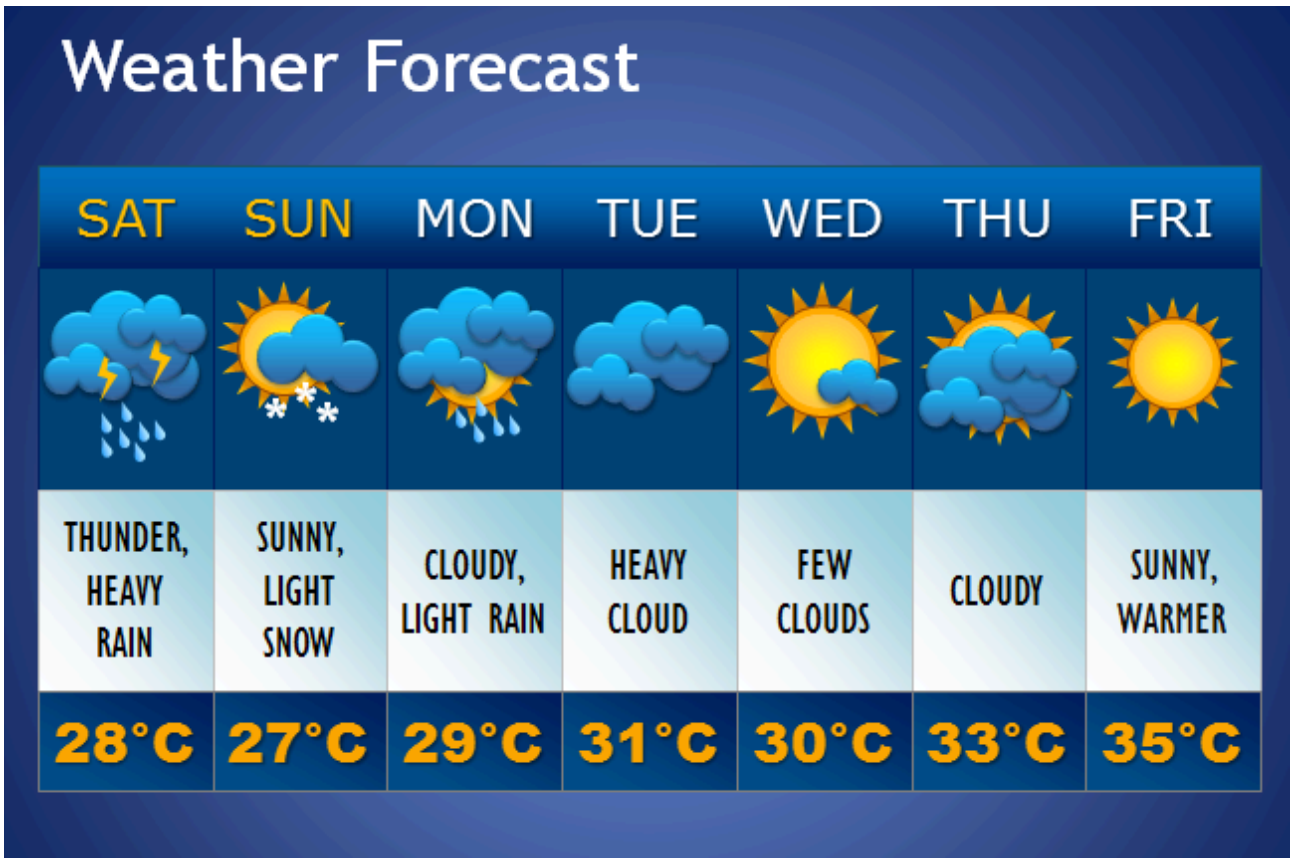


Figure 1: Statistics are all around us from many different sources, such as the weather forecast

Let's revise the terms that you should know by doing Activity 1.1.

Activity 1.1: Define the key terms used in data handling

Time required: 10 minutes

What you need:

- a pen and paper

What to do:

Read the following and answer the questions:
We want to study the average amount of money first year learners spend at ABC College on supplies that do not include books. We randomly surveyed 100 first year learners at the college. Three of those learners spent R 1000, R 850 and R 500.

1. What is the population in the study?
2. What is the sample in the study?
3. State the parameter.
4. State the statistic.

5. What are the possible variables being studied?
6. Write another word for 'average'.
7. What are the data values?

What did you find?

1. A **population** is a collection of persons, things or objects under study. The population in this study is all first year learners attending ABC College.
2. To study the population, we select a **sample**. The sample could be all enrolled in a beginning statistics course at ABC College (although this sample may not represent the entire population). In this study 100 learners make up the sample. The bigger the sample size is the more closely it will represent the population.
3. A **parameter** is a number used to represent a population characteristic and cannot be easily found hence it is estimated by a statistic. The parameter in this study is the average amount of money spent (excluding books) by all first year learners at ABC College.
4. The **statistic** is the average amount of money spent (excluding books) by first year college learners in the sample.
5. The **variable** could be the amount of money spent (excluding books) by one first year learner. For example, let $X =$ the amount of money spent (excluding books) by one first year learner at the college.
6. The 'average' is also called the 'mean'; a number that describes the central tendency of data.
7. Data are the actual values of the variable. The data are the rand amounts spent by the first year learners, or example $\mathrm{R}\ 1000,\ \mathrm{R}\ 850$ and $\mathrm{R}\ 500$.

Data are either **qualitative** or **quantitative**. Qualitative data describe features. Car colour, race and blood type are examples of qualitative data.

Quantitative data are numbers and the result of counting or measuring. Weight, pulse rate and number of people are examples of quantitative data. Quantitative data is further divided into:

- discrete data – this data takes on only certain numerical values and can be counted, for example the number of days.
- continuous data – this data can be measured and include fractions, decimals or irrational numbers, for example time.

Data in its raw form is called ungrouped data. Once the data has been organised using frequencies we call this grouped data. We use different methods to calculate measures of central tendency for ungrouped and grouped data

Measures of central tendency for ungrouped data

We can summarise data by calculating measures of central tendency and dispersion. In this unit we will focus on the measures of central tendency, which are the **mean**, **median** and **mode**.

The mean

The mean (average) and the median are the two measures of central tendency that are used most widely. Of the three measures of central tendency, the mean is most heavily influenced by any outliers or skewness

in data. When there are outliers in the data, the median is often the preferred measure of central tendency because the median is more resistant to outliers than the mean.

To calculate the mean we add all the data values and divide by the total number of values. For example, to calculate your average mark for three maths tests you would add the three marks together and divide by three. We can write this as a formula as shown below.

Mean for ungrouped data:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\bar{x}$ called ' $x$bar' indicates the mean

n is the total number of data values

$\sum$ is the sum of the data

x_i are the data values

Let's look at an example to better understand these measures of central tendency.



Example 1.1

Exam scores for 11 learners are as follows:

$50;\ 58;\ 69;\ 42;\ 63;\ 72;\ 52;\ 80;\ 65;\ 72;\ 91$

1. Calculate the mean mark.
2. How many learners received better than average marks?
3. Adam got 63%he believes that his mark is better than the average because it is greater than 50%. Is he correct? Explain.

Solution

1. Using the formula for the mean, we find the sum of the exam scores and then divide by the number of values.

$$\begin{aligned}\bar{x} &= \frac{\sum_{i=1}^{n} x_i}{n} \\ &= \frac{50 + 58 + 69 + 42 + 63 + 72 + 52 + 80 + 65 + 72 + 91}{11} \\ &= \frac{714}{11} \\ &= 64.9\%\end{aligned}$$

How to use the Casio fx-82ZA Plus to find the mean for ungrouped data:
Step 1: Start with the calculator turned off. Turn the calculator on then set the calculator to statistics (MODE 2) and press 1 for 1 variable statistics. If the calculator has a frequency column showing turn the frequency off by pressing SHIFT, MODE (SET UP), REPLAY down, 3 for statistics then 2 for

'off'. For ungrouped data make sure frequency is turned off.

Step 2: Enter the data. Press 50 then $=58=69=$ and so on until you enter $91=$. If you make a mistake, don't worry! Simply scroll up to the wrong data value and type the correct value over it.

Step 3: After you have finished entering the scores, press AC to indicate the completion of the data entering stage. Don't panic when the scores disappear! The data entering screen will disappear but can be brought back if needed.

Step 4: To bring up the value for the mean press SHIFT 1 then $4:\text{VAR}$ next press $2:\overline{x}=$ and the value of 64.9 is displayed.

2. Five learners scored more than 64.9% on the test.
3. No he is not correct. The average test mark is 64.9% not 50% and with a mark of 63% he scored below the average.



Exercise 1.1

1. AIDS data indicating the number of months a patient with AIDS lives after taking a new antibody drug are as follows:
   $3;\ 4;\ 8;\ 8;\ 10;\ 11;\ 12;\ 13;\ 14;\ 15;\ 15;\ 16;\ 16;\ 17;\ 17;\ 18;\ 21;\ 22;\ 22;\ 24;\ 24;\ 25;\ 26;\ 26;\ 27;\ 27;$
   $29;\ 29;\ 31;\ 32;\ 33;\ 33;\ 34;\ 34;\ 35;\ 37;\ 40;\ 44;\ 44;\ 47$

   Calculate the mean and explain what the statistic you have found represents.
2. Statistics can be used to compare anything, in this case authors. The following shows a simple random sample that compares the letter counts for the first words used by three authors in ten articles they wrote.

   Terry: $7;\ 9;\ 3;\ 3;\ 3;\ 4;\ 1;\ 3;\ 2;\ 2$
   Davis: 3; 3; 3; 4; 1; 4; 3; 2; 3; 1
   Maris: 2; 3; 4; 4; 4; 6; 6; 6; 8; 3

   Calculate the mean letter count for each author.

The full solutions are at the end of the unit.

The median

The median is the middle value of an ordered data set. To find the median, we first sort the data in increasing order and then pick out the value in the middle of the sorted list. The median is the best measure of the centre when a data set contains outliers or extreme values.



Example 1.2

We will once again use the exam scores for 11 learners, which are as follows:

$50;\ 58;\ 69;\ 42;\ 63;\ 72;\ 52;\ 80;\ 65;\ 72;\ 91$

Find the median and state how many learners scored marks below the median value and how many scored marks above the median.

Solution

To find the median, we must first order the data.

$42;\ 50;\ 52;\ 58;\ 63;\ 65;\ 69;\ 72;\ 72;\ 80;\ 91$

You can quickly find the location of the median by using: $\frac{n+1}{2}$

If n is an odd number, the median is the middle value of the ordered data (ordered smallest to largest). If n is an even number, the median is equal to the two middle values added together and divided by two after the data has been ordered.

Since there are an odd number of values the median will be the middle value after ordering the data.

Position of the median is $\frac{11+1}{2} = 6$

$42;\ 50;\ 52;\ 58;\ 63;\ \underline{65};\ 69;\ 72;\ 72;\ 80;\ 91$

Count six places from the beginning of the ordered list and you will find that the sixth value is 65 therefore the median mark is 65.

The median divides the data into two equal parts so 50% of data values lie below the median and 50% of values lie above it. There are five marks below 65 and five marks above 65.



Exercise 1.2

1. Looking again at the same AIDS data which indicate the number of months a patient with AIDS lives after taking a new antibody drug. The data are as follows:
   $3;\ 4;\ 8;\ 8;\ 10;\ 11;\ 12;\ 13;\ 14;\ 15;\ 15;\ 16;\ 16;\ 17;\ 17;\ 18;\ 21;\ 22;\ 22;\ 24;\ 24;\ 25;\ 26;\ 26;\ 27;\ 27;$
   $29;\ 29;\ 31;\ 32;\ 33;\ 33;\ 34;\ 34;\ 35;\ 37;\ 40;\ 44;\ 44;\ 47$
   Calculate the median.
2. Suppose that in a small town of 50 people, one person earns $\mathrm{R}5\ 000\ 000$ per year and the other 49 each earn $\mathrm{R}30\ 000$ a year. Which is the better measure of the 'centre'; the mean or the median? Explain your reasoning.
3. In a sample of 60 houses, one house is worth $\mathrm{R}2\ 500\ 000$. Half of the rest are worth $\mathrm{R}280\ 000$, and all the others are worth $\mathrm{R}350\ 000$. Without doing any calculations, which is the better measure of the "centre" house value; the mean or the median?

The full solutions are at the end of the unit.

The mode

The mode is the most frequent value. There can be more than one mode in a data set. A data set with two modes is called bimodal.

Example 1.3

The number of books checked out from the library by 25 learners are as follows:

$0;\ 0;\ 0;\ 1;\ 2;\ 3;\ 3;\ 4;\ 4;\ 5;\ 5;\ 7;\ 7;\ 7;\ 7;\ 8;\ 8;\ 8;\ 9;\ 10;\ 10;\ 11;\ 11;\ 12;\ 12$

Find the mode.

Solution

It is easier to pick out the mode if the data are ordered. The mode is 7 as it is repeated the most often (4 times).

Exercise 1.3

1. Look at the following set of numbers:
   $6;\ 7;\ 11;\ 10;\ 15;\ 13;\ 14;\ 8$
   a. Notice in this data set there are an even number of values. What should you do first to find the average of these numbers?
   b. Write down the steps you would use to find the median.
   c. Find the mean of the data.
   d. Is it possible to calculate the mode of the data? Will it be a useful measure in this case?
2. What word describes a distribution that has two modes?

The full solutions are at the end of the unit.

Note

To review measures of central tendency and the shape of distributions you can watch this video when you have an internet connection.

Summarizing Distributions: What is Central Tendency (Duration: 10.13)

Measures of dispersion for ungrouped data

Measures of dispersion tell us how spread out a data set is. If a measure of dispersion is small, the data are clustered in a small region. If a measure of dispersion is large, the data are spread out over a large region. In this section we revise the measures of dispersion, the range and interquartile range.

The simplest measure of dispersion is the range. The range is the difference between the maximum and minimum values of the data set.

Quartiles divide data into four equal parts. The quartiles are computed in a similar way to the median. The median is halfway into the ordered data set and is also called the second quartile.

The first quartile (Q_1) is one quarter of the way into the ordered data set; whereas the third quartile (Q_3) is three quarters of the way into the ordered data set. The data must be ordered before you can calculate the quartiles.

The inter-quartile range is the difference between the first and third quartiles of the data set.

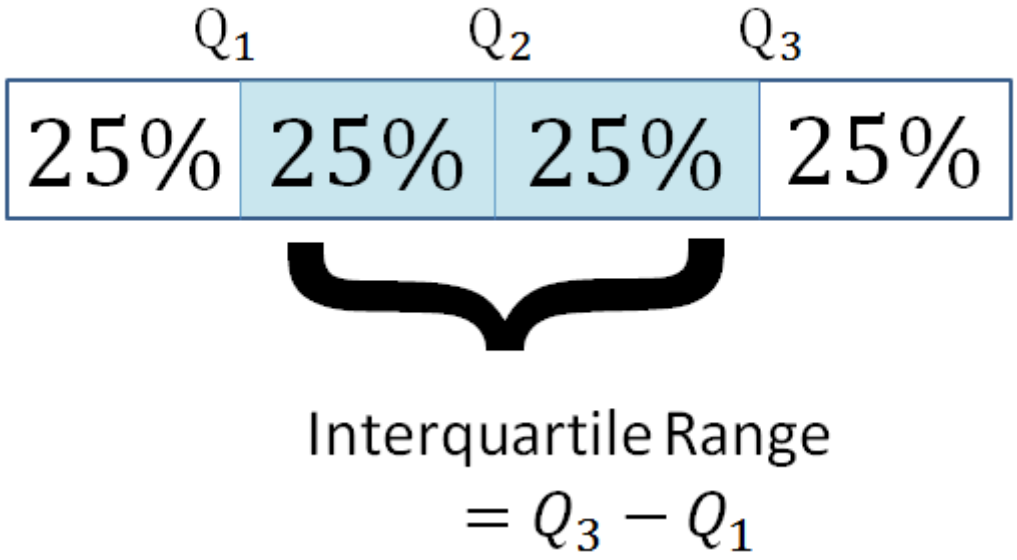


Figure 2: The inter-quartile range

Activity 1.2: Calculate the range and interquartile range (IQR)

Time required: 25 minutes

What you need:

- a pen and paper

What to do:

Answer these questions and show all the necessary steps.

The data below gives the heights (in mm) of seedlings 4 weeks after germinating:
$47;\ 52;\ 56;\ 62;\ 71;\ 74;\ 78;\ 86;\ 89;\ 92;\ 93;\ 95$

1. Calculate the median.
2. Divide the data into an upper half and lower half.
3. Find the median of the lower half of the data. What have you found?
4. Find the median of the upper half of the data. What have you found?
5. How many parts has the data been divided into?
6. Find the difference between the values in 3 and 4 above. What have you found?
7. Between what heights do 50% of the seedlings lie?
8. Find the difference between the tallest and shortest seedling. What do we call this?

What did you find?

1. The data is already arranged from lowest to highest and has 12 entries. To find the position of the median: $\frac{n+1}{2} = \frac{12+1}{2} = 6.5$.
   To find the median we must add the 6th and 7th data values and divide by 2:
   $\text{M} = \frac{74+78}{2} = 76 \text{ mm}$
   The median divides the data into an upper and lower half and is located between 74 and 78.
   $\text{Lower half: } 47;\ 52;\ 56;\ 62;\ 71;\ 74$
   $\text{Upper half: } 78;\ 86;\ 89;\ 92;\ 93;\ 95$
2. We must find the median of the lower half of the data, which is called the first quartile and represented by Q_1. The first quartile (also called the lower quartile) has 25% of data (scores) below it; it is the median of the lower 50% of the data. It is found between the 3rd and 4th values.
   $\begin{aligned}\text{Q}_1 &= \text{first quartile} \\ &= \frac{56+62}{2} \\ &= 59 \text{ mm}\end{aligned}$
3. The median of the upper half of the data is called the upper or third quartile and is represented as Q_3. The third quartile has 75% of the data (scores) below it. It will be located between the 9th and 10 thvalue of the original data set.
   $\begin{aligned}\text{Q}_3 &= \text{third quartile} \\ &= \frac{89+92}{2} \\ &= 90.5 \text{ mm}\end{aligned}$
4. The data has been divided into four parts.
5. We can use the first quartile and third quartile to compute the IQR.
   $\begin{aligned}\text{IQR} &= \text{Q}_3 - \text{Q}_1 \\ &= 90,5 - 59 \\ &= 31.5 \text{ mm}\end{aligned}$
6. 50% of the seedlings are between 59 mm and 90.5 mm in height.
7. $\text{Tallest} - \text{shortest} = 95 - 47 = 48 \text{ mm}$
8. The difference between the largest and smallest values in a data set is called the range.

An outlier is a data point that is significantly different from the other data points. The IQR can help to determine potential outliers.

A value is suspected to be a potential outlier if it is less than $1.5 \times \text{IQR}$ below the first quartile or more than $1.5 \times \text{IQR}$ above the third quartile. Potential outliers always require further investigation, for example, investigating the effect that it has on the data set as a whole.

Exercise 1.4

1. For the following 13 house prices, calculate the IQR and determine if any prices are potential outliers.
   $389\ 950;\ 230\ 500;\ 158\ 000;\ 479\ 000;\ 639\ 000;\ 114\ 950;\ 5\ 500\ 000;\ 387\ 000;$
   $659\ 000;\ 529\ 000;\ 575\ 000;\ 488\ 800;\ 1\ 095\ 000$
2. The table below summarises two data sets of test scores for an evening statistics class and a daytime statistics class.

	Min	(Q_1)	Median	(Q_3)	Maximum
Day	32	56	74.5	82.5	99
Night	25.5	78	81	89	98

   a. Find the IQR for both classes. Compare and comment on the two IQRs.
   b. Are there outliers in either of the classes? Provide reasons for your answer.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the mean for ungrouped data by hand and by using a calculator.
- How to find the median and mode for ungrouped data.
- How to find the range and interquartile range for ungrouped data.

Unit 1: Assessment

Suggested time to complete: 30 minutes

1. Sixty-five randomly selected car salespeople were asked the number of cars they generally sell in one week. Fourteen people answered that they generally sell three cars; nineteen generally sell four cars; twelve generally sell five cars; nine generally sell six cars; eleven generally sell seven cars.
   a. Calculate the sample mean.
   b. Find the median.
   c. Find the mode.
2. Of the three measures, which tends to reflect skewing the most; the mean, the mode, or the median? Why?
3. Sharpe School is applying for a grant that will be used to add fitness equipment to the school gym. The principal surveyed 15 anonymous learners to determine how many minutes a day the learners spend exercising. The results from the learners are shown:

0 minutes; 40 minutes; 60 minutes; 30 minutes; 60 minutes;
10 minutes; 45 minutes; 30 minutes; 300 minutes; 90 minutes;
30 minutes; 120 minutes; 60 minutes; 0 minutes; 20 minutes

a. Determine the following five values of the data:
minimum, first quartile, median, third quartile and maximum.

b. If you were the principal, would you be justified in purchasing new fitness equipment? Calculate the IQR and any potential outliers and note any pitfalls the principal should be aware of as part of your answer.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $$\begin{aligned}\bar{x} &= \frac{3+4+2(8)+10+11+\ldots+44(2)+47}{40} \\ &= \frac{944}{40} \\ &= 23.6\end{aligned}$$
On average a patient will live 23.6 months after taking the new antibody drug.

2. Terry's mean:
$$\frac{7+9+4(3)+4+1+2(2)}{10} = 3.7$$
Davis' mean:
$$\frac{5(3)+2(4)+2(1)+2}{10} = 2.7$$
Maris' mean:
$$\frac{2+2(3)+3(4)+3(6)+8}{10} = 4.6$$

Back to Exercise 1.1

Exercise 1.2

1. To find the median, M, first use the formula for the location. The location is: $\frac{n+1}{2} = \frac{41}{2} = 20.5$. Starting at the smallest value, the median is located between the 20th and 21st values (the two 24s):
$3;\ 4;\ 8;\ 8;\ 10;\ 11;\ 12;\ 13;\ 14;\ 15;\ 15;\ 16;\ 16;\ 17;\ 17;\ 18;\ 21;\ 22;\ 22;\ \underline{24;\ 24};\ 25;\ 26;\ 26;\ 27;\ 27;$
$29;\ 29;\ 31;\ 32;\ 33;\ 33;\ 34;\ 34;\ 35;\ 37;\ 40;\ 44;\ 44;\ 47$
$$\begin{aligned}\text{M} &= \frac{24+24}{2} \\ &= 24\end{aligned}$$

2. Calculate the mean then the median and compare.
$$\begin{aligned}\bar{x} &= \frac{5\ 000\ 000 + 49(30\ 000)}{50} \\ &= 129\ 400\end{aligned}$$
The mean is $\text{R}129\ 400$

$\text{M} = 30\ 000$. The median is $\text{R}30\ 000$ (since there are 49 people who earn $\text{R}30\ 000$ and one person who earns $\text{R}5\ 000\ 000$ and the median is in position 25.5).
The median is a better measure of the centre of the data than the mean because 49 of the values are $\text{R}30\ 000$ and only one is $\text{R}5\ 000\ 000$. The $\text{R}5\ 000\ 000$ is an outlier. The $\text{R}30\ 000$ gives us a better sense of the middle of the data.

3. The median will be a better indication of the middle of the data as most of the data values lie between $\text{R}280\ 000$ and $\text{R}350\ 000$. $\text{R}2\ 500\ 000$ is an outlier and will cause the mean value to be increased and give an incorrect indication of the centre of the data.

Back to Exercise 1.2

Exercise 1.3

1.
   a. Find the sum of the data values.
   b. **Step 1:** Order the data set from least to greatest.
      Step 2: Find the position of the median using the formula $\frac{n+1}{2}$.
      Since there is an even number of items in the data set, the median is found by taking the mean (average) of the two middlemost numbers.

      Step 3: Write down the median.
   c. $$\bar{x} = \frac{6+7+11+10+15+13+14+8}{8}$$
      $$= 10.5$$
   d. There is no mode as no number occurs more than once. The mode will not be useful in this case, the mean and median are equal therefore the distribution would be symmetric.
2. Bimodal.

Back to Exercise 1.3

Exercise 1.4

1. Order the data from smallest to largest value.
   $114\ 950;\ 158\ 000;\ 230\ 500;\ 387\ 000;\ 389\ 950;\ 479\ 000;\ 488\ 800;$
   $529\ 000;\ 575\ 000;\ 639\ 000;\ 659\ 000;\ 1\ 095\ 000;\ 5\ 500\ 000;$

   Find the median.
   $\text{M} = 488\ 800$
   $$\text{Q}_1 = \frac{230\ 500 + 387\ 000}{2}$$
   $$= 308\ 750$$
   $$\text{Q}_3 = \frac{639\ 000 + 659\ 000}{2}$$
   $$= 649\ 000$$

$$\begin{aligned}\text{IQR} &= 649\ 000 - 308\ 750\\ &= 340\ 250\\ 1.5 \times \text{IQR} &= 1.5 \times 340\ 250\\ &= 510\ 375\\ \text{Q}_1 - 1.5 \times \text{IQR} &= 308\ 750 - 510\ 375\\ &= -201\ 625\\ \text{Q}_3 + 1.5 \times \text{IQR} &= 649\ 000 + 510\ 375\\ &= 1\ 159\ 375\end{aligned}$$

No house price is less than $\text{R} - 201\ 625$. However, $\text{R}5\ 500\ 000$ is more than $\text{R}1\ 159\ 375$. Therefore, $\text{R}5\ 500\ 000$ is a potential outlier.

2.

a. The IQR for the day group is:
$$\begin{aligned}\text{Q}_3 - \text{Q}_1 &= 82.5 - 56\\ &= 26.5\end{aligned}$$

The IQR for the evening group is:
$$\begin{aligned}\text{Q}_3 - \text{Q}_1 &= 89 - 78\\ &= 11\end{aligned}$$

The interquartile range (the spread or variability) for the day class is larger than the evening class IQR. This suggests more variation will be found in the day class's test scores.

b. Day class outliers are found using:
$$\begin{aligned}\text{Q}_1 - (1.5)\text{IQR} &= 56 - 1.5(26.5)\\ &= 16.25\\ \text{Q}_3 + (1.5)\text{IQR} &= 82.5 + 1.5(26.5)\\ &= 122.25\end{aligned}$$
Since the minimum and maximum values for the day class are greater than 16.25 and less than 122.25, there are no outliers in the day class.
Evening class outliers are found using:
$$\begin{aligned}\text{Q}_1 - (1.5)\text{IQR} &= 78 - 1.5(11)\\ &= 61.5\\ \text{Q}_3 + (1.5)\text{IQR} &= 89 + 1.5(11)\\ &= 105.5\end{aligned}$$
For the evening class, any test score less than 61.5 is an outlier: this includes the score of 25.5. Since no test score is greater than 105.5, there is no upper end outlier.

Back to Exercise 1.4

Unit 1: Assessment

1.

a.
$$\begin{aligned}\bar{x} &= \frac{14(3) + 19(4) + 12(5) + 9(6) + 11(7)}{65}\\ &= \frac{309}{65}\\ &= 4.75\end{aligned}$$

b. The median is in the 33rd position. Therefore, the median number of cars sold is 4.

c. The mode is 4 as it appears nineteen times.

2. The mean reflects skewness the most as it is calculated using every value in a data set.

3. Order the data first.

a.

$$\begin{aligned}\text{Min} &= 0 \text{ minutes}\\ \text{Q}_1 &= 20\\ \text{Median} &= 40\\ \text{Q}_3 &= 60\\ \text{Max} &= 300\end{aligned}$$

b. Because 75% of the learners exercise for 60 minutes or less daily, and the IQR is 40 minutes, we know that half of the learners surveyed exercise between 20 minutes and 60 minutes daily. This seems a reasonable amount of time spent exercising, so without looking at potential outliers it seems the principal would be justified in purchasing the new equipment.
Next, let's look for outliers. The value 300 appears to be a potential outlier.

$$\begin{aligned}\text{Q}_3 + 1.5(\text{IQR}) &= 60 + (1.5)(40)\\ &= 120\end{aligned}$$

The value 300 is greater than 120 so it is a potential outlier. If we delete it and calculate the five values, we get the following values:

$$\begin{aligned}\text{Min} &= 0\\ \text{Q}_1 &= 20\\ \text{Q}_3 &= 60\\ \text{Max} &= 120\end{aligned}$$

We still have 75% of the learners exercising for 60 minutes or less daily and half of the learners exercising between 20 and 60 minutes a day. However, 15 learners is a small sample and the principal should survey more learners to be more confident in his survey results.

Back to Unit 1: Assessment

Media Attributions

- Fig 1 Weather forecast © https://slidehunter.com is licensed under a CC BY (Attribution) license
- Fig 2 Interquartile range © DHET is licensed under a CC BY (Attribution) license

Unit 2: The five-number summary and box-and-whisker diagram

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Combine measures of central tendency and dispersion to work out the five-number summary.
- Construct the box-and-whisker plot.
- Interpret the box-and-whisker plot.

What you should know

Before you start this unit, make sure you can:

- Calculate central tendencies and dispersion of data and represent data effectively. Revise the following level 2, topic 4 units on data handling:
  - Subject outcome 4.1, unit 1: data collecting.
  - Subject outcome 4.1, unit 2: measures of central tendency of ungrouped data.
  - Subject outcome 4.1, unit 3: measures of dispersion of ungrouped data.
  - Subject outcome 4.2, unit 1: data representation.
  - Subject outcome 4.2, unit 2: bar graphs and histograms.
  - Subject outcome 4.2, unit 3: frequency polygon and line graphs.

Introduction

To have a good understanding of data, and for a better overall picture of what it's telling us, we must combine the measures of central tendency with the measures of dispersion. The **five-number summary** combines a measure of central tendency (the median) with measures of dispersion, the range and the inter-quartile range. A **box-and-whisker-plot** or box plot is a visual display of the five-number summary that helps us see the trends in data.

Box plots show how the quartiles divide the data into sections where each section contains approximately 25% of the data in that set. The five-number summary is listed in the following order: minimum, first quartile, median, third quartile and maximum.

Minimum value: The lowest value in a data set.

Lower quartile: (Q_1): The median of the lower half of an ordered data set.

Median: (Q_2): The median is the middle value of an ordered data set.

Upper quartile: (Q_3): The median of the upper half of an ordered data set.

Maximum value: The highest value in a data set.

Work through Activity 2.1 to get a better understanding of how to represent the five-number summary using a box-and-whisker diagram.

Activity 2.1: Constructing a box plot

Time required: 15 minutes

What you need:

- a pen and paper

What to do:

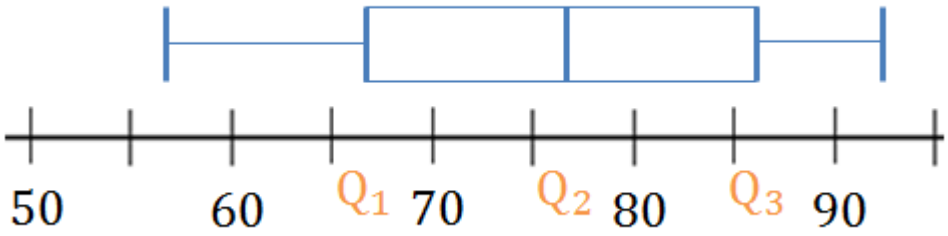
The following data set shows the maths marks of eight learners:

$62;\ 56;\ 71;\ 78;\ 89;\ 92;\ 86;\ 74$

1. Rewrite the data set in ascending order.
2. Find the maximum and minimum marks.
3. Calculate the median mark.
4. Calculate the lower and upper quartiles.
5. Write down the values for the five-number summary.
6. Use a number line to show the information you have calculated. Indicate where the answers to question 5 are located. Draw a box around the IQR and join it to the minimum and maximum values.
7. Indicate if there are any outliers in this data set.

What did you find?

1. The values arranged from lowest to highest: $56;\ 62;\ 71;\ 74;\ 78;\ 86;\ 89;\ 92$
2. The maximum mark is 92 and the minimum mark is 56.
3. The median, also called the second quartile, will lie between the 4th and 5th values.
   $\text{M} = \frac{74 + 78}{2} = 76.$
4. $$\begin{aligned} \text{Q}_1 &= \frac{62 + 71}{2} \\ &= 66.5 \\ \text{Q}_3 &= \frac{86 + 89}{2} \\ &= 87.5 \end{aligned}$$
5. Minimum $= 56$
   First quartile $= 66.5$
   Median $= 76$
   Third quartile $= 87.5$
   Maximum $= 92$
6. Mark off the points above the number line and draw a box around the IQR, join it to the minimum and maximum values to get the box-and-whisker plot from the number line.



The box shows the distance between Q_1 and Q_3 (IQR). A line inside the box shows the median. The lines extending outside the box (the whiskers) show where the minimum and maximum values are located. Note: this graph can be drawn vertically too.

7. To check for outliers use the $1.5 \times \text{IQR}$ rule.

$$\begin{aligned} \text{IQR} &= 87.5 - 66.5 \\ &= 21 \\ 1.5 \times \text{IQR} &= 1.5(21) \\ &= 31.5 \\ Q_1 - 31.5 &= 35 \\ Q_3 + 31.5 &= 119 \end{aligned}$$

There are no outliers in this data set.

Take note!

To identify outliers, **upper** and **lower fences** are used to set limits of data values. The formula for the upper fence is $Q_3 + 1.5 \times \text{IQR}$ and the formula for the lower fence is $Q_1 - 1.5 \times \text{IQR}$ where IQR is the interquartile range. The lower fence is the lower limit and the upper fence is the upper limit. Any value outside of these fences is considered an outlier.

The diagram below shows the main parts of a box and whisker plot.

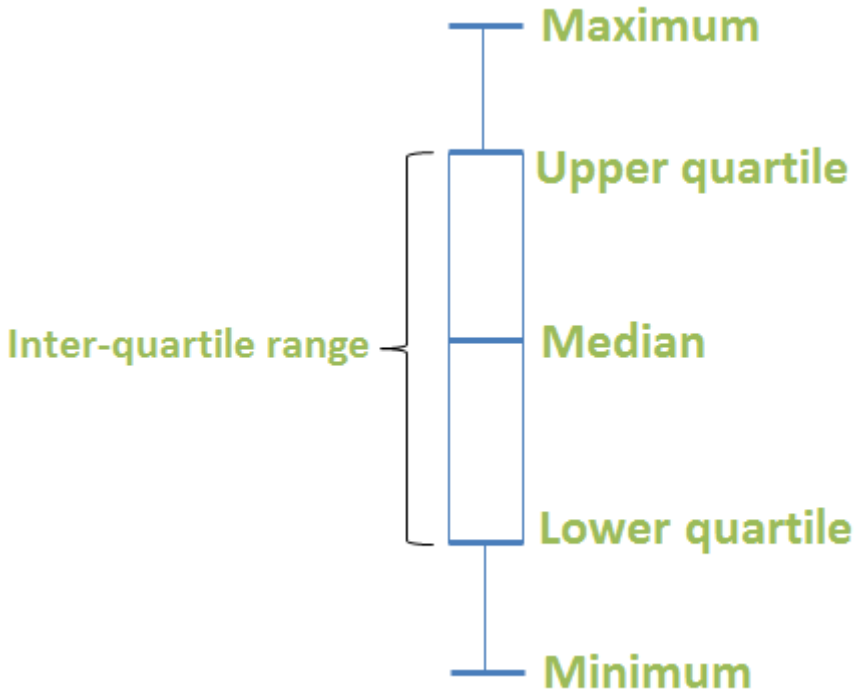


Figure 1: The main parts of a box and whisker plot

Take note!

It is important to start a box plot with a scaled number line. Otherwise the box plot may not be useful.

Example 2.1

Douglas works as a telesales person. He keeps a record of the number of sales he makes each month. The data below show how much he sells each month.

$\{49;\ 70;\ 22;\ 35;\ 68;\ 45;\ 60;\ 48;\ 19;\ 120;\ 43;\ 12\}$

1. Calculate his median number of sales.
2. Give the five-number summary.
3. Draw a box-and-whisker plot of the sales. Find the upper and lower fences and show any outliers on your box-and-whisker plot.

Solution

1. Order the data: $\{12;\ 19;\ 22;\ 35;\ 43;\ 45;\ 48;\ 49;\ 60;\ 68;\ 70;\ 120\}$

$$\begin{aligned} \text{M} &= \frac{45+48}{2} \\ &= 46.5 \end{aligned}$$

2.

$$\begin{aligned} \text{Min} &= 12 \\ \text{Q}_1 &= 28.5 \\ \text{Q}_2\,(\text{median}) &= 46.5 \\ \text{Q}_3 &= 64 \\ \text{Max} &= 120 \end{aligned}$$

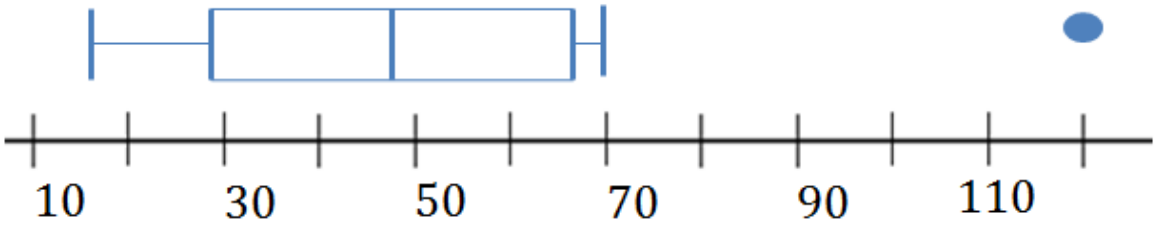
3.

$$\begin{aligned} \text{IQR} &= 64 - 28.5 \\ &= 35.5 \\ 1.5 \times \text{IQR} &= 1.5(35.5) \\ &= 53.25 \end{aligned}$$

$$\text{Lower fence} = \text{Q}_1 - 53.25 = -24.75$$

$$\text{Upper fence} = \text{Q}_3 + 53.25 = 117.25$$

120 is an outlier.



If a data value is very far away from the quartiles, instead of being shown using the whiskers of the box-and-whisker plot, outliers are shown as separately plotted points. The whiskers are then drawn to the minimum and maximum values that are not outliers.

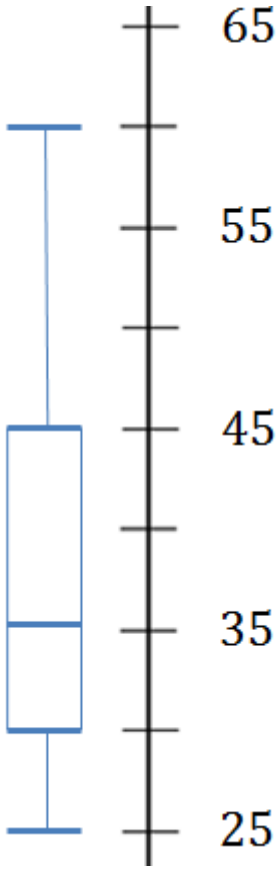
Note

When you have access to the internet watch this video, which explains how to draw a box-and-whisker plot for a data set.

Constructing a box and whisker plot (Duration: 08.18)

Example 2.2

Determine the five-number summary and inter-quartile range from the box-and-whisker plot below.



Solution

We can read off the values for the five-number summary from the box plot.

$$\begin{aligned} \text{Min} &= 25 \\ \text{Q}_1 &= 30 \\ \text{Q}_2\,(\text{median}) &= 35 \\ \text{Q}_3 &= 45 \\ \text{Max} &= 60 \end{aligned}$$

The upper quartile minus the lower quartile gives the inter-quartile range.

$$\begin{aligned} \text{IQR} &= 45 - 30 \\ &= 15 \end{aligned}$$

Exercise 2.1

1. The stem-and-leaf diagram below shows the pulse rate per minute of ten learners.

7	8
8	1 3 5 5
9	0 1 1
10	3 5

a. Determine the mean and the range of the data.

b. Give the five-number summary and create a box plot for the data.

2. The following is a list of data: $3;\ 8;\ 8;\ 5;\ 9;\ 1;\ 4;\ x$
In each case, determine the value of x if the:

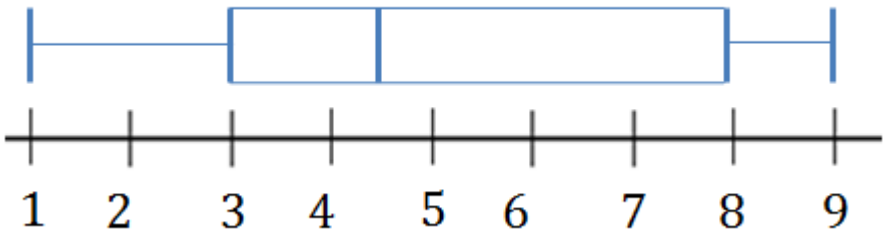
a. range $= 16$

b. mode $= 8$

c. median$= 6$

d. mean$= 6$

e. box plot looks as follows:



3. Given the following data set:
$1.25\ ;\ 1.5\ ;\ 2.5\ ;\ 2.5\ ;\ 3.1\ ;\ 3.2\ ;\ 4.1\ ;\ 4.25\ ;\ 4.75\ ;\ 4.8\ ;\ 4.95\ ;\ 5.1$

a. Draw the box plot.

b. Find the upper and lower fences of the data.

The full solutions are at the end of the unit.

Activity 2.2: Interpreting the box-and whisker plot

Time required: 20 minutes

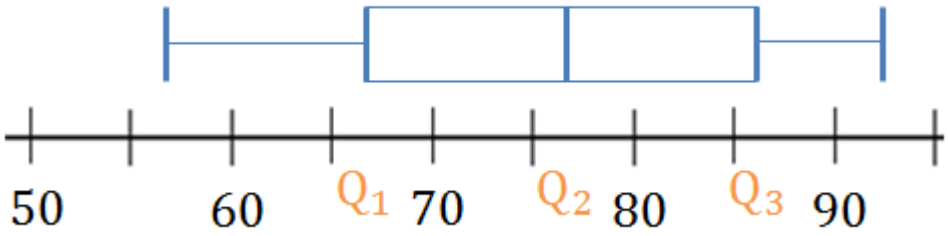
What you need:

- a pen and paper

What to do:

In Activity 2.1 you saw that the marks of eight learners gave the following box plot:

1.



What percentage of learners scored below 66.5?

2. How many learners scored below 66.5?
3. Between what two marks does the middle $50\%$of the data lie?
4. What mark did 50% of learners score less than?
5. Complete this sentence: Twenty-five percent of learners got a mark greater than_____.
6. What percentage of learners scored below 87.5?
7. Comment on the symmetry of the data values.
8. Comment on the variability of the data.

What did you find?

1. Twenty-five percent of scores fall below the lower quartile value.
2. $0.25 \times 8 = 2$ So 2 learners got less than 66.5.
3. The IQR shows the middle 50% of scores, therefore 50% of the data lies between 66.5 and 87.5.
4. Half the scores are greater than or equal to the median and half are less than the median. 50% of learners got less than 76.
5. Twenty-five percent of learners got a mark greater than 87.5.
6. Seventy-five percent of learners scored less than 87.5.
7. The data appears to be more or less symmetric as the median lies close to the middle of the box.
8. Consider the range and IQR. The range is easily influenced by extreme values and outliers so it is less reliable than the IQR, which takes into account only the middle 50% of the data.

$$\begin{aligned}\text{IQR} &= 87.5 - 66.5 \\ &= 21 \\ \text{Range} &= 92 - 56 \\ &= 36\end{aligned}$$

Low variability is ideal because it means that you can make better predictions about the population based on sample data. High variability means that the values are less consistent, so it's more difficult to make predictions.
We can say that the marks are moderately variable.

When data are skewed, the majority of the data values are located either on the high or low ends. The shape of a box-and-whisker plot will show if a data set is normally distributed or skewed.

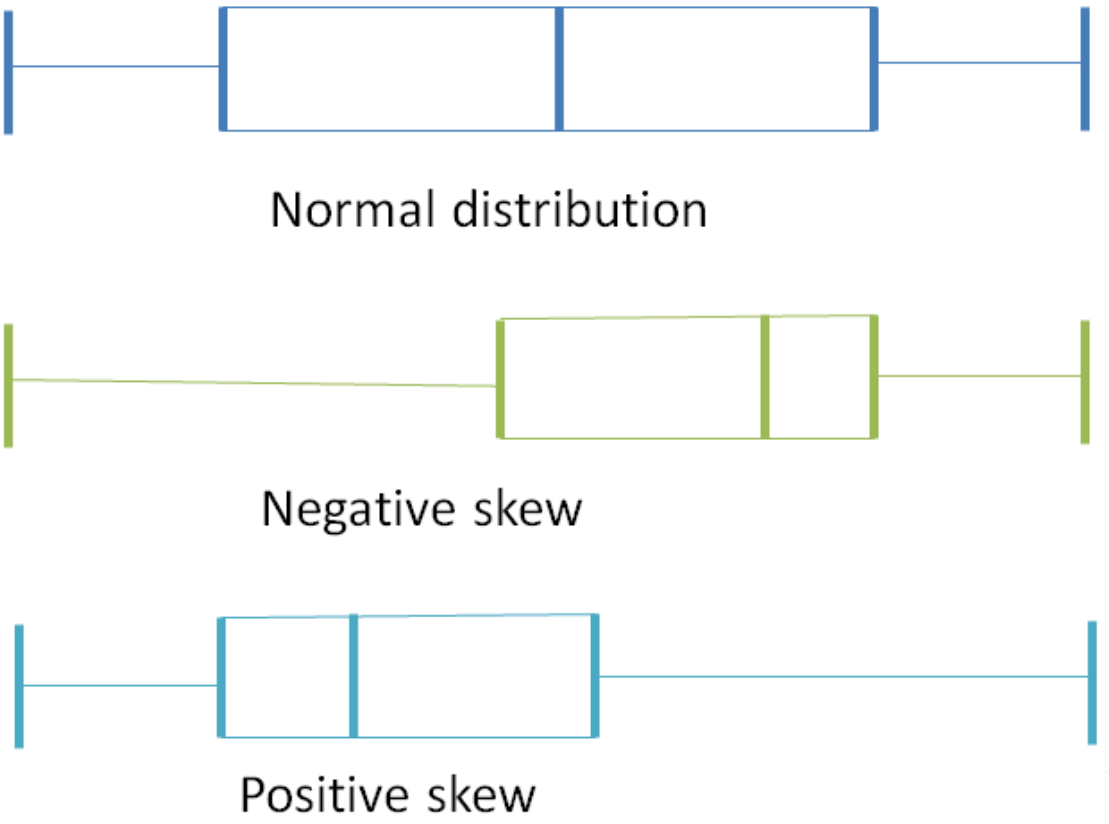


Figure 2: Box-and-whisker plots showing if a data set is normally distributed or skewed

When the median is in the middle of the box, and the whiskers are about the same length on either side of the box, then the distribution is symmetric or normal.

When the median is pulled toward the upper quartile, and the whisker is shorter on the upper end of the box, then the distribution is negatively skewed (skewed left).

When the median is pulled toward the lower quartile, and the whisker is shorter on the lower end of the box, then the distribution is positively skewed (skewed right).

Note

For more on interpreting box plots watch this video called “Interpreting box plots”.

Interpreting box plots (Duration: 07.43)

Summary

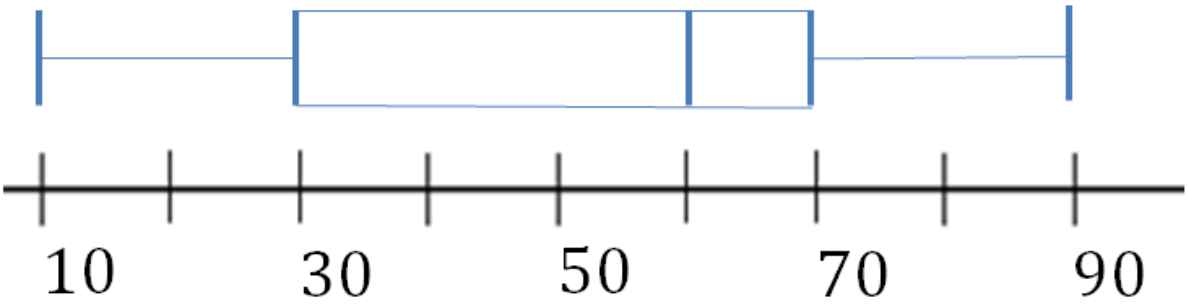
In this unit you have learnt the following:

- How to define the five-number summary.
- How to represent the five-number summary using a box-and-whisker plot.
- How to interpret a box plot.
- How the median is affected by negatively and positively skewed data.

Unit 2: Assessment

Suggested time to complete: 30 minutes

1. Two mathematics classes, A and B, are in competition to see which class performed best in the June examination. The marks of the learners in class A are given below and the box-and-whisker plot for class B illustrates the results of class B. Both classes have 25 learners. Marks are given as percentages.
   Marks of class A:
   9; 14; 14; 19; 21; 23; 33; 35; 37; 37; 42; 45; 55; 56; 57; 59; 68; 75; 75; 75; 77; 78; 80; 81; 92
   The box and whisker diagram for the learners in class B is:



   a. Write down the five-number summary for class A.
   b. Are there any outliers in the data for class A? Explain.
   c. Draw the box-and-whisker diagram (box plot) for class A. Show all relevant values.
   d. Determine which class did better in the June Examination and give reasons for your conclusion.
2. The ages of 28 people whose birthday coincides with that of one of their children are shown below.
   78; 53; 70; 97; 37; 68; 48; 35; 71; 63; 47; 60; 63; 58;
   74; 39; 67; 64; 42; 52; 38; 54; 60; 75; 69; 77; 65; 72
   a. Determine the median age of the group by first determining its position.
   b. Determine the upper and lower quartiles of the data by first determining their positions.
   c. Determine the upper and lower fence values.
   d. Construct a box-and-whisker diagram for the above information showing any outliers.
3. Test scores for a college statistics class held during the day are:
   99; 56; 78; 55.5; 32; 90; 80; 81; 56; 59; 45; 77; 84.5; 84; 70; 72; 68; 32; 79; 90
   Test scores for a college statistics class held during the evening are:
   98; 78; 68; 83; 81; 89; 88; 76; 65; 45; 98; 90; 80; 84.5; 85; 79; 78; 98; 90; 79; 81; 25.5
   a. For each data set, what percentage of the data is between:
      - the smallest value and the first quartile?

- the first quartile and the median?
- the median and the third quartile?
- the third quartile and the largest value?
- the first quartile and the largest value?

b. Create a box plot for each set of data. Use one number line for both box plots.

c. Which box plot has the widest spread for the middle 50% of the data? What does this mean for that set of data in comparison to the other set of data?

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

a.

$$\begin{aligned}\bar{x} &= \frac{78+81+83+2(85)+90+(2)91+103+105}{10} \\ &= \frac{892}{10} \\ &= 89.2 \\ \text{Range} &= 105-78 \\ &= 27\end{aligned}$$

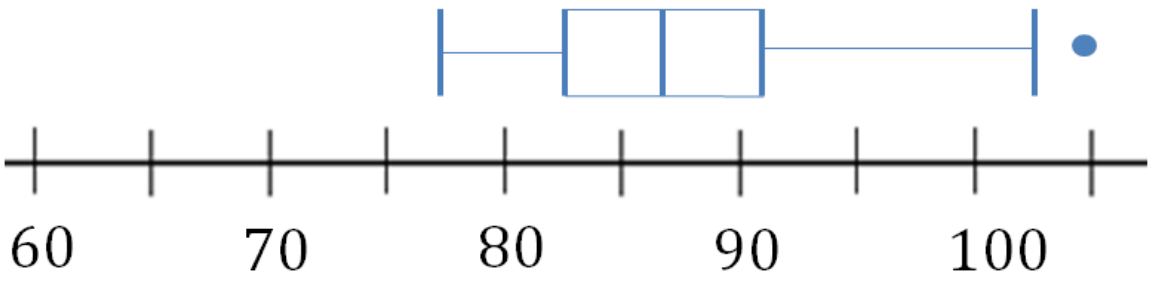
b. Five number summary:

$$\begin{aligned}&\{78;\ 81;\ 83;\ 85;\ 85;\ 90;\ 91;\ 91;\ 103;\ 105\} \\ \text{Min} &= 78 \\ \text{Q}_1 &= 83 \\ \text{Median} &= 87.5 \\ \text{Q}_3 &= 91 \\ \text{Max} &= 105\end{aligned}$$

There is an outlier in this data set:

$$\begin{aligned}\text{IQR} &= 91-83 \\ &= 8 \\ \text{Upper fence} &= \text{Q}_3+1.5(\text{IQR}) \\ &= 91+1.5(8) \\ &= 103\end{aligned}$$

105 is an outlier.



2.

a.

$$x - 1 = 16$$
$$x = 17$$

b. $x \in \mathbb{Z},\ x \neq 1; 3; 4; 5; 9$

x can be any integer value that will not change the mode of 8 so we must exclude all other values in the data set.

c. The median will be between positions 4 and 5

$$\frac{5 + x}{2} = \text{M}$$
$$x = 6 \times 2 - 5$$
$$= 7$$

d.

$$\text{Mean} = \frac{1 + 3 + 4 + 5 + 8 + 8 + 9 + x}{8}$$
$$= 6$$
$$\frac{x + 38}{8} = 6$$
$$x = 10$$

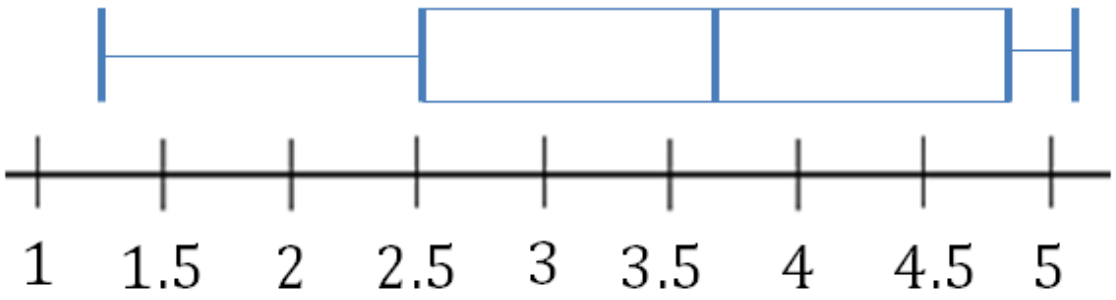
e.

$$\text{Median} = 4.5$$
$$\frac{x + 5}{2} = 4.5$$
$$x = 4$$

3.

a.

$$\text{Min} = 1.25$$
$$\text{Q}_1 = 2.5$$
$$\text{Q}_2 = 3.65$$
$$\text{Q}_3 = 4.78$$
$$\text{Max} = 5.1$$



b.

$$\text{IQR} = 4.78 - 2.5$$
$$= 2.28$$
$$\text{Lower fence} = \text{Q}_1 - 1.5(2.28)$$
$$= 2.5 - 1.5(2.28)$$
$$= -0.92$$
$$\text{Upper fence} = \text{Q}_3 + 1.5(2.28)$$
$$= 4.78 + 1.5(2.28)$$
$$= 8.2$$

Back to Exercise 2.1

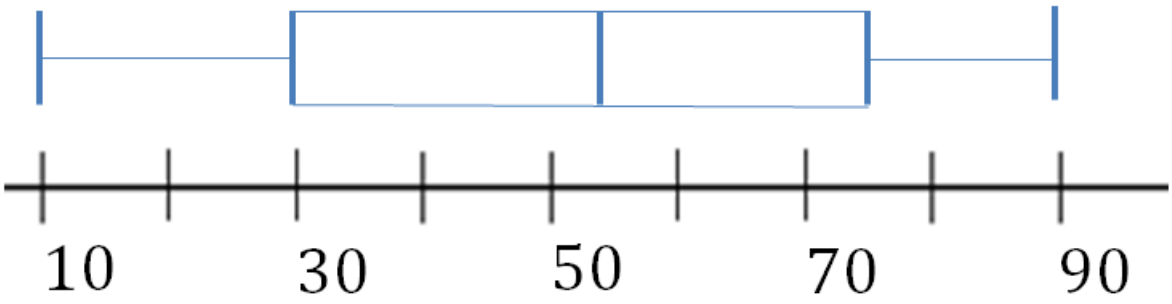
Unit 2: Assessment

1.

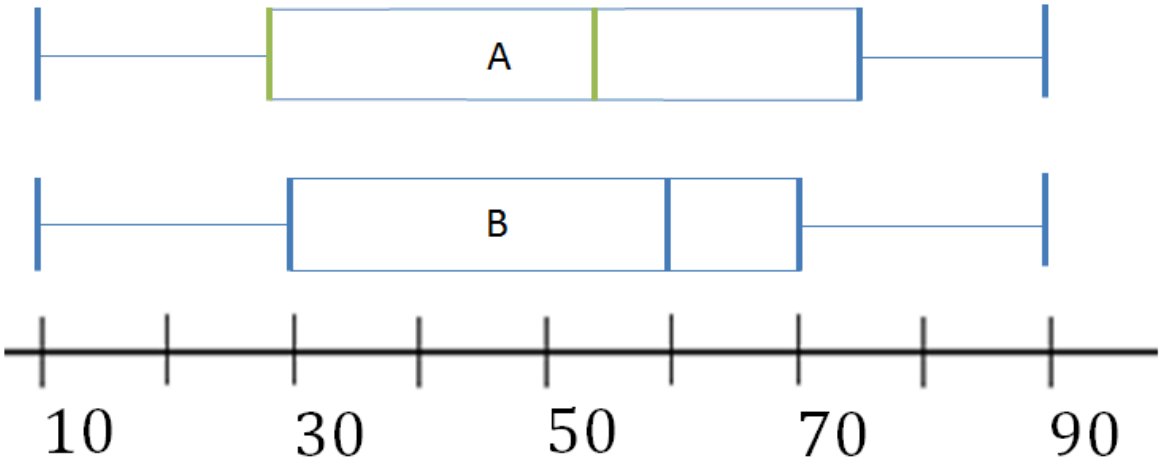
a. $\text{min} = 9;\ \text{first quartile} = 28;\ \text{median} = 55;\ \text{third quartile} = 75;\ \text{max} = 92$

b. All values fall within the lower and upper fences so there are no outliers.

c.



d. Compare the box-and-whisker plots for the classes.



Class B did marginally better than class A. Its median is 60 while class A's is 55. Furthermore, the first quartile for class B is higher than that of class A.

2.

a. First sort the data then calculate the position of the median age.
$35;\ 37;\ 38;\ 39;\ 42;\ 47;\ 48;\ 52;\ 53;\ 54;\ 58;\ 60;\ 60;\ 63$
$63;\ 64;\ 65;\ 67;\ 68;\ 69;\ 70;\ 71;\ 72;\ 74;\ 75;\ 77;\ 78;\ 97$
The median will be between positions 14 and 15.
$\text{Median age } = 63$

b. The lower quartile lies between positions 7 and 8.

$$\begin{aligned} Q_1 &= \frac{48 + 52}{2} \\ &= 50 \end{aligned}$$

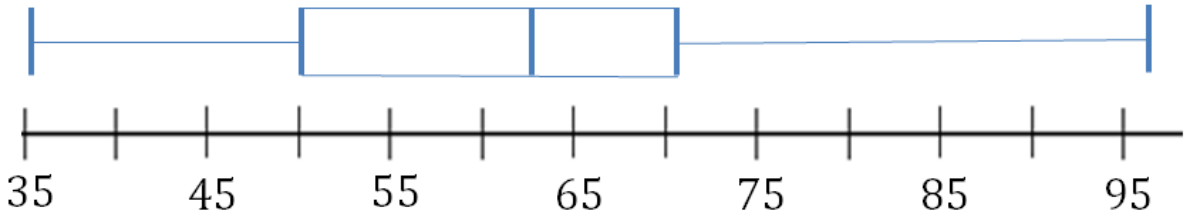
The upper quartile lies between positions 21 and 22.

$$\begin{aligned} Q_3 &= \frac{70 + 71}{2} \\ &= 70.5 \end{aligned}$$

c.

$$\begin{aligned}\text{IQR} &= 70.5 - 50 \\ &= 20.5 \\ \text{Upper fence} &= \text{Q}_3 + 1.5(\text{IQR}) \\ &= 70.5 + 1.5(20.5) \\ &= 101.25 \\ \text{Lower fence} &= \text{Q}_1 - 1.5(\text{IQR}) \\ &= 50 - 1.5(20.5) \\ &= 19.25\end{aligned}$$

d.



There are no outliers.

3.

a. Order the data for each group.
Day class:
$32;\ 32;\ 45;\ 55.5;\ 56;\ 56;\ 59;\ 68;\ 70;\ 72;\ 77;\ 78;\ 79;\ 80;\ 81;\ 84;\ 84.5;\ 90;\ 90;\ 99$

There are twenty data values for the day class and the five-number-summary is:

$$\begin{aligned}\text{Min} &= 32 \\ \text{Q}_1 &= 56 \\ \text{Q}_2 &= 74.5 \\ \text{Q}_3 &= 82.5 \\ \text{Max} &= 99\end{aligned}$$

- There are six data values ranging from the minimum to the first quartile so 30% of data values lie between 32 and 56.
- There are six data values ranging from 56 to 74.5 so 30% of the data values lie between the first quartile and the median.
- There are five data values ranging from 74.5 to 82.5 so 25% of data values lie between the median and the third quartile.
- There are five data values ranging from 82.5 to 99, which is 25% of the data.
- There are 15 data values between the first quartile, 56, and the largest value, 99, this is 75% of the data.

Night class:
$25.5;\ 45;\ 65;\ 68;\ 76;\ 78;\ 78;\ 79;\ 79;\ 80;\ 81;\ 81;\ 83;\ 84.5;\ 85;\ 88;\ 89;\ 90;\ 90;\ 98;\ 98;\ 98$

There are twenty-two data values for the evening class and the five-number-summary is:

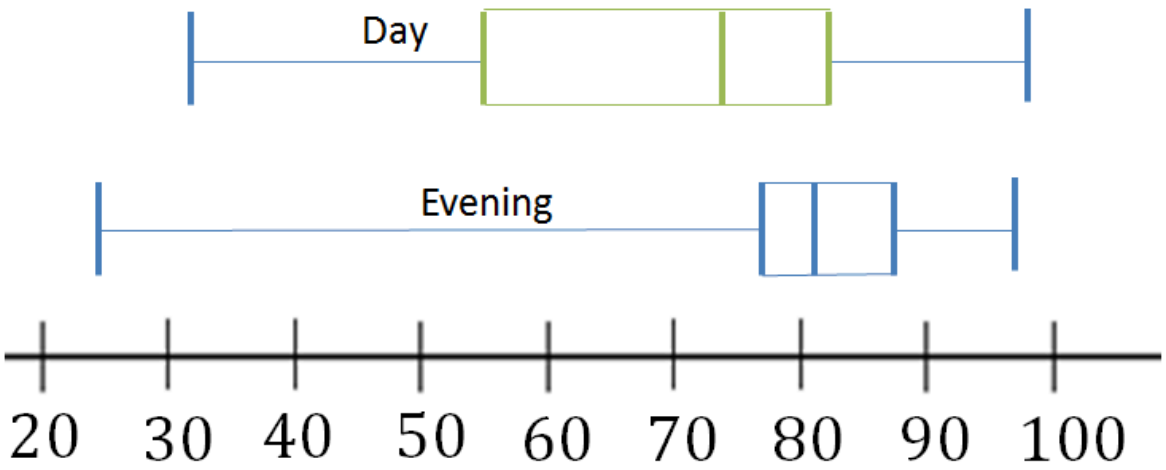
$$\begin{aligned}\text{Min} &= 25.5 \\ \text{Q}_1 &= 78 \\ \text{Q}_2 &= 81 \\ \text{Q}_3 &= 89 \\ \text{Max} &= 98\end{aligned}$$

- There are six data values ranging from the minimum to the first quartile so approximately 27%

of data values.

- There are five data values ranging from the first quartile to the median so approximately 23% of data values.
- There are six data values ranging from the median to the third quartile so approximately 27% of data values.
- There are five data values ranging from the third quartile to the maximum so approximately 23% of data values.
- There are sixteen data values ranging from the first quartile to the maximum so approximately 73% of data values.

b.



c. The day class has the wider spread for the middle 50% of the data. The IQR for the day class data is greater than the IQR for the evening class. This means that there is more variability in the day class data.

Back to Unit 2: Assessment

Media Attributions

- Fig 1_Activity 2.1 Box and whisker © DHET is licensed under a CC BY (Attribution) license
- Fig 2_Key parts of box and whisker diagram © DHET is licensed under a CC BY (Attribution) license
- Fig 3_Example 2.1_Draw box and whisker © DHET is licensed under a CC BY (Attribution) license
- Fig 4_Example 2.2 Interpret Box and whisker © DHET is licensed under a CC BY (Attribution) license
- Fig 5_Ex 2.1 Q1eBox and whisker © DHET is licensed under a CC BY (Attribution) license
- Fig 8_Shape of box plots © DHET is licensed under a CC BY (Attribution) license
- Fig 9_Assessment Q1 © DHET is licensed under a CC BY (Attribution) license
- Fig 6_Exercise 2.1 Q1 Ans b © DHET is licensed under a CC BY (Attribution) license
- Fig 7_Ex 2.1 Q3Ans © DHET is licensed under a CC BY (Attribution) license
- Fig10_Assessment Q1c © DHET is licensed under a CC BY (Attribution) license
- Fig 11 Assessment Q1d © DHET is licensed under a CC BY (Attribution) license
- Fig 12_Assessment Ans 2d © DHET is licensed under a CC BY (Attribution) license
- Fig 13_Assessment Ans 3b © DHET is licensed under a CC BY (Attribution) license

SUBJECT OUTCOME XII

DATA HANDLING: CALCULATE, REPRESENT AND INTERPRET MEASURES OF CENTRAL TENDENCY AND DISPERSION IN UNIVARIATE NUMERICAL GROUPED DATA

Subject outcome

Subject outcome 4.2: Calculate, represent and interpret measures of central tendency and dispersion in univariate numerical grouped data.

Learning outcomes

- Construct a frequency distribution table by grouping data into classes.
- Calculate the cumulative frequency and plot the ogive curve.
- Use the ogive curve to estimate quartile values.
- Construct histograms using tabulated grouped data.
- Calculate the mean ($\bar{x}$), median (M_e) and modal (M_o) values of grouped data using the formulae:
  - $\bar{x} = \frac{\sum f_i x_i}{n}$
  - $M_e = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times C$
  - $M_o = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c$

Unit 1 outcomes

By the end of this unit you will be able to:

- Construct a frequency distribution by grouping data into classes.

Unit 2 outcomes

By the end of this unit you will be able to:

- Work out class intervals.
- Plot the ogive.
- Interpret the ogive.

Unit 3 outcomes

By the end of this unit you will be able to:

- Construct a histogram.
- Interpret a histogram.

Unit 4 outcomes

By the end of this unit you will be able to:

- Calculate the mean of grouped data using the formula $\bar{x} = \frac{\sum f_i x_i}{n}$.
- Calculate the median of grouped data using the formula $\mathrm{M}_e = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c$.
- Calculate the modal value of grouped data using the formula $M_o = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c$.

Unit 1: Grouping data using frequency distributions

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Construct a frequency distribution by grouping data into classes.

What you should know

Before you start this unit, make sure you can:

- Work with data and represent data effectively. You can revise the following subject outcomes in Maths level 2:
  - Subject outcome 4.1: Calculate central tendencies and dispersion of data.
  - Subject outcome 4.2: Represent data effectively.

Introduction

To make sense of data we need to condense it or organise it in a way that makes interpretation easier. We have worked with ungrouped data, which is unorganised or raw data. Now, we will use frequencies to organise and group data. There are different methods to calculate statistics for ungrouped and grouped data. But, before you summarise data, it is important to understand how to organise data using frequency distributions. By counting how often data values appear we can make a frequency distribution.

Frequency is how often something occurs. A **frequency distribution**, also called a frequency table, is used to organise qualitative and quantitative raw data

Work through Activity 1.1 to learn how to organise data using frequency tables.

Activity 1.1: Organising data using a frequency distribution

Time required: 25 minutes

What you need:

- a pen and paper

What to do:

Answer the questions based on the given data. Show all the necessary steps.

The following data on the heights (in cm) of netball players has been collected.

$$
\begin{array}{cccccccccccccccc}
165 & 148 & 158 & 150 & 160 & 165 & 150 & 156 & 155 & 164 & 162 & 160 & 158 & 148 & 158 \\
140 & 146 & 160 & 148 & 152 & 139 & 165 & 148 & 160 & 156 & 158 & 170 & 155 & 160 & 148 \\
155 & 158 & 179 & 170 & 158 & 161 & 155 & 160 & 163 & 178 & 138 & 172 & 170 & 156 & 160 \\
160 & 171 & 140 & 160 & 170 & 175 & 148 & 170 & 177 & 155 & 167 & 154 & 160 & 170 & 155 \\
136 & 179 & 150 & 167 & 148 & 160 & 164 & 167 & 157 & 165 & 163 & 140 & 162 & 178 & 160 \\
170 & 163 & 162 & 165 & 175 & 165 & 152 & 147 & 180 & 148 & 170 & 165 & 167 & 165 & 165
\end{array}
$$

1. How many netball players were surveyed?
2. Is there a simple way to organise the data?
3. Rewrite the data values from smallest to biggest.
4. How many netball players are between $146 - 155$ cm in height?
5. Group the heights in intervals of 9 cm starting with a first interval of $136 - 145$ cm. Draw a table to show the number of netball players that fall within each interval. Label the first column 'height' and the second column 'frequency'.
6. Can the intervals overlap? Give a reason for your answer.
7. What is the sum of the frequencies equal to?
8. From the table you drew for question 5, how many players were shorter than 166 cm?

What did you find?

1. There were 90 players surveyed.
2. Even answering a simple question such as 'how many netball players were surveyed?' is a long process and can lead to mistakes using ungrouped data. We must organise the data to make it easier to work with. We can arrange these values by rewriting and grouping heights together using intervals.
3. It is very time consuming to order large sets of raw data and in reality there is computer software that can do this work for us.

   $$
   \begin{array}{cccccccccccccccc}
   136; & 138; & 139; & 140; & 140; & 140; & 146; & 147; & 148; & 148; \\
   148; & 148; & 148; & 148; & 148; & 148; & 150; & 150; & 150; & 152; \\
   152; & 154; & 155; & 155; & 155; & 155; & 155; & 155; & 156; & 156; \\
   156; & 157; & 158; & 158; & 158; & 158; & 158; & 158; & 160; & 160; \\
   160; & 160; & 160; & 160; & 160; & 160; & 160; & 160; & 160; & 160; \\
   161; & 162; & 162; & 162; & 163; & 163; & 163; & 164; & 164; & 165; \\
   165; & 165; & 165; & 165; & 165; & 165; & 165; & 165; & 167; & 167; \\
   167; & 167; & 170; & 170; & 170; & 170; & 170; & 170; & 170; & 170; \\
   171; & 172; & 175; & 175; & 177; & 178; & 178; & 179; & 179; & 180
   \end{array}
   $$
4. There are 22 players between $146 - 155$ cm in height.
5.

Height intervals (in cm)	Frequency
$136 - 145$	6
$146 - 155$	22
$156 - 165$	40
$166 - 175$	16
$176 - 185$	6

6. No the intervals cannot overlap. You cannot have a height that falls into two different intervals as you will be double counting.
7. The total frequency is the same as the total number of observations. $\text{Sum of frequency} = 90$.
8. We must add the frequencies for the intervals $136 - 145$; $146 - 155$ and $156 - 165$. There are 68 players shorter than 166 cm.

Activity 1.1 showed how to construct a frequency distribution. We also saw that it is much easier to answer questions about data from a frequency table rather than using ungrouped data. As the size of the data set grows it will become even more difficult to handle in its raw form, so we need to organise data before analysing it.



Take note!

Summarising the data into class intervals leads to a loss of detail, in other words we cannot return to the original data set from the grouped intervals. In Activity 1.1, for example, you cannot tell how many players had a height of exactly 160 cm from the grouped frequency table.

Constructing a frequency distribution

Things to note when creating class intervals for a frequency distribution:

- Create equal intervals. First, decide how many class intervals you would like to use then calculate the range of data values. Next, divide the range by the number of classes to get an approximate interval width.
- The first class (lower limit) should start below the smallest observation.
- Classes must not overlap and must cover the entire range of the data set.

In the next example you will learn how to create class intervals.



Example 1.1

The following data shows the ages of people attending a public reading by an author at a library.

$10;\ 10;\ 11;\ 12;\ 12;\ 13;\ 13;\ 14;\ 14;\ 14;\ 15;\ 15;\ 15;\ 16;\ 16;\ 16;\ 17;\ 17;\ 17;\ 17;$
$18;\ 18;\ 18;\ 18;\ 18;\ 19;\ 19;\ 19;\ 19;\ 19;\ 19;\ 20;\ 20;\ 21;\ 22;\ 22;\ 22;\ 22;\ 22;\ 22;$
$23;\ 25;\ 25;\ 25;\ 26;\ 26;\ 27;\ 30$

Construct a frequency distribution for the data.

Solution

There are 48 observations. The minimum age is 10 and the maximum age is 30 therefore the range of data values is 20. If we would like to create 7 class intervals, then the equal interval widths will be 3 ($20 \div 7 = 3$).

We start the first class interval below the smallest observation so we can start at 9 and go up in intervals of 3. You can use a variable to represent age, in this example we have used x.

Age intervals	**Frequency**
$9 < x \leq 12$	5
$12 < x \leq 15$	8
$15 < x \leq 18$	12
$18 < x \leq 21$	9
$21 < x \leq 24$	7
$24 < x \leq 27$	6
$27 < x \leq 30$	1
Total	48

You will notice that we include the upper limit of each interval in the lower class and exclude it from the next class. For example, in the first interval we have included 12 $(9 < x \leq 12)$ as the upper limit and we then exclude it $(12 < x \leq 15)$ as the lower limit of the next interval.

In the frequency column we count the number of observations that fall into each interval and list it. Make sure that the sum of the frequency column is equal to the total number of observations.



Exercise 1.1

The following shows a sample of raw data collected on the number of customers entering a printing shop over a period of time.

14	14	15	24	22	26
20	22	14	11	13	24
23	11	12	12	23	16
12	18	20	21	20	15
21	16	12	23	22	14

1. Construct a frequency table for the data given.

2. For how many days were customers observed?
3. The shop owner is deciding if he should close the shop down. He will only stay open for business if he has 16 or more customers on 50% of days. Based on the frequency table, should he stay open or close down?

The full solutions are at the end of the unit.

Note

When you have an internet connection watch this video called "Frequency tables and dot plots" to learn more about frequency distributions.

Frequency tables and dot plots (Duration: 07.18)

Cumulative frequencies

Some graphs that represent data from a grouped frequency distribution need cumulative frequencies. Cumulative means to add up or collect values together.

You can think of cumulative frequency as the running total of frequencies in a frequency distribution. This means you are adding a value to all of the values that came before it.

To find the cumulative frequencies, add all the previous frequencies to the frequency for the current row, as shown in the Table 1.1.

Table 1.1 Cumulative frequency of ages

Age intervals	Frequency	Cumulative frequency
$9 < x \leq 12$	5	5
$12 < x \leq 15$	8	$5 + 8 = 13$
$15 < x \leq 18$	12	$13 + 12 = 25$
$18 < x \leq 21$	9	$25 + 9 = 34$
$21 < x \leq 24$	7	$34 + 7 = 41$
$24 < x \leq 27$	6	$41 + 6 = 47$
$27 < x \leq 30$	1	$47 + 1 = 48$
Total	48	

The last value in the cumulative frequency column will always be equal to the total sum of all observations, since the sum of all frequencies will already have been added to the previous total.

Cumulative frequency graphs are often used to represent and compare data. You will see more of this in unit 2 of this subject outcome.

Summary

In this unit you have learnt the following:

- How to group data using frequencies.
- How to construct a frequency distribution.
- How to create equal class intervals for a frequency distribution.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. The maths marks, out of 50, for 35 learners are given below:

   $$46;\ 40;\ 12;\ 10;\ 47;\ 23;\ 26;\ 8;\ 29;\ 34;\ 37;\ 17;\ 40;\ 50;\ 18;\ 23;\ 33;$$
   $$23;\ 24;\ 15;\ 35;\ 23;\ 19;\ 22;\ 28;\ 35;\ 27;\ 42;\ 29;\ 26;\ 46;\ 33;\ 27;\ 19;\ 28$$

   Organise the data by using a frequency table.
2. The frequency table shows the test marks for a statistics course. Find the missing values.

Test score intervals	Frequency	Cumulative frequency
$49.5 < x \leq 59.5$	5	A
$59.5 < x \leq 69.5$	10	15
$69.5 < x \leq 79.5$	B	45
$79.5 < x \leq 89.5$	40	85
$89.5 < x \leq 99.5$	15	C

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

Intervals for the number of customers	Frequency
$10 - 15$	13
$16 - 21$	8
$22 - 27$	9

2. 30 days
3. He has 16 or more customers on 17 days, which is more than 50% of the total number of days observed so he should not close his shop down.

Back to Exercise 1.1

Unit 1: Assessment

1.

Score intervals	Frequency
$0 - 10$	2
$11 - 20$	6
$21 - 30$	14
$31 - 40$	8
$41 - 50$	5

2.

Test score intervals	Frequency	Cumulative frequency
$49.5 < x \leq 59.5$	5	$\text{A} = 5$
$59.5 < x \leq 69.5$	10	15
$69.5 < x \leq 79.5$	$\text{B} = 45 - 15 = 30$	45
$79.5 < x \leq 89.5$	40	85
$89.5 < x \leq 99.5$	15	$\text{C} = 100$

Back to Unit 1: Assessment

Unit 2: Construct the ogive

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Work out class intervals.
- Plot the ogive.
- Interpret the ogive.

What you should know

Before you start this unit, make sure you can:

- Construct a frequency distribution. Work through unit 1 of this subject outcome if you need to revise frequency distributions.

Introduction

In unit 1 of this subject outcome we learnt that when consecutive frequencies are added together we get cumulative frequencies. A cumulative frequency can be plotted as its own graph, called the ogive (pronounced 'o-jive') or the cumulative frequency graph. We use the data and intervals from a grouped frequency distribution to plot the ogive. As you will see it is easier to get information by looking at an ogive rather than working through the data in a frequency table.

The ogive is used to approximate the number of observations less than or equal to a specific value and is useful when you want to see what has happened up to a particular point. The following activity will take you through the process of constructing an ogive.

Activity 2.1: Use a frequency distribution to draw an ogive

Time required: 20 minutes

What you need:

- a pen and paper

What to do:

Below is the cumulative frequency table of the ages of people attending a public reading by an author at a library.

Class intervals	Class midpoint	Frequency	Cumulative frequency
$9 < x \leq 12$	10.5	5	
$12 < x \leq 15$		8	13
$15 < x \leq 18$		13	
$18 < x \leq 21$		15	41
$21 < x \leq 24$	22.5	24	
$24 < x \leq 27$		17	82
$27 < x \leq 30$		18	

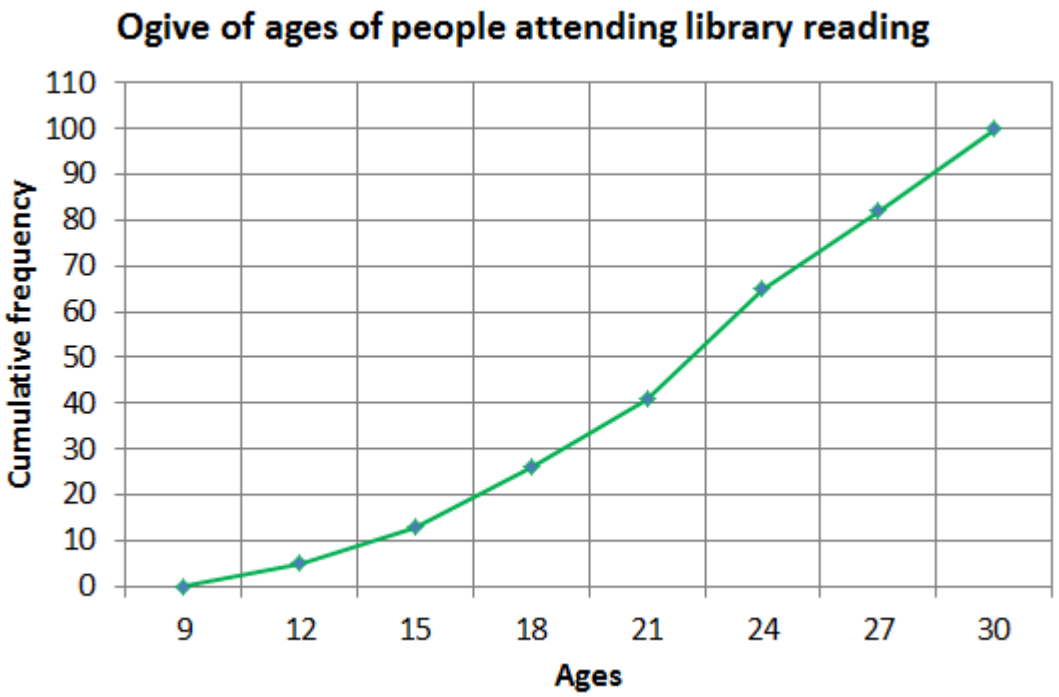
1. Rewrite and complete the table.
2. If you were to plot a graph of the data, what values would go on the horizontal axis and on the vertical axis?
3. How can you make sure the graph starts from the x-axis?
4. On a set of labelled axes plot the cumulative frequency against the class midpoints and name your graph.
5. What percentage of attendees was less than 21 years old?

What did you find?

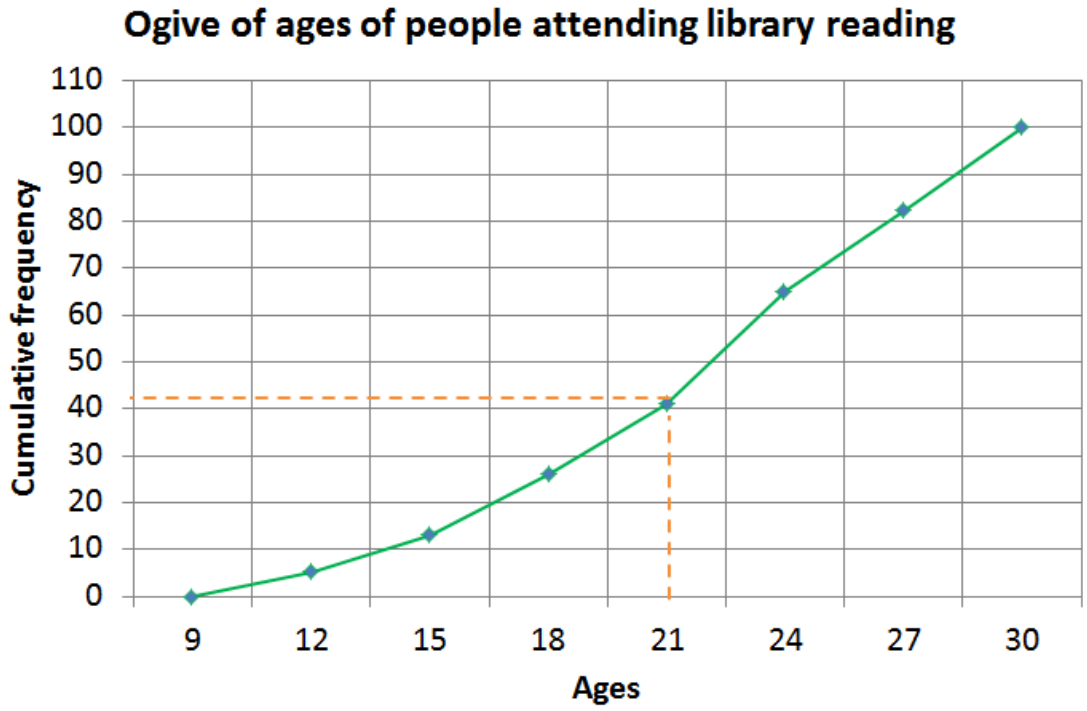
1. Find the class midpoint by adding the lower and upper class limits and dividing by 2. Your completed table should have the following values. You will not use the class midpoint to construct the ogive but it will be useful when we calculate central tendency and dispersion in later units.

Class intervals	Class midpoint	Frequency	Cumulative frequency
$9 < x \leq 12$	10.5	5	5
$12 < x \leq 15$	13.5	8	13
$15 < x \leq 18$	16.5	13	$\mathbf{26}$
$18 < x \leq 21$	19.5	15	41
$21 < x \leq 24$	22.5	24	$\mathbf{65}$
$24 < x \leq 27$	25.5	17	82
$27 < x \leq 30$	28.5	18	100

   To determine the cumulative frequency, we add up the frequencies going down the table. The first cumulative frequency is the same as the frequency, because we are adding it to zero. The final cumulative frequency is always equal to the sum of all the frequencies.
2. Class intervals in numbers of years is plotted on the x-axis and cumulative frequency plotted on the vertical axis (y-axis).
3. For the graph to start on the x-axis, the first lower class boundary must be plotted with a cumulative frequency of zero. There are no data values between zero and 9 so the graph must reflect no data values in that range.
4. Plot a dot at each class interval endpoint against the cumulative frequency. Join the dots using line segments.



5. Read off from the graph as shown below. Find 21 on the x-axis and draw a vertical line upward until you touch the ogive, then draw a horizontal line from that point to meet the y-axis. The corresponding value on the y-axis tells you how many people were less than 21.



We see that $\frac{41}{100} = 41\%$ of attendees were less than 21 years old.

How to construct a cumulative frequency graph/ogive

In Activity 2.1 we saw that these are the steps to construct an ogive:

1. The class intervals are labelled on the horizontal axis (x-axis) and the cumulative frequencies along the vertical axis (y-axis).
2. Every ogive will start at the lower boundary of the first-class interval on the horizontal axis with a frequency of zero.
3. Plot a dot at each class interval endpoint against the cumulative frequency. In this way, the end of the final interval will always be the total number of data elements that we will have added up across all intervals.
4. Join the dots using straight lines.

Note: this is not a continuous curve.



Example 2.1

1. The table shows the marks out of 60 for a physics test.
   Complete the table.

Intervals	Frequency	Cumulative frequency
$10 < n \leq 20$	5	
$20 < n \leq 30$	7	
$30 < n \leq 40$	12	
$40 < n \leq 50$	10	
$50 < n \leq 60$	6	

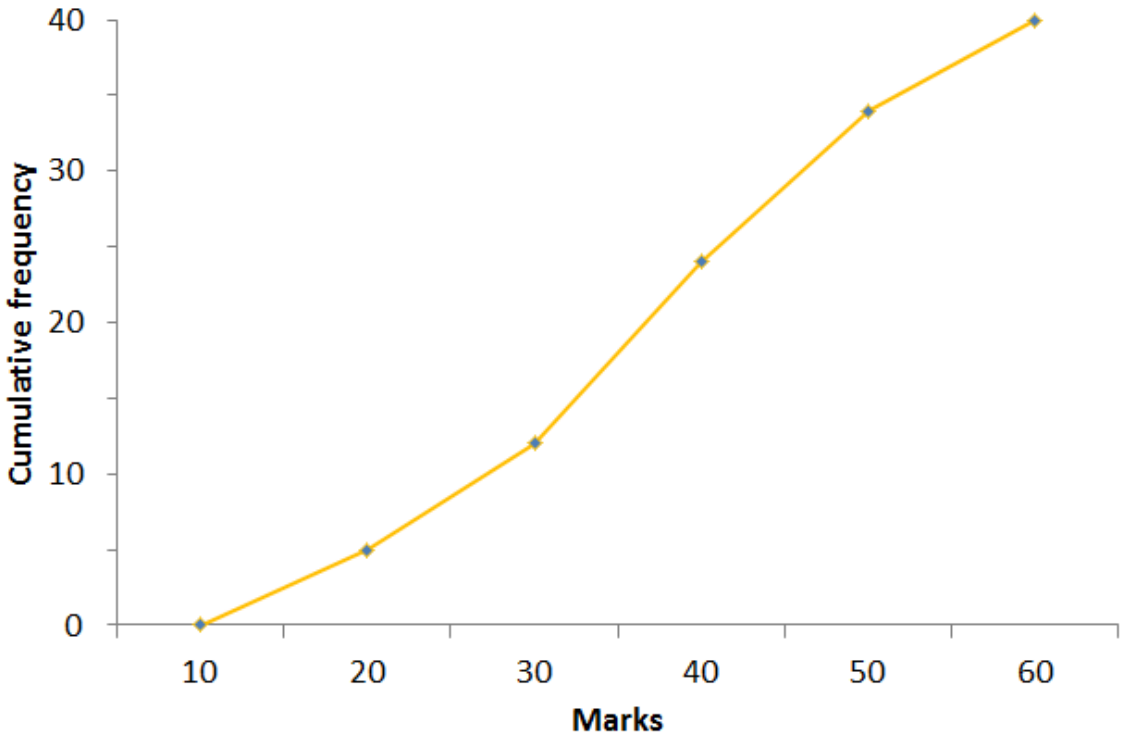
2. Draw the ogive for the data.
3. What percentage of learners got a mark higher than 40?

Solution

1. We complete the cumulative frequency column by adding frequencies as we go down the table.
   The completed table looks like this:

Intervals	Frequency	Cumulative frequency
$10 < n \leq 20$	5	5
$20 < n \leq 30$	7	12
$30 < n \leq 40$	12	24
$40 < n \leq 50$	10	34
$50 < n \leq 60$	6	40

2. Now we use the completed frequency table to draw the ogive by plotting the endpoint of each interval against the cumulative frequency for that interval.



3. Find 40 on the x-axis and draw a vertical line to meet the ogive, all values to the right of that point are marks that are higher than 40. We can see that 16 learners $(40 - 24) = 16$ scored 40 or more marks. So $\frac{16}{40} \times 100 = 40\%$ of learners got more than 40 on the test.

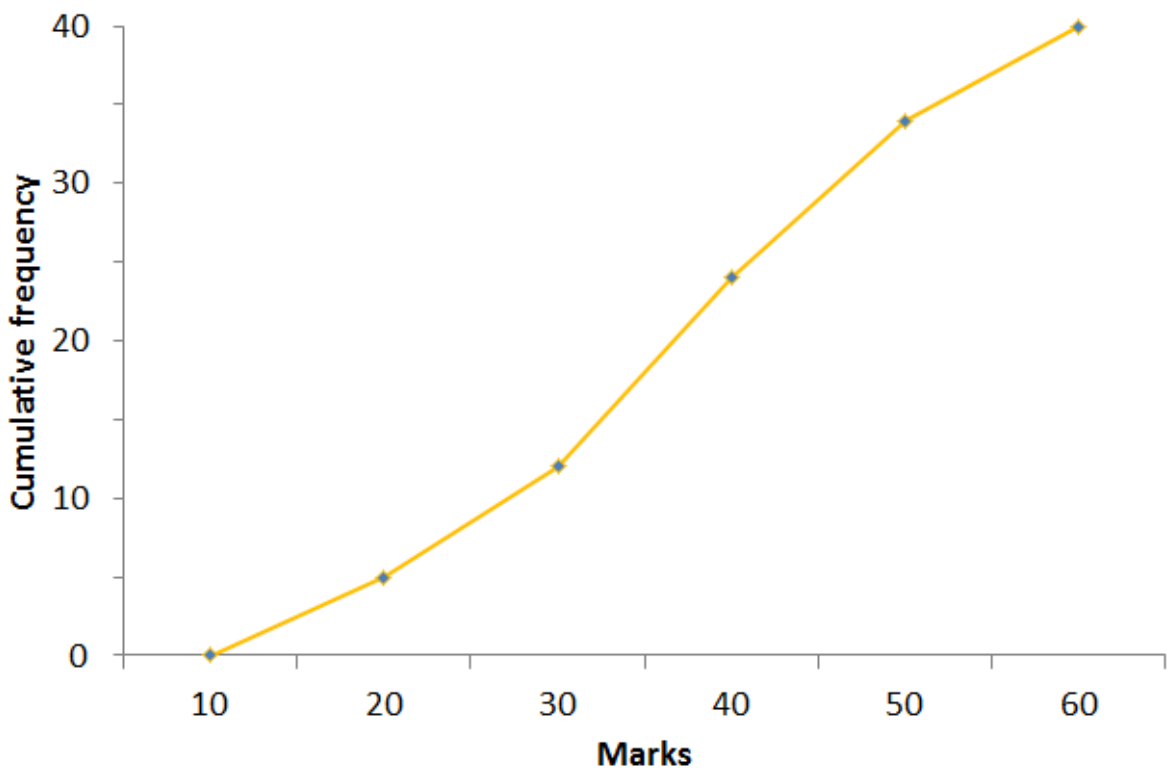
Interpreting the ogive

Ogives are useful for determining the quartiles and five-number summary of data. Remember that the median is simply the value in the middle when we order the data. A quartile is simply a quarter of the way from the beginning or the end of an ordered data set. With an ogive we already know how many data values are above or below a certain point, so it is easy to find the middle or a quarter of the data set. Remember that the answers you find for the quartiles will be approximations as you are dealing with grouped data.



Example 2.2

Use the following ogive to compute the five-number summary:



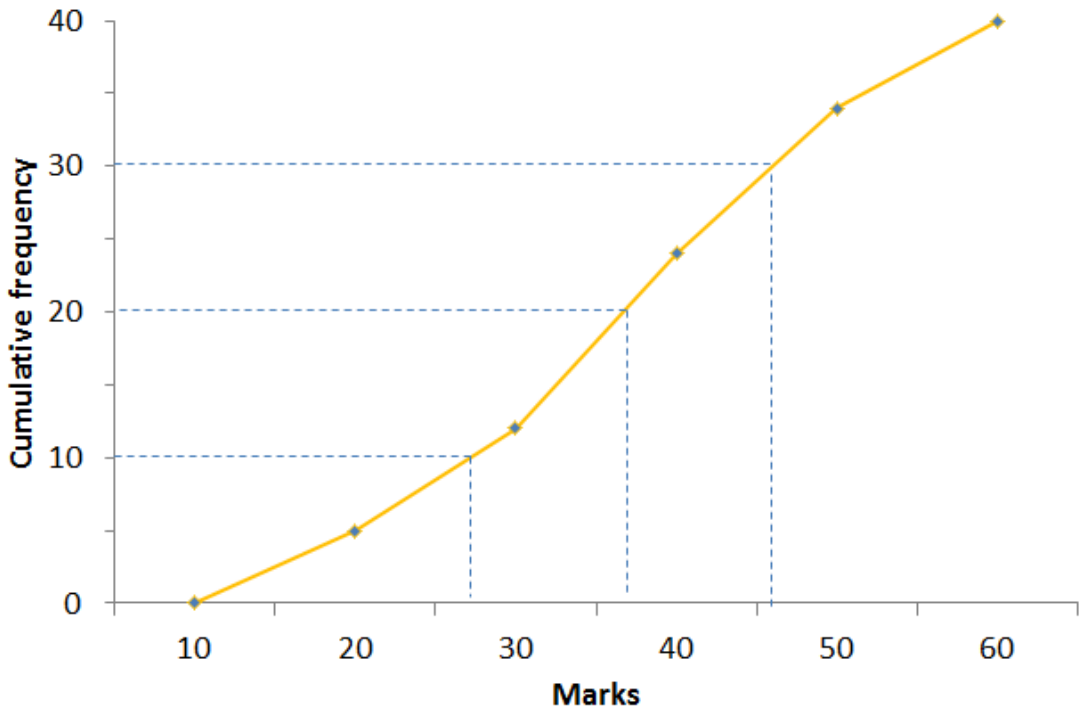
Solution

Remember that the five-number summary consists of the minimum, all the quartiles including the median, which is the second quartile, and the maximum value.

Step 1: Find the minimum and maximum
The minimum value in the data set is 10 as this is where the ogive starts on the horizontal axis. The maximum value in the data set is 60 as this is where the ogive stops on the horizontal axis.

Step 2: Find the quartiles
The quartiles are the values that are $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ of the way into the ordered data set. Here the cumulative frequency goes up to 40, so we can find the quartiles by looking at the values corresponding to counts of 10 $(\frac{1}{4} \times 40)$, 20 $(\frac{1}{2} \times 40)$ and 30 $(\frac{3}{4} \times 40)$. On the ogive, a count of:

- 10 corresponds to a value of approximately 26 (first quartile)
- 20 corresponds to a value of approximately 36 (median)
- 30 corresponds to a value of approximately 45 (third quartile).



Step 3: Write down the five-number summary

$\text{Minimum} = 10$

$Q_1 \approx 26$

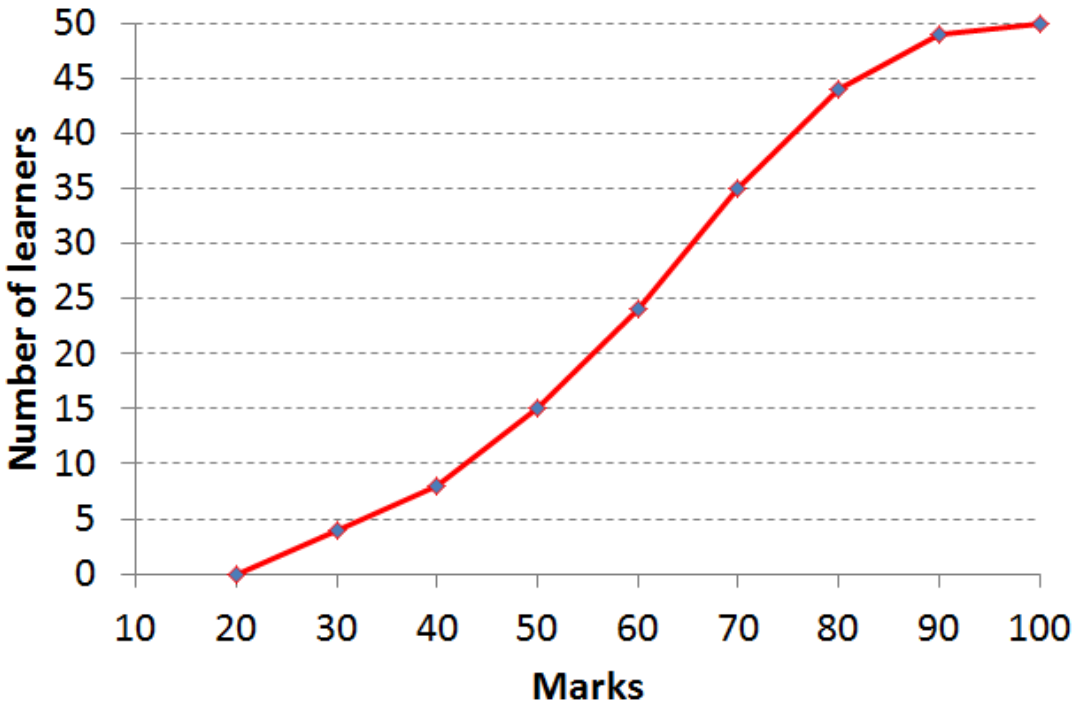
$Q_2 \approx 36$

$Q_3 \approx 45$

$\text{Maximum} = 60$

Exercise 2.1

1. The following data set lists the ages of 24 people.
   $2;\ 5;\ 1;\ 76;\ 34;\ 23;\ 65;\ 22;\ 63;\ 45;\ 53;\ 38$
   $4;\ 28;\ 5;\ 73;\ 79;\ 17;\ 15;\ 5;\ 34;\ 37;\ 45;\ 56$
   Use the data to answer the following questions:
   a. Using an interval width of 8, construct a cumulative frequency plot.
   b. Below what value do the bottom 50% of the ages fall? Give a reason for your answer.
   c. Below what value do the bottom 40% fall?
2. Use the ogive to answer the questions below. Note that marks are given as a percentage.



a. How many learners got between 50% and 70%?
b. How many learners got at least 70%?
c. Find the median mark for this class, rounded to the nearest integer.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to construct the ogive from a grouped frequency distribution.
- How to interpret information from an ogive.

Unit 2: Assessment

Suggested time to complete: 25 minutes

1. The marks (as a percentage) obtained in an NCV Maths level 3 examination are shown in the table below:

Intervals	Frequency	Cumulative frequency
$0 < x \leq 10$	0	0
$10 < x \leq 20$	2	
$20 < x \leq 30$	6	8
$30 < x \leq 40$	7	
$40 < x \leq 50$	14	29
$50 < x \leq 60$	20	
$60 < x \leq 70$	35	84
$70 < x \leq 80$	29	
$80 < x \leq 90$	6	
$90 < x \leq 100$	1	120

a. How many total observations are there?

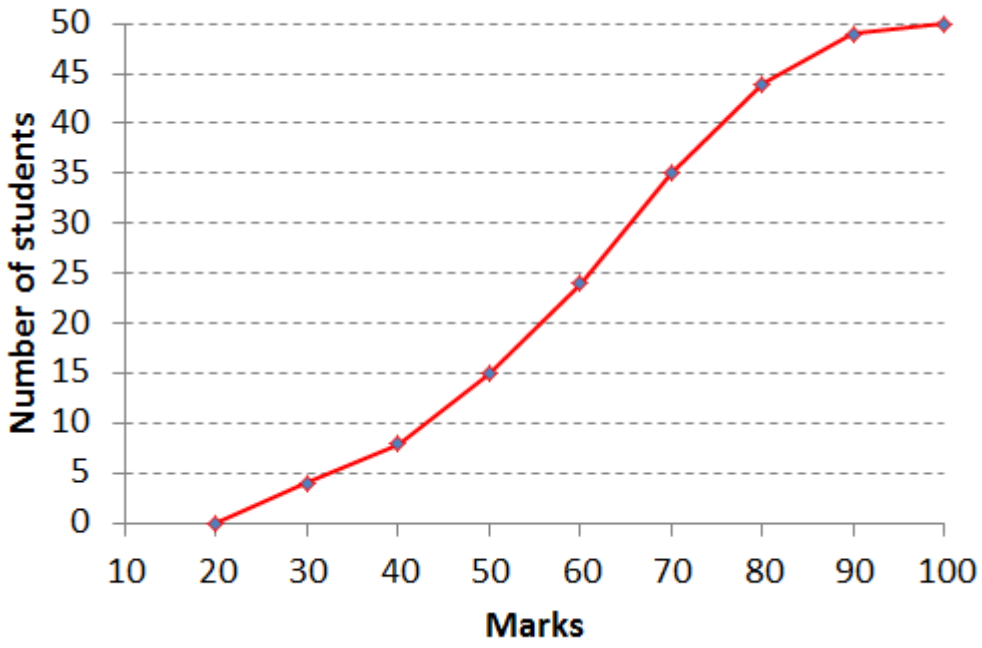
b. Copy and complete the table.

c. Draw the ogive for the data.

d. In which class interval will the median be located?

e. Estimate the median.

2. Use the ogive below to draw a box-and-whisker plot:



3. The ages of 28 people whose birthday coincides with that of one of their children are shown below.
$78;\ 53;\ 70;\ 97;\ 37;\ 68;\ 48;\ 35;\ 71;\ 63;\ 47;\ 60;\ 63;\ 58;$
$74;\ 39;\ 67;\ 64;\ 42;\ 52;\ 38;\ 54;\ 60;\ 75;\ 69;\ 77;\ 65;\ 72$

a. Construct the frequency distribution showing cumulative frequency.

b. Draw the ogive for the data.

c. Estimate the median.

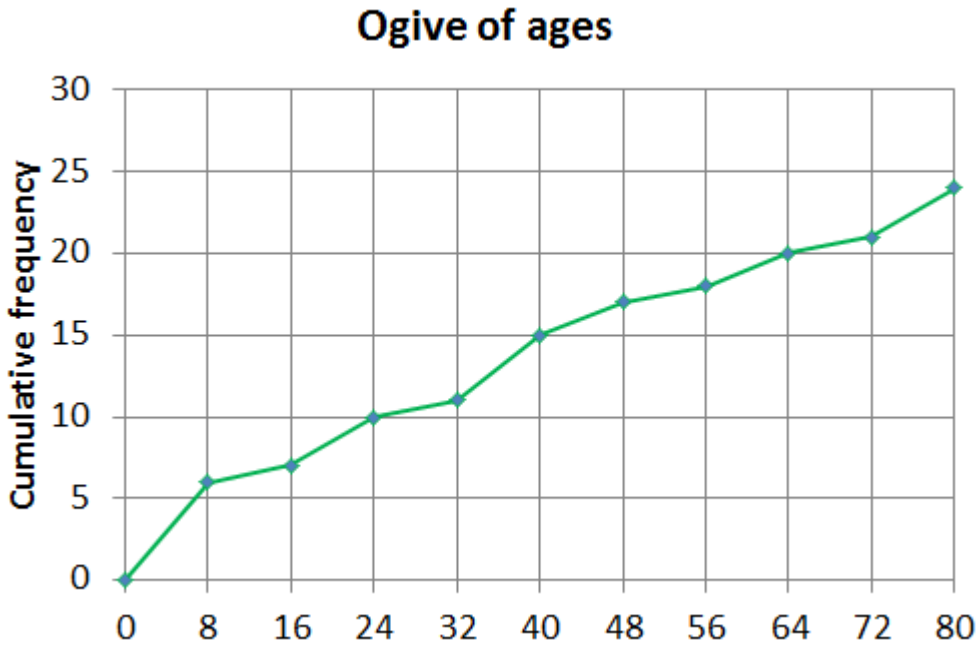
The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1. Construct a frequency distribution before drawing the cumulative frequency graph.
   a.

Age intervals	Frequency	Cumulative frequency
$0 < x \leq 8$	6	6
$8 < x \leq 16$	1	7
$16 < x \leq 24$	3	10
$24 < x \leq 32$	1	11
$32 < x \leq 40$	4	15
$40 < x \leq 48$	2	17
$48 < x \leq 56$	1	18
$56 < x \leq 64$	2	20
$64 < x \leq 72$	1	21
$72 < x \leq 80$	3	24



   b. Below 34: as there are 24 values the median will be between positions 11 and 12. Reading off the graph that gives us approximately 34.
   c. Below 24.

2.
   a. 20 learners. Read off from the graph and subtract $(35 - 15 = 20)$.
   b. 15 learners. At least 70 means 70 or more, so subtract the number of learners at 70 from the number at 100 $(50 - 35 = 15)$.
   c. The median mark will be approximately 60%.

Back to Exercise 2.1

Unit 2: Assessment

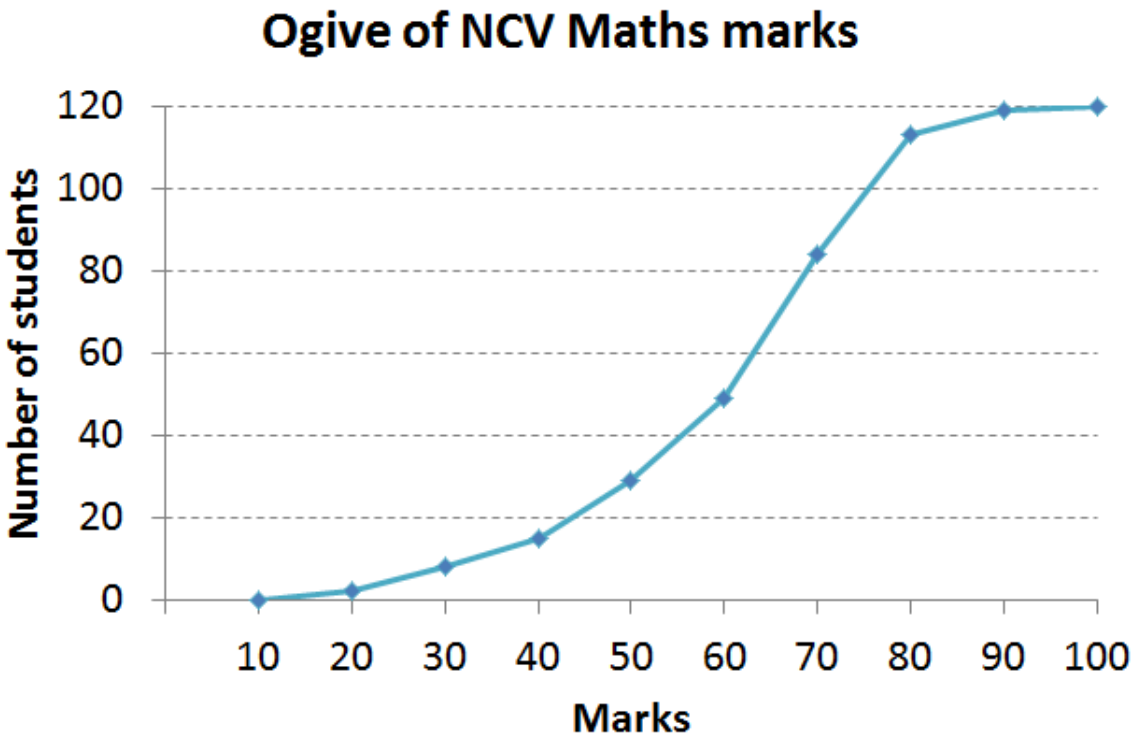
1.

a. 120 observations in total.

b.

Intervals	Frequency	Cumulative frequency
$0 < x \leq 10$	0	0
$10 < x \leq 20$	2	$\mathbf{2}$
$20 < x \leq 30$	6	8
$30 < x \leq 40$	7	$\mathbf{15}$
$40 < x \leq 50$	14	29
$50 < x \leq 60$	20	$\mathbf{49}$
$60 < x \leq 70$	35	84
$70 < x \leq 80$	29	$\mathbf{113}$
$80 < x \leq 90$	6	$\mathbf{119}$
$90 < x \leq 100$	1	120

c.

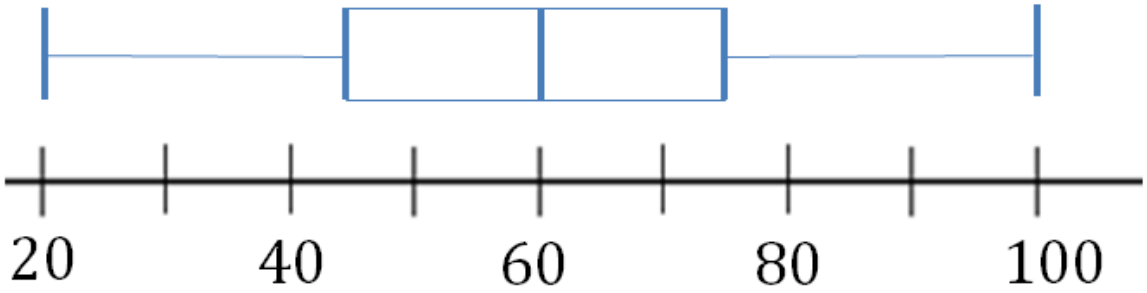


d. There are 120 data values so the median will lie between position 60 and 61. Looking at the cumulative frequency we see that the position of the median places it in the interval $60 < x \leq 70$.

e. Reading off from the ogive, the median is approximately 63.

2.

$$\begin{aligned}\text{Min} &= 20\\ Q_1 &\approx 45\\ Q_2 &\approx 60\\ Q_3 &\approx 74\\ \text{Max} &= 100\end{aligned}$$

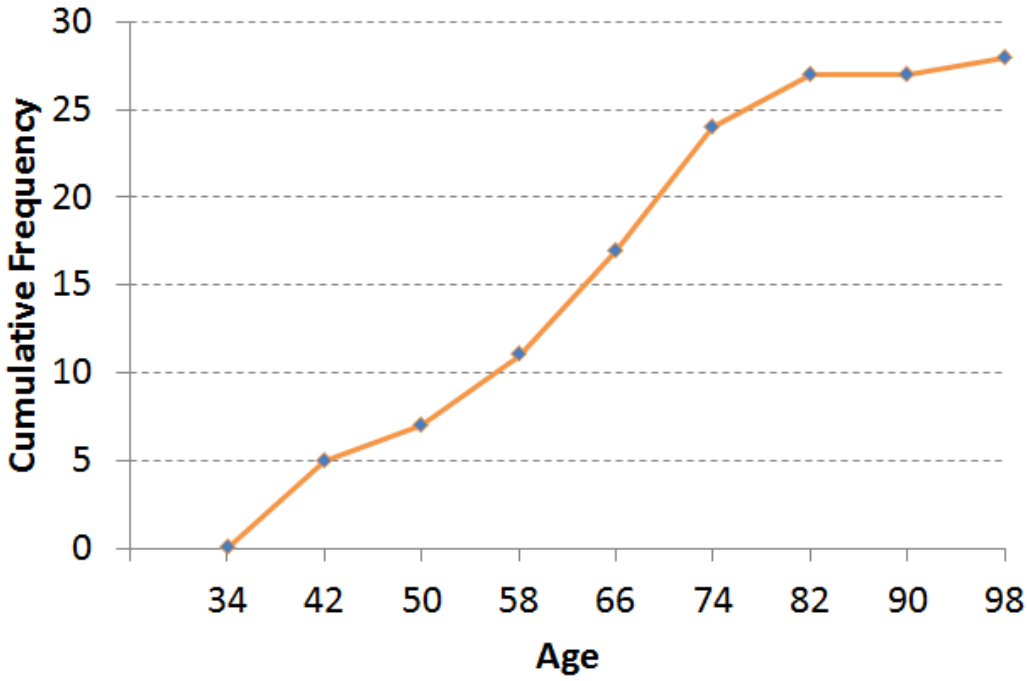


3.

a.

Age intervals	Frequency	Cumulative frequency
$34 < x \leq 42$	5	5
$42 < x \leq 50$	2	7
$50 < x \leq 58$	4	11
$58 < x \leq 66$	6	17
$66 < x \leq 74$	7	24
$74 < x \leq 82$	3	27
$82 < x \leq 90$	0	27
$90 < x \leq 98$	1	28

b.



c. Median age is approximately 63.

Back to Unit 2: Assessment

Media Attributions

- Img 01_Ogive of ages Activity 2.1 © DHET is licensed under a CC BY (Attribution) license
- Activity 2.1 part E © DHET is licensed under a CC BY (Attribution) license
- Img 03_Example 2.1 © DHET is licensed under a CC BY (Attribution) license
- Example 2.2 © DHET is licensed under a CC BY (Attribution) license
- Exercise 2.1 Q2 © DHET is licensed under a CC BY (Attribution) license
- Assessment Q2 © DHET is licensed under a CC BY (Attribution) license
- Exercise 2.1 Q1a Answer © DHET is licensed under a CC BY (Attribution) license
- Assessment Q1c Ans © DHET is licensed under a CC BY (Attribution) license
- Assessment Q2 Ans © DHET is licensed under a CC BY (Attribution) license
- Assessment Q3b Ans © DHET is licensed under a CC BY (Attribution) license

Unit 3: Constructing histograms

NATASHIA BEARAM-EDMUNDS



Unit outcomes

By the end of this unit you will be able to:

- Construct a histogram.
- Interpret a histogram.

What you should know

Before you start this unit, make sure you can:

- recall the basics of representing data. You can revise the following subject outcomes in level 2, subject outcome 4.2.

Introduction

Histograms are often used to display **grouped** data. One advantage of a histogram is that it can easily be used for large data sets. In 'real life' applications we generally use a histogram when the data set consists of 100 values or more.

A histogram consists of rectangles drawn next to each other so that they touch. The horizontal axis shows what the data represents, for example, distance. The vertical axis shows either frequency or relative frequency. In this unit and for your purposes you will only need to plot frequency graphs and not relative frequency. The histogram shows the shape of the data, the centre and the spread of the data.

Constructing a histogram

If you are not given a frequency distribution you can construct a histogram as shown in the following example.



Example 3.1

Create a histogram for the number of books bought by 50 part-time college learners.

$1;\ 1;\ 1;\ 1;\ 1;\ 1;\ 1;\ 1;\ 1;\ 1;\ 1$
$2;\ 2;\ 2;\ 2;\ 2;\ 2;\ 2;\ 2;\ 2;\ 2$
$3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3;\ 3$
$4;\ 4;\ 4;\ 4;\ 4;\ 4$
$5;\ 5;\ 5;\ 5;\ 5$
$6;\ 6$

Solution

Note that 'number of books' is discrete data, since books are counted.

To draw the histogram you must decide how many bars or intervals, also called classes, represent the data.

Calculate the number of bars as follows: $\frac{\text{largest value} - \text{smallest value}}{\text{number of bars}} = \text{width of bar/interval}$.

Before you can calculate the number of bars you must choose a starting point for the first interval to be less than the smallest data value.

With discrete data, all the data values happen to be integers. In this example the smallest value is 1, so a convenient starting point will be $1 - 0.5 = 0.5$. The largest value is 6 so we add 0.5 to 6 to get the end point of 6.5.

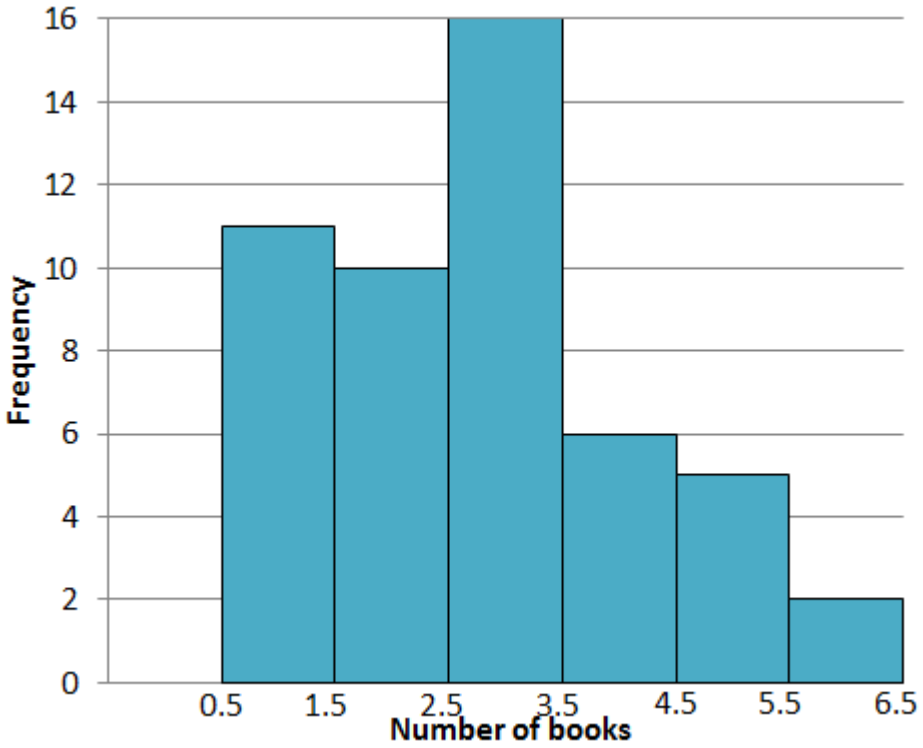
Next, calculate the width of each bar or class interval. If the data are discrete and there are not too many different values, a width that places the data values in the middle of the bar is the most convenient.

Since the data consist of the numbers $1;\ 2;\ 3;\ 4;\ 5;\ 6$ and the starting point is 0.5, a width of one places the 1 in the middle of the interval from $0.5$to 1.5, the 2 in the middle of the interval from 1.5 to 2.5, and so on.

Therefore, $\frac{6.5 - 0.5}{\text{number of bars}} = 1$ so the number of bars is 6.

Intervals	**Frequency**
$0.5 < x \leq 1.5$	11
$1.5 < x \leq 2.5$	10
$2.5 < x \leq 3.5$	16
$3.5 < x \leq 4.5$	6
$4.5 < x \leq 5.5$	5
$5.5 < x \leq 6.5$	2

The histogram displays the number of books on the x-axis and the frequency on the y-axis.



The height of the bars gives the frequency of each interval and the intervals are listed on the x-axis. Now that you've drawn the histogram, you get a better overall picture of the data because you can visualise it. We can also see at a glance some important things that the data reveals.

The tallest bar shows that 3 books was the most popular observation, with a frequency of 16.

Take note!

At this maths level, the interval width will be given in exam questions.

Histograms are often used to display continuous data. You will recall that this is data that can be measured. The next example uses continuous data to construct a histogram.

Example 3.2

The following data represent the heights of 16 people, in centimetres.

$162\ ;\ 168\ ;\ 177\ ;\ 147\ ;\ 189\ ;\ 171\ ;\ 173\ ;\ 168;\ 178\ ;\ 184\ ;\ 165\ ;\ 173\ ;\ 179\ ;\ 166\ ;\ 168\ ;\ 165$

Use class intervals that start at $140\ \text{cm}$ and end at $190\ \text{cm}$ to draw a histogram.

Solution

Step 1: Determine intervals

We need intervals of the same length between $140\ \text{cm}$ and $190\ \text{cm}$ so we could go up by 10 centimetres each time. Calculate the number of intervals, which corresponds to the number of bars, using:

$$\frac{\text{largest value} - \text{smallest value}}{\text{number of bars}} = \text{width of bar/interval}$$

$$\begin{aligned}\frac{190 - 140}{\text{number of bars}} &= 10\\ 10 \times \text{number of bars} &= 50\\ \therefore \text{number of bars} &= 5\end{aligned}$$

Note: If you know how many bars you'd like to use you can use the method above to calculate the bar width.

We can use round and square brackets to show which values are excluded and included in the intervals. Round brackets are used to show values that are not included and square brackets are used to show included values. Interval $(140; 150]$ means all values greater than $140\ \text{cm}$and less than or equal to $150\ \text{cm}$.

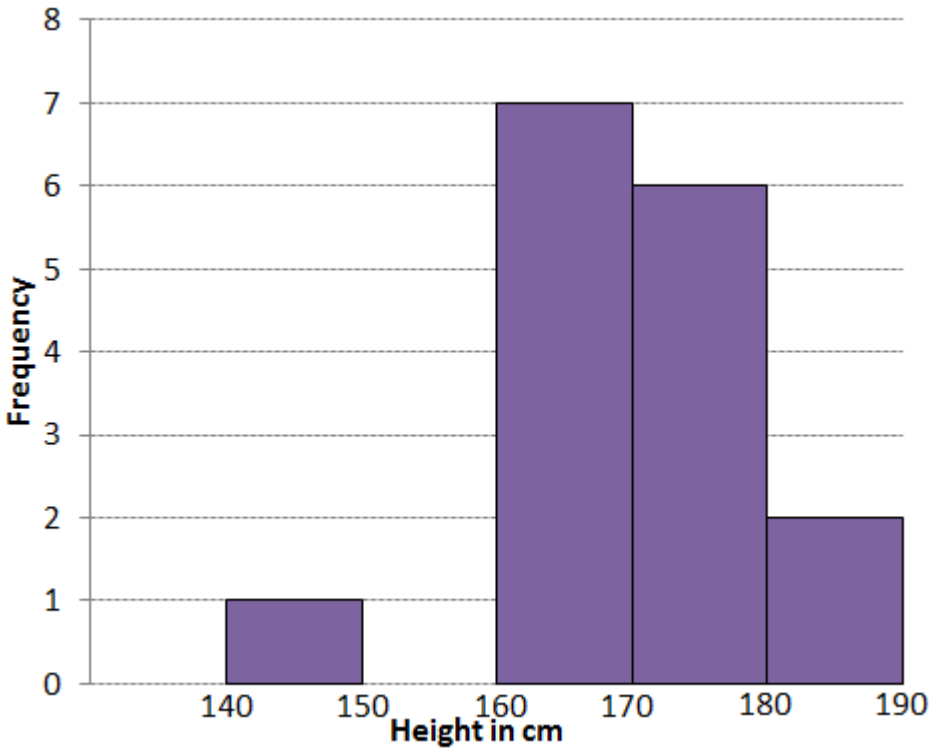
Intervals
$(140; 150]$
$(150; 160]$
$(160; 170]$
$(170; 180]$
$(180; 190]$

Step 2: Include the frequency in each interval

The following frequency table summarises the number of data values in each of the intervals.

Intervals	Frequency
$(140; 150]$	1
$(150; 160]$	0
$(160; 170]$	7
$(170; 180]$	6
$(180; 190]$	2

Step 3: Draw the histogram



The interval $(150; 160]$ has no entries so we do not draw a bar in that interval.

Exercise 3.1

1. Below is the cumulative frequency table of the ages of people attending a public reading by an author at a library.

Class intervals	Frequency	Cumulative frequency
$9 < x \leq 12$	5	5
$12 < x \leq 15$	8	13
$15 < x \leq 18$	13	26
$18 < x \leq 21$	15	41
$21 < x \leq 24$	24	65
$24 < x \leq 27$	17	82
$27 < x \leq 30$	18	100

   Use the table to construct:

   a. a histogram.

   b. an ogive.

2. The following data represent the number of employees at various restaurants. Use this data to cre-

ate a histogram.
$22;\ 35;\ 15;\ 26;\ 40;\ 28;\ 18;\ 20;\ 25;\ 34;\ 39;\ 42;\ 24;\ 22;\ 19;\ 27;\ 22;\ 34;\ 40;\ 20;\ 38;\ 28$

Use $10 < x \leq 19$ as the first interval.

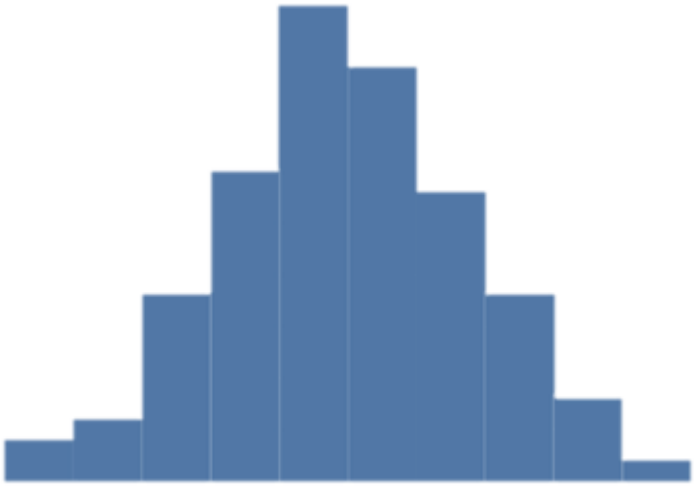
The full solutions are at the end of the unit.

Interpreting histograms

The shape of a histogram can tell you a lot about the distribution of the data and gives information about the mean, median and mode of the data set. The most common shapes of histograms are symmetric and skewed.

You will recall that the mean is the measure of central tendency that is highly influenced by extreme values. So, by looking at the position of the mean in relation to the median you can tell if a graph is skewed or symmetric.

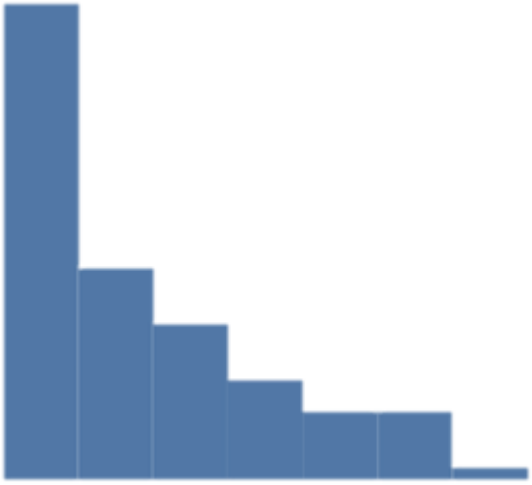
A **symmetric histogram** is more or less identical on both sides of the mean.



For symmetric distributions, the mean is approximately equal to the median and the left and right tails are equally balanced, meaning that they have about the same length.

In a skewed histogram the data seems to extend more to one side than the other. We say the tail is longer on the one side than the tail on the other side. There are two types of skewedness:

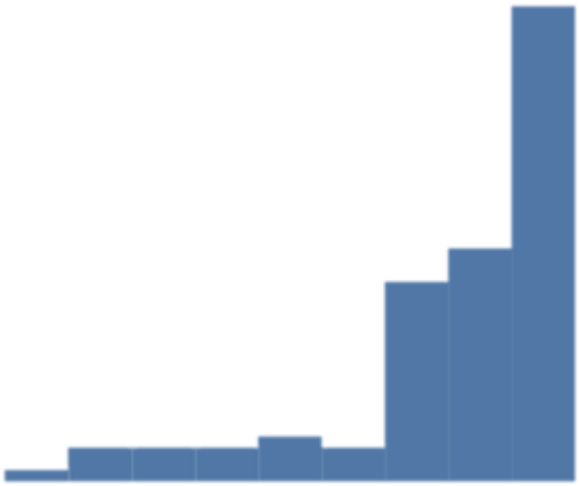
1. **Skewed right** (positive)
   A histogram skewed to the right has a longer tail on the right side.



mean > median

For positively skewed data the mean >median. If there are extreme values towards the positive end of a distribution, these will increase the value of the mean.

2. **Skewed left** (negative)
   A histogram skewed to the left has a longer tail on the left side.



mean < median

For negatively skewed data, you will notice that mean <median. This is due to the presence of extreme lower values to the left (or negative) side of the distribution that will decrease the value of the mean.

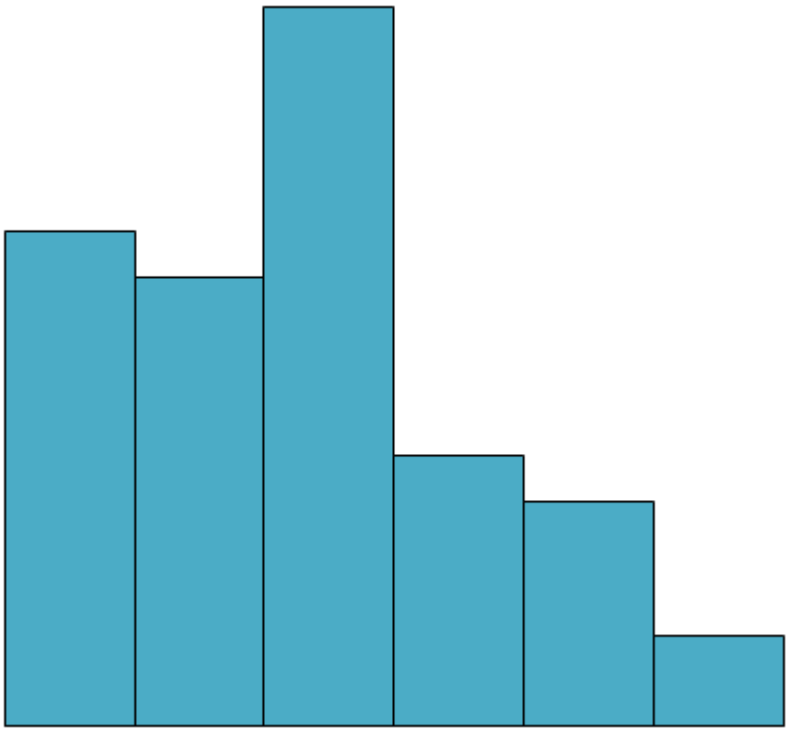
Note

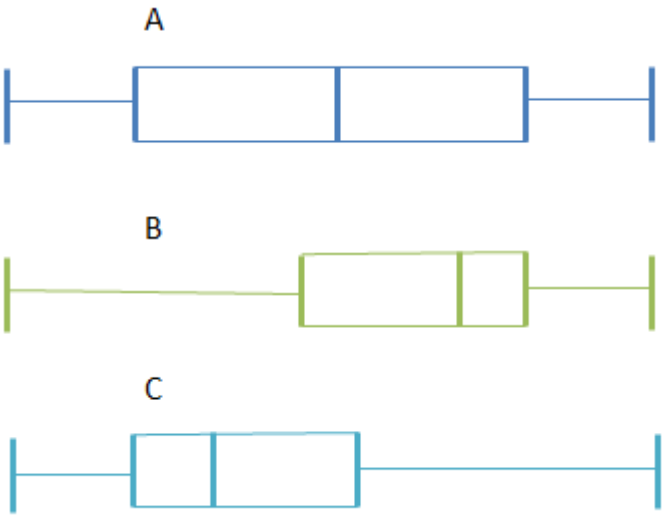
Watch this video called “Classifying shapes of distributions” for a summary of types of histograms.

Classifying shapes of distributions (Duration: 04.27)

Example 3.3

Which box-and-whisker plot represents the shape of the histogram below?





Solution

The histogram has a longer tail on the right so it is positively skewed, which shows that the mean is greater than the median.

Box plot A: The median is in the middle of the box, and the whiskers are about the same length on either side of the box so the distribution is symmetric or normal.

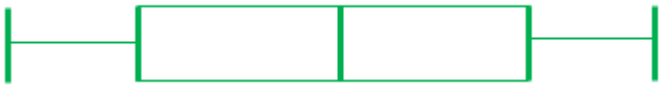
Box plot B: The median is pulled towards the upper quartile, and the whisker is shorter on the upper end of the box so the distribution is negatively skewed (skewed left).

Box plot C: The median is pulled towards the lower quartile, and the whisker is shorter on the lower end of the box so the distribution is positively skewed (skewed right).

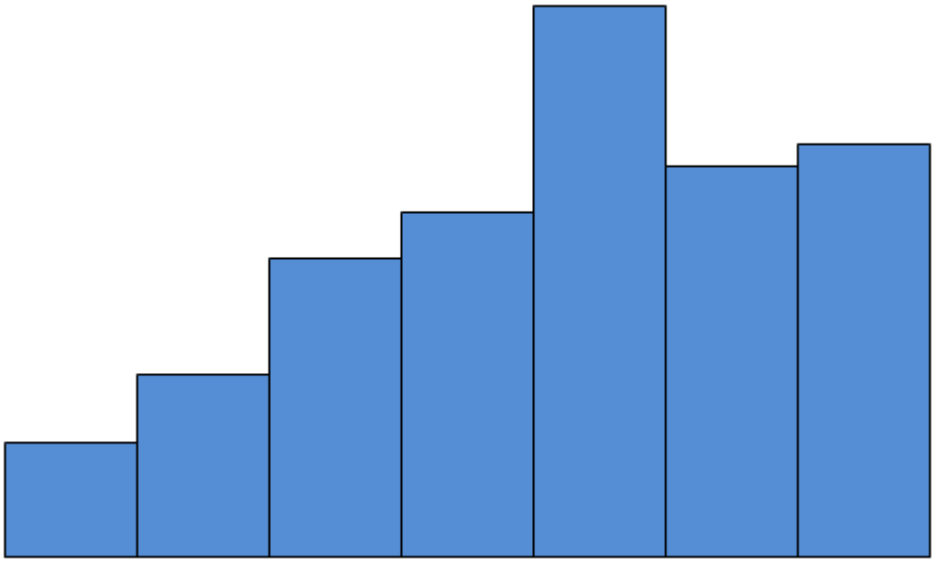
Therefore, box plot C is a possible representation of the histogram in this example.

Exercise 3.2

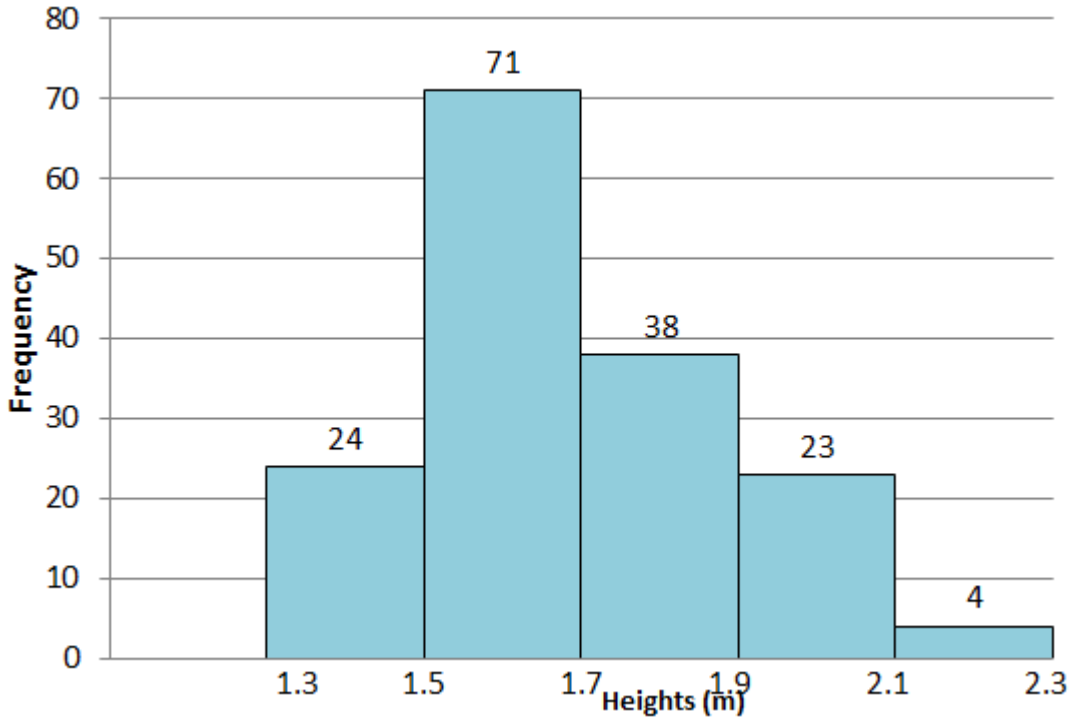
1. State whether the following data sets are symmetric, skewed right or skewed left and comment on the position of the mean in relation to the median.
   a. A data set with this box-and whisker plot:



   b. A data set with this histogram:



2. If the mean $>$median$>$mode will the graph skew to the right or left?
3. The heights of learners in a college are recorded and are shown in the histogram below:



a. How many heights were recorded?

b. What was the most common interval of recorded heights?

c. Comment on the shape of the distribution. What is the cause of this?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to construct a histogram from discrete data.
- How to construct a histogram from a frequency distribution.
- How to interpret the shape of a histogram.

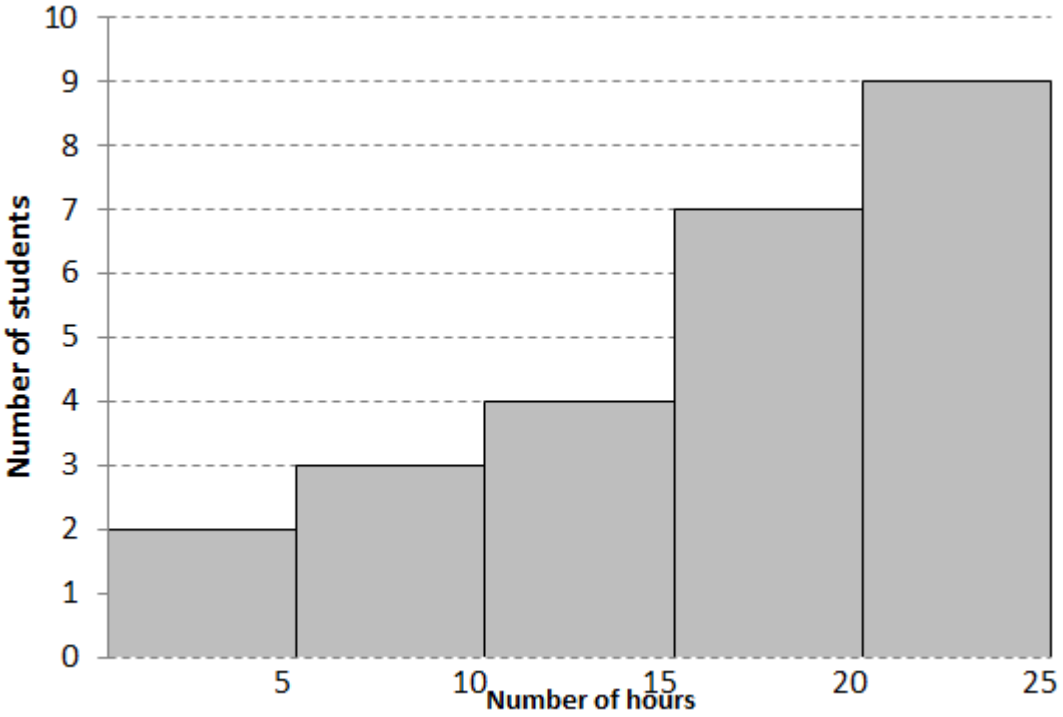
Unit 3: Assessment

Suggested time to complete: 25 minutes

1. In a traffic survey a random sample of 50 motorists were asked what distance they drove to work daily. The results of the survey are shown in kilometres in the table below. Draw a histogram to represent the data.

Distance	$0 < d \leq 10$	$10 < d \leq 20$	$20 < d \leq 30$	$30 < d \leq 40$	$40 < d \leq 50$
f	9	17	15	5	4

2. The histogram below shows the number of hours learners in a class spent playing video games over a weekend.



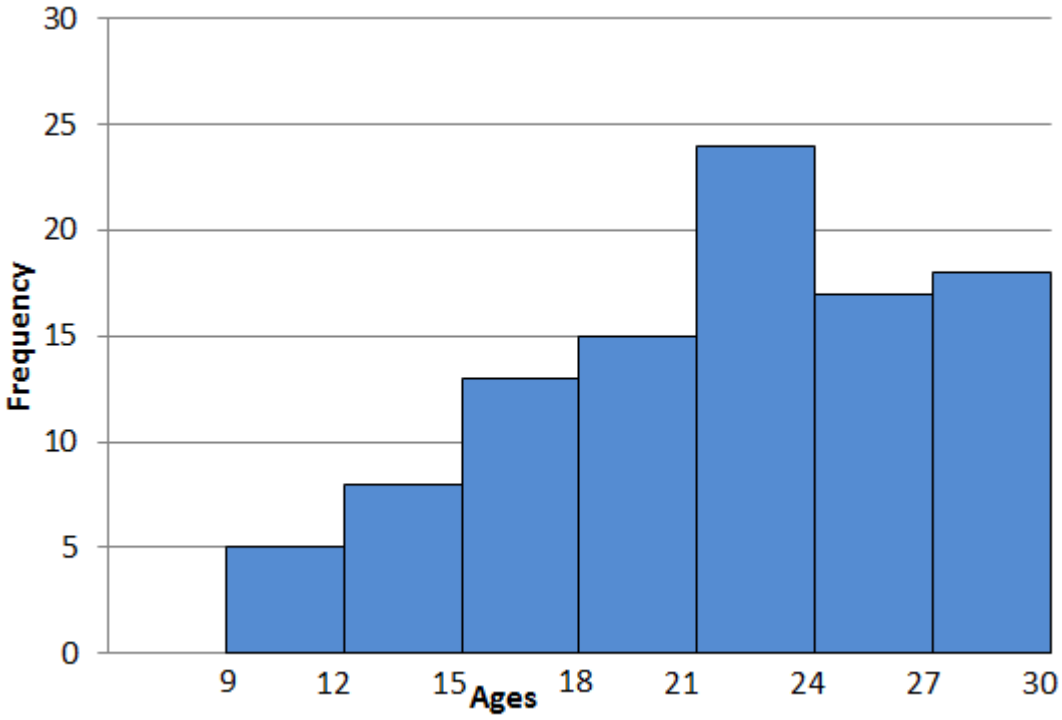
   a. How many learners were in the survey?
   b. What was the maximum number of hours spent playing video games?
   c. How many learners spent 10 or less hours playing video games?
   d. How many learners spent more than 15 hours playing video games?
   e. How many hours did most learners spend playing video games?
   f. Comment on the shape of the histogram.

The full solutions are at the end of the unit.

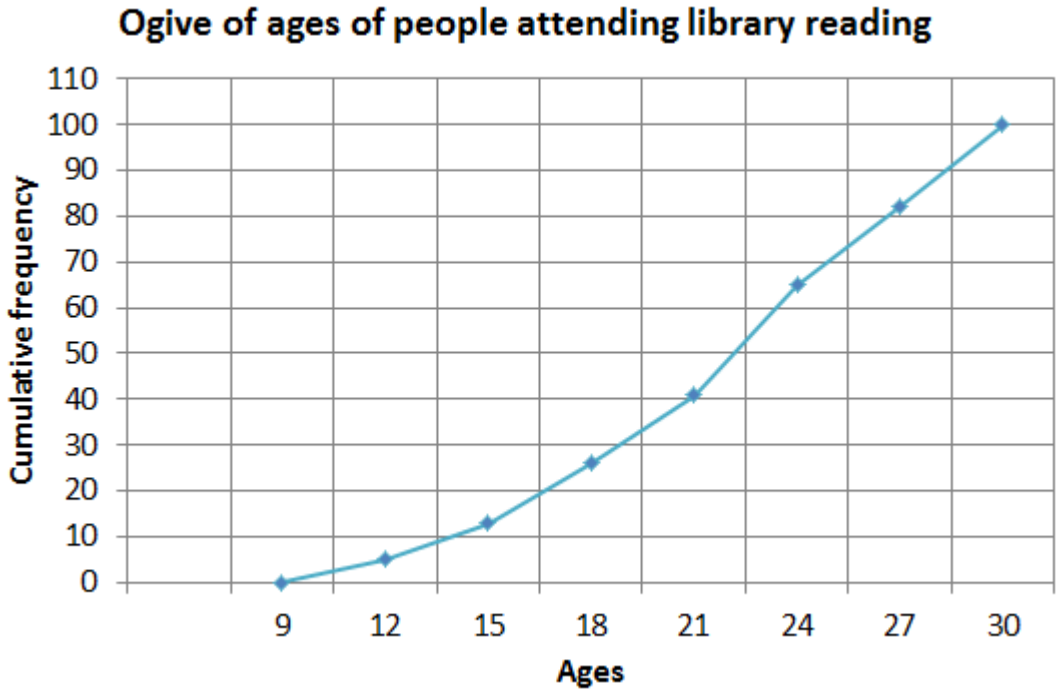
Unit 3: Solutions

Exercise 3.1

1.
   a. The histogram

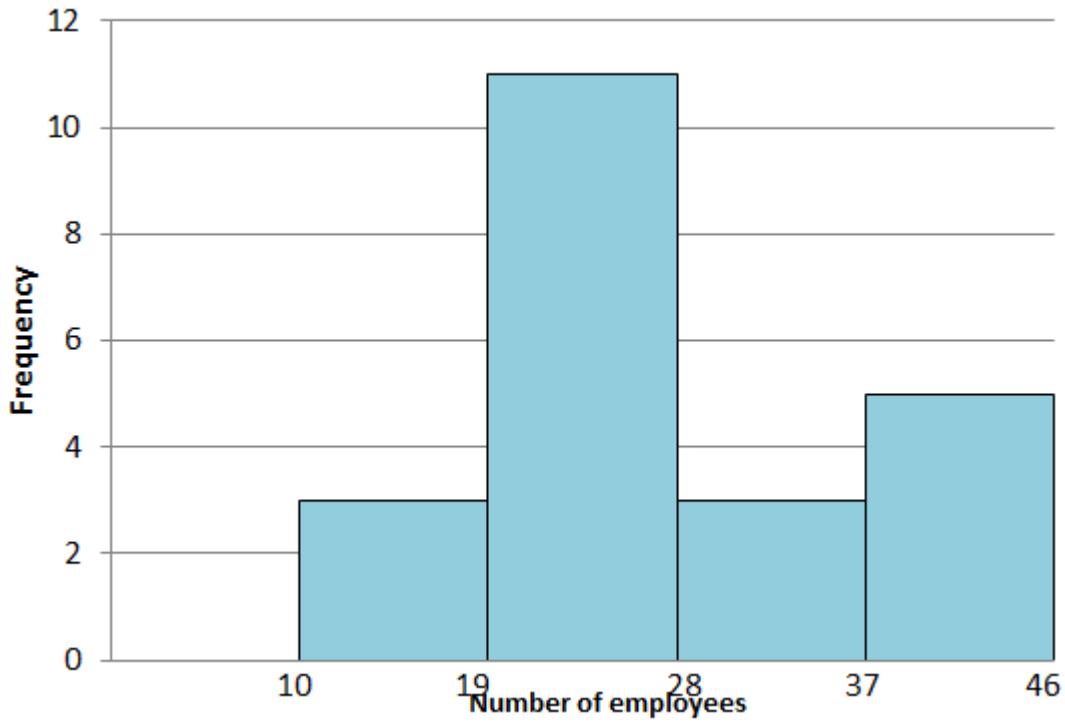


b. The ogive



2. Draw a frequency table first to make it easier to draw the histogram.

Intervals	Frequency
$10 < x \leq 19$	3
$19 < x \leq 28$	11
$28 < x \leq 37$	3
$37 < x \leq 46$	5



Back to Exercise 3.1

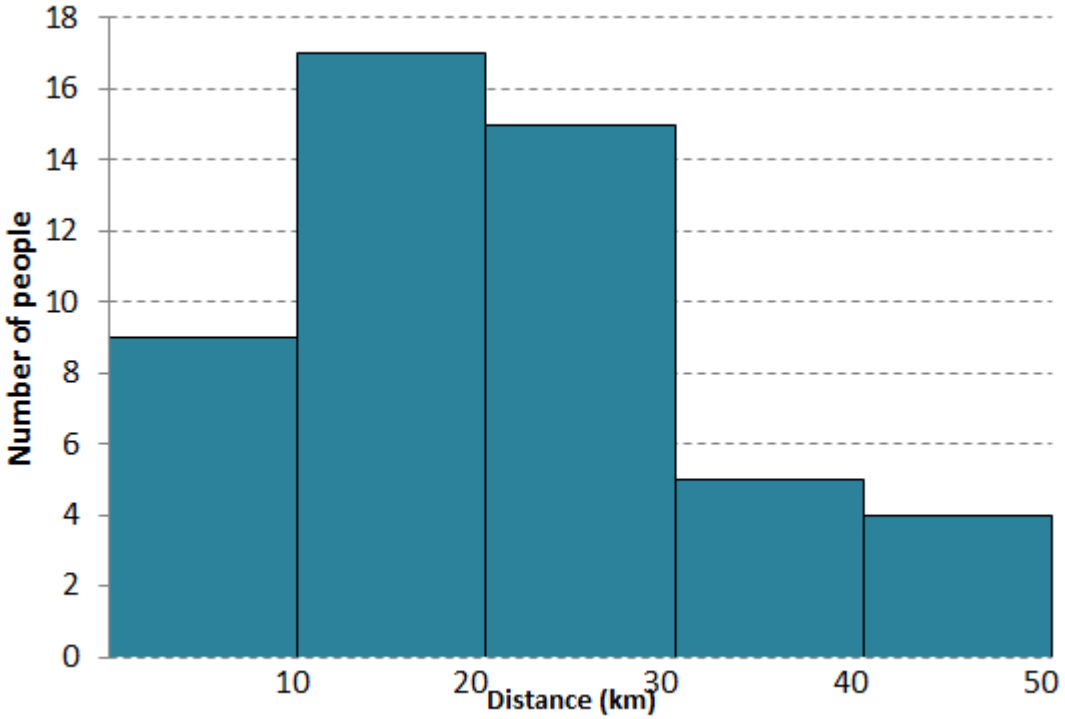
Exercise 3.2

1.
    a. Symmetric box-and-whisker plot with mean approximately equal to the median.
    b. Negatively skewed histogram with mean less than the median.
2. Positively skewed data (skewed to the right) since the mean $>$median$>$ mode.
3.
    a. Adding the height of the bars you get 160 observations.
    b. Most heights were in the $1.5\ \text{m}$ to $1.7\ \text{m}$ interval as this is shown as the tallest bar.
    c. The shape of the histogram indicates that the data are skewed to the right. This is due to a few very tall people with a height of over $2.1\ \text{m}$.

Back to Exercise 3.2

Unit 3: Assessment

1.



2.

a. By adding the height of the bars together you will calculate that 25 learners made up the survey.
b. 25 hours
c. 5 learners
d. 16 learners
e. Most learners spent 20 to 25 hours playing video games.
f. The histogram is negatively skewed (it has a longer tail on the left).

Back to Unit 3: Assessment

Media Attributions

- Histogram 1 Example 3.1 © DHET is licensed under a CC BY (Attribution) license
- Histo 2 Ex 3.2 © DHET is licensed under a CC BY (Attribution) license
- Histo 5 symmetric © DHET is licensed under a CC BY (Attribution) license
- Histo 6 Skewed right © DHET is licensed under a CC BY (Attribution) license
- Histo 7 Skewed left © DHET is licensed under a CC BY (Attribution) license
- Histo 8 Example 3.3 © DHET is licensed under a CC BY (Attribution) license
- Fig 9 Example 3.3 © DHET is licensed under a CC BY (Attribution) license
- Fig 10 Exercise 3.2 Q1a © DHET is licensed under a CC BY (Attribution) license
- Fig 10 Exercise 3.2 Q1b © DHET is licensed under a CC BY (Attribution) license
- Graph 11 Exercise 3.2 Q3 © DHET is licensed under a CC BY (Attribution) license
- Graph 12 Assess Q 2 © DHET is licensed under a CC BY (Attribution) license
- Histo 3 Exercise 3.1 Q1 © DHET is licensed under a CC BY (Attribution) license
- Exercise 3.1 Q1b © DHET is licensed under a CC BY (Attribution) license
- Histo 4 Exercise 3.1 Q2 Answ © DHET is licensed under a CC BY (Attribution) license
- Graph 13 Assess Q1 Ans © DHET is licensed under a CC BY (Attribution) license

Unit 4: Central tendency and dispersion for grouped data

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Calculate the mean of grouped data using the formula $\bar{x} = \frac{\sum f_i x_i}{n}$.
- Calculate the median of grouped data using the formula $\mathrm{M}_e = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c$.
- Calculate the modal value of grouped data using the formula $M_o = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c$.

What you should know

Before you start this unit, make sure you can:

- Calculate central tendencies and dispersion of ungrouped data. To revise this topic you can review level 2, subject outcome 4.1.

Introduction

When only grouped data is available, we do not know the individual data values; we only know the intervals and interval frequencies, therefore we cannot compute exact measures of central tendency and dispersion for the data set. For grouped data we calculate estimated measures of central tendency and dispersion.

Take note!

Measures of central tendency or averages are single numbers that provide summary information about an entire set of data, without listing every single data value.

Measures of dispersion provide information on how spread out the data are around the centre values.

Measures of central tendency for grouped data

There are three main measures of central tendency: the mean, the median and the mode. In the following activity we learn how to find the mean, median and mode for grouped data using special formulae.

Activity 4.1: Calculate the mean, median and mode for grouped data

Time required: 30 minutes

What you need:

- a pen
- paper
- a calculator

What to do:

Answer the following questions and show all the necessary steps.

A survey was conducted among 50 recyclers. The mass of the waste each of them recycled in a month was recorded in the frequency table below.

Mass of recycling (in kg)	Midpoint of interval	Frequency	Midpoint $\times$ frequency
$40 < m \leq 45$	$\frac{40+45}{2} = 42.5$	7	$7 \times 42.5 = 297.5$
$45 < m \leq 50$		10	
$50 < m \leq 55$		15	
$55vm \leq 60$		12	
$60 < m \leq 65$		6	
	Total	50	$2\ 625$

1. Copy and complete the table.
2. Find the estimated mean using the formula $\bar{x} = \frac{\sum f_i x_i}{n}$, (where $f_i x_i$ is the product of the class or interval midpoint and the frequency).
3. Why do we call it the estimated mean when working with grouped data?
4. What is the modal class interval of the data?
5. Use the following formula to calculate the mode: $M_o = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c$ (where l is the lower limit of the modal class, c is the class width, f_m is the frequency of the modal class, f_{m-1} is the frequency of the class before the modal class and f_{m+1} is the frequency of the class after the modal class).
6. In which class interval will you find the median?
7. Use the following formula to calculate the median: $\mathrm{M}_e = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c$ (where l is the lower

limit of the median class, n is the number of observations, F is the cumulative frequency of the class before the median class, f is the frequency of the median class and c is the class width).

What did you find?

1. Below is the completed table.

Mass of recycling (in kg)	Midpoint of interval	Frequency	Midpoint $\times$ frequency
$40 < m \leq 45$	$\frac{40+45}{2} = 42.5$	7	$7 \times 42.5 = 297.5$
$45 < m \leq 50$	47.5	10	$10\,(47.5) = 475$
$50 < m \leq 55$	52.5	15	$15\,(52.5) = 787.5$
$55 < m \leq 60$	57.5	12	$12\,(57.5) = 690$
$60 < m \leq 65$	62.5	6	$6\,(62.5) = 375$
	Total	50	$2\ 625$

2. The estimated mean is calculated as:
$$\begin{aligned}\bar{x} &= \frac{7\,(42.5) + 10\,(47.5) + 15\,(52.5) + 12\,(57.5) + 6\,(62.5)}{50} \\ &= \frac{2\ 625}{50} \\ &= 52.5 \text{ kg}\end{aligned}$$
3. We are working with grouped data rather than the original ungrouped data hence some information has been lost when the grouped intervals were created. So the best we can do is approximate the mean.
4. The modal class is the interval with the highest number of data values (frequency). Therefore, the modal interval is $50 < m \leq 55$, as the highest frequency is recorded in that interval.
5. We can estimate the mode using the formula:
$$\begin{aligned}M_o &= 50 + \frac{15 - 10}{2(15) - 10 - 12} \times 5 \\ &= 50 + \frac{25}{8} \\ &= 53.125 \text{ kg}\end{aligned}$$
6. The median will be found in the central interval when the intervals are arranged in order. Since there are 50 values altogether the median will be found between the 25th and 26th values. By adding the frequencies of the intervals, we see that the median will be located in the $50 < m \leq 55$ interval.
7. $$\begin{aligned}\text{M}_e &= 50 + \frac{\left(\frac{50}{2} - 17\right)}{15} \times 5 \\ &= 50 + \frac{8}{3} \\ &= 52.67\end{aligned}$$

Below are the formulae you will use to calculate the estimated mean, median and mode for grouped data.

Mean:

$$\bar{x} = \frac{\sum f_i x_i}{n}$$

$f_i x_i$ is the class midpoint multiplied by the frequency
n is the total number of observations

Median:

$$\mathrm{M}_e = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c$$

l is the lower limit of the median class
n is the total number of observations
F is cumulative frequency of the class before the median class
f is the frequency of the median class
c is the class width

Mode:

$$M_o = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c$$

l is the lower limit of the modal class
f_m is the frequency of the modal class
f_{m-1} is the frequency of the class before the modal class
f_{m+1} is the frequency of the class after the modal class
c is the class width



Example 4.1

A survey was conducted among 50 recyclers. The mass of the waste each of them recycled in a month was recorded in the frequency table below.

Mass of recycling (in kg)	Midpoint of interval	Frequency	Midpoint $\times$ frequency
$40 < m \leq 45$	$\frac{40+45}{2} = 42.5$	7	$7 \times 42.5 = 297.5$
$45 < m \leq 50$	47.5	10	$10\,(47.5) = 475$
$50 < m \leq 55$	52.5	15	$15\,(52.5) = 787.5$
$55 < m \leq 60$	57.5	12	$12\,(57.5) = 690$
$60 < m \leq 65$	62.5	6	$6\,(62.5) = 375$
	Total	50	$2\ 625$

Use a calculator to find the mean for the data.

Solution

It is very rare in real life applications that statistics will be computed by hand. In this example we will use the Casio fx-82ZA calculator to find the mean for the grouped data from Activity 4.1. Make sure that you consult your calculator manual for steps specific to your calculator.

Step 1: As you are working with grouped data we need to tell the calculator that we are using frequencies. To do that: Press SHIFT MODE (SET UP). Scroll down using the arrows and Select 3: STAT

The calculator display says frequency 1: On or 2: Off. Select 1: On

Step 2: Press MODE and 2: STAT then Select 1: 1-VAR

Step 3: Enter the class midpoint under the 'X' column and press $=$. Next enter the corresponding frequency under the FREQ column and press $=$. You can use the arrows to scroll and correct any mistakes.

After you have finished entering the data, press AC. Don't panic when the values disappear! The data entering screen will disappear but the data has been stored and can be brought back if required.

Step 4: To find the summary statistics including the mean press SHIFT '1' (STAT).

Press 4: VAR: all the summary statistics will be found here.

Press 2: $\overline{x} =$ for the estimated mean. Note: your answer should be the same as the value $(52.5\ \text{kg})$ calculated manually in Activity 4.1. If not then you have made a mistake.

We will use these same calculator steps to find the variance and standard deviation for data in Maths Level 4 so make sure you understand how to use your calculator.

Exercise 4.1

1. The following data shows the number of hours over the weekend that teenagers play video games.

Hours spent on video games	Frequency
$(0; 3.5]$	3
$(3.5; 7.5]$	7
$(7.5; 11.5]$	12
$(11.5; 15.5]$	7
$(15.5; 19.5]$	9

What is the best estimate for the mean number of hours spent playing video games?

2. In a traffic survey a random sample of 50 motorists were asked what distance they drove to work daily. The results of the survey are shown in kilometres in the table below.

Distance	$0 < d \leq 10$	$10 < d \leq 20$	$20 < d \leq 30$	$30 < d \leq 40$	$40 < d \leq 50$
Frequency	9	17	15	5	4
Midpoint x_i					
$f_i \times x_i$					

   a. Copy and complete the table.
   b. Calculate the mean distance.
   c. Calculate the mode.
   d. Calculate the median distance.
   e. Draw the ogive for the data using cumulative frequencies.

The full solutions are at the end of the unit.

Measures of dispersion

Variation is present in any set of data. For example, a $300\ \text{ml}$ can of cooldrink may contain slightly more or slightly less than $300\ \text{ml}$. Manufacturers regularly run tests to determine if the amount of beverage in a can falls within the desired range.

To find out how scattered the data values are from the mean we must calculate the measures of variability, called measures of dispersion.

The range, interquartile range and standard deviation are the three commonly used measures of dispersion.

Be aware that when you collect data, your data may vary somewhat from the data someone else is collecting for the same purpose. This is completely natural. However, if two or more of you are collecting the same data and get very different results, it is time to re-evaluate the data methods and accuracy.

Take note!

Measures of dispersion show how much the data vary from the average value of a data set.

Interquartile range

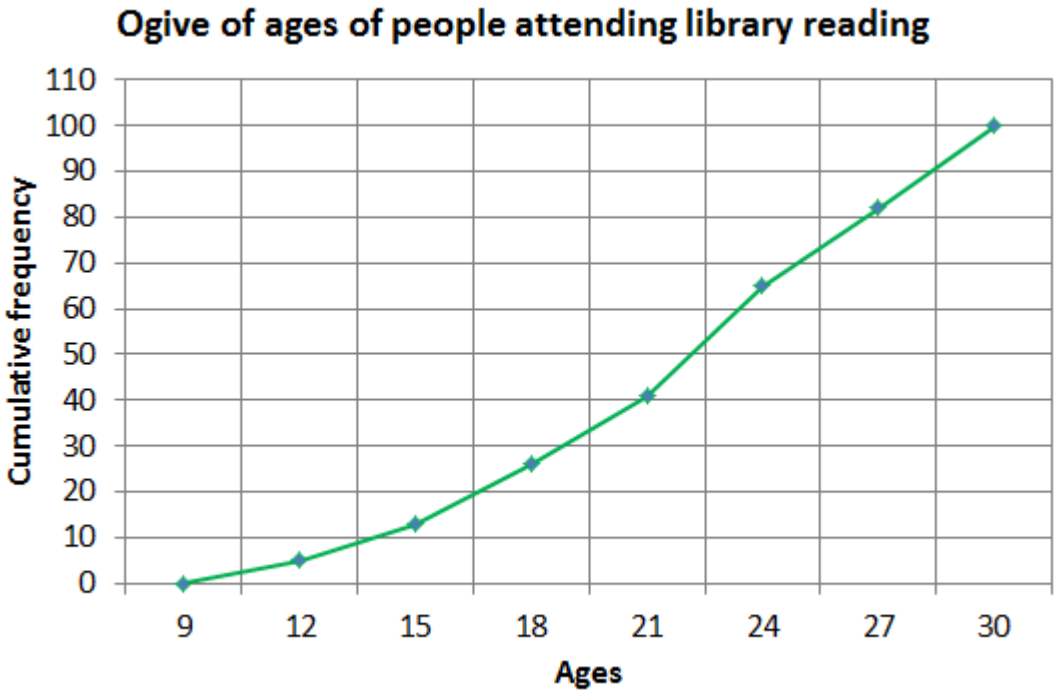
The interquartile range (IQR) is a number that shows the spread of the middle half or the middle 50% of the data. It is the difference between the third quartile (Q_3) and the first quartile (Q_1). The IQR is a good measure of the spread of data as it is not affected by outliers.

Interquartile range: $\mathrm{IQR} = \mathrm{Q}_3 - \mathrm{Q}_1$



Example 4.2

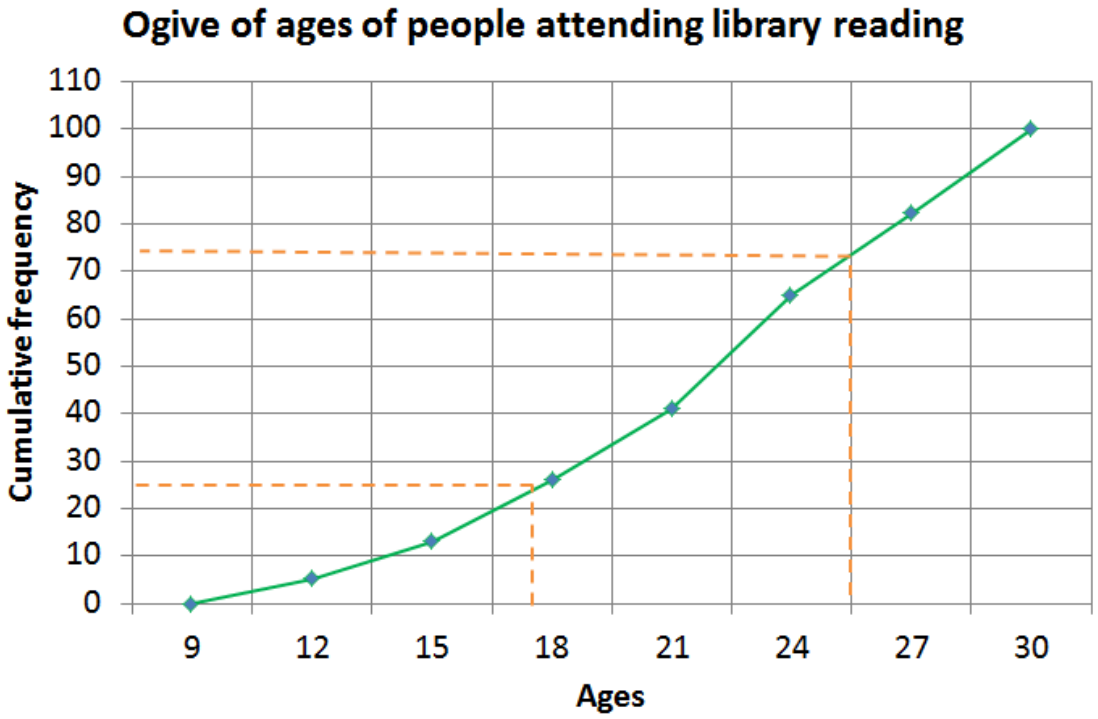
Below is the ogive of the ages of people attending a public reading by an author at a library.



1. Find Q_3.
2. What age are 75% of the people younger than?
3. Find Q_1 .
4. 25% of the people are less than _____ years old.
5. Find the IQR. What does this value tell us?

Solution

We can read the answers from the graph shown below.



1. There are 100 people attending the reading therefore, Q_3 is in position 75 $(\frac{3}{4} \times 100 = 75)$. We can read the value from the graph, $\mathrm{Q}_3 \approx 25$.
2. Q_3 is the value that 75% of the data values are less than. Therefore, 75% of the people are younger than 25.
3. Q_1 is in position 25 $(\frac{1}{4} \times 100 = 25)$, from the graph this corresponds to approximately 17.
4. 25% of the people are less than 17 years old.
5. $\mathrm{IQR} = 25 - 17 = 8$
   The IQR tells us that the middle 50% of ages vary by about 8 years. We can also tell that 50% of the people are aged between 17 and 25 years old.

Note

For more practise on IQRs you can watch the video "Calculating interquartile ranges".

Calculating interquartile ranges (Duration: 6.11)

The interquartile range is very useful when you compare different sets of data. A larger IQR means the data are more spread out (varied) and a lower IQR shows less variability.

Example 4.3

We need to compare two sets of maths marks for Class A and Class B. We are given the following information.

Class A:

$\text{Mean} = \text{median} = \text{mode} = 55\%$

$\text{IQR} = 45\%$

$\text{Range} = 75\%$

Class B:

$\text{Mean} = \text{median} = \text{mode} = 55\%$

$\text{IQR} = 35\%$

$\text{Range} = 80\%$

Which class performed better in the test?

Solution

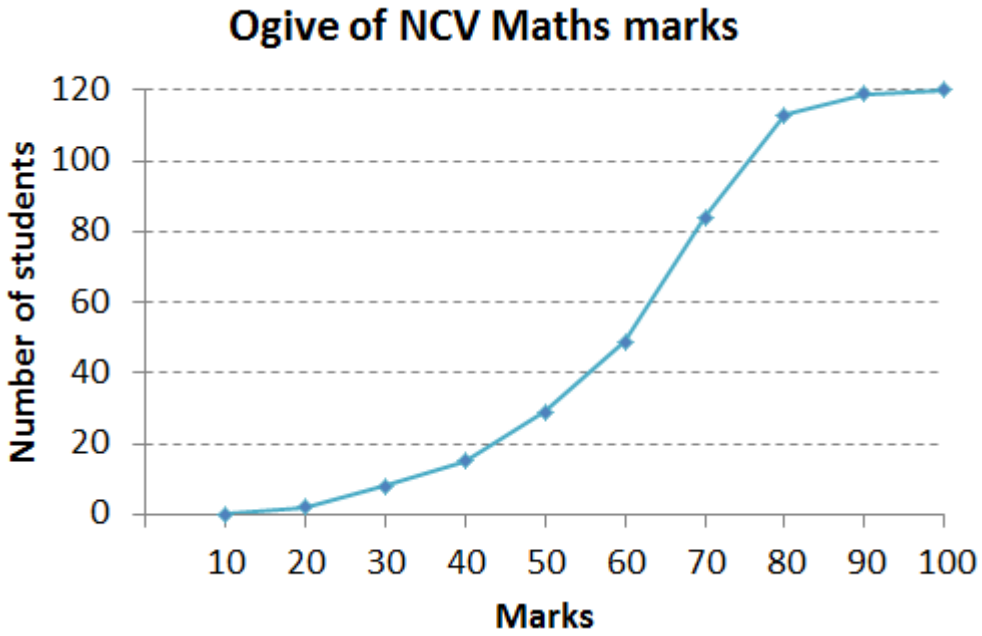
Both data sets have the same mean, median and mode but the data values may be very different. So by just focusing on the measures of central tendency you may arrive at the incorrect conclusion that both classes performed the same.

The range of Class B is higher than the range of Class A but the range is easily affected by outliers so it is not the best measure of variability of the data.

The IQR for Class B is lower than the IQR for Class A. This means there is less variability in the middle 50% of the marks for Class B, therefore, Class B performed better than Class A.

Exercise 4.2

The marks (as a percentage) obtained in an NCV Maths Level 3 examination are shown in the ogive.



1. How many learners' marks were recorded?
2. 75% of learners scored less than what mark?
3. 25% of learners scored less than what mark?
4. 75% of learners scored more than what mark?
5. Find the interquartile range.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the mean, median and mode for grouped data.
- How to use the ogive to find the interquartile range.
- How to interpret the interquartile range.

Unit 4: Assessment

Suggested time to complete: 30 minutes

1. The following table shows the amount (in Rand) that 50 families, who live in apartments, spent on pre-paid electricity.

Electricity cost (Rand amount)	Number of families
$450 < \text{R} \leq 600$	4
$600 < \text{R} \leq 750$	12
$750 < \text{R} \leq 900$	16
$900 < \text{R} \leq 1050$	10
$1050 < \text{R} \leq 1150$	8

a. Complete the frequency table.

Electricity cost (Rand amount)	Class midpoint	Number of families	Frequency $\times$ class midpoint
$450 < \text{R} \leq 600$		4	
$600 < \text{R} \leq 750$		12	
$750 < \text{R} \leq 900$		16	
$900 < \text{R} \leq 1050$		10	
$1050 < \text{R} \leq 1150$		8	

b. What was the average amount spent on electricity?

c. Calculate the median amount.

d. Compare the mean amount to the median and comment on the skewness of the data.

2. The table below shows the ages of people attending a local gym on one particular day. Study the table and answer the questions that follow.

Ages	Frequency	Cumulative frequency
< 30	2	2
$30 - 34$	10	12
$35 - 39$	8	20
$40 - 44$	9	29
$45 - 49$	6	35
$50 - 54$	5	40

a. Sketch the ogive curve of the data.

b. Use the ogive curve to find the interquartile range by first estimating the values for the first and third quartiles.

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. Find the class midpoint and frequency multiplied by class midpoint first.

Hours spent on video games	Class midpointx_i	Frequency	$f_i \times x_i$
$(0; 3.5]$	1.75	3	$1.75 \times 3 = 5.25$
$(3.5; 7.5]$	5.5	7	$5.5 \times 7 = 38.5$
$(7.5; 11.5]$	9.5	12	$9.5 \times 12 = 114$
$(11.5; 15.5]$	13.5	7	$13.5 \times 7 = 94.5$
$(15.5; 19.5]$	9	9	$17.5 \times 9 = 157.5$
	Total	38	$\sum f_i \times x_i = 409.75$

$$\begin{aligned}\bar{x} &= \frac{\sum f_i x_i}{n} \\ &= \frac{409.75}{38} \\ &= 10.78 \text{ hours}\end{aligned}$$

2.

a.

Distance	$0 < d \leq 10$	$10 < d \leq 20$	$20 < d \leq 30$	$30 < d \leq 40$	$40 < d \leq 50$
Frequency	9	17	15	5	4
Midpoint x_i	5	15	25	35	45
$f_i \times x_i$	45	255	375	175	180

b. The mean distance is 20.6 km.

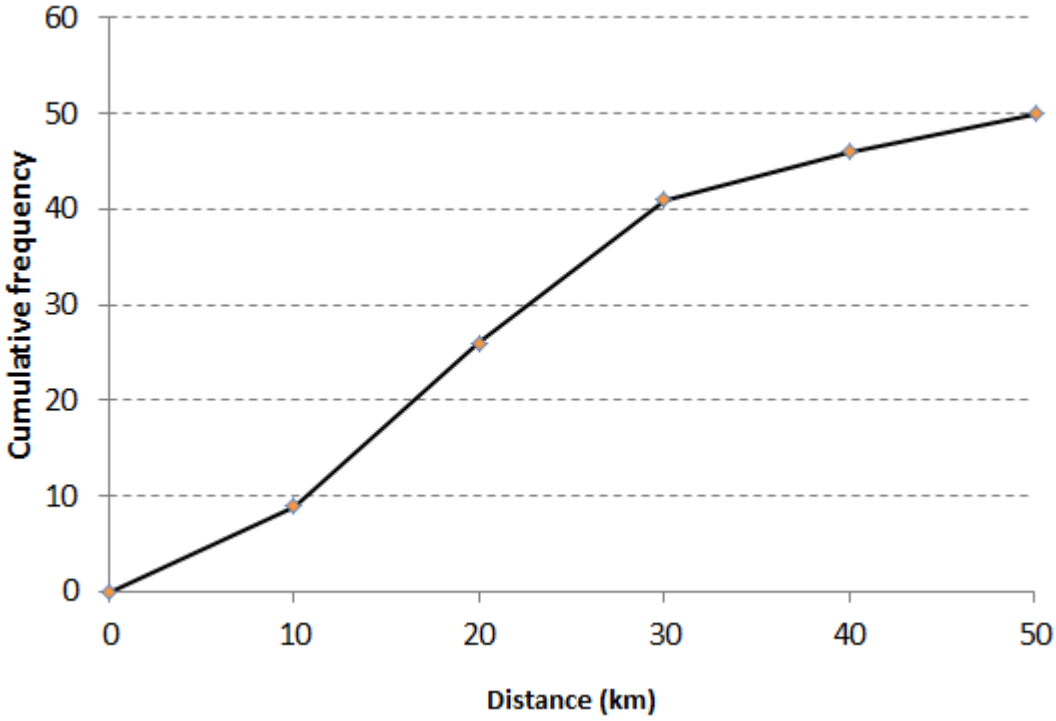
c. The modal interval is $10 < d \leq 20$ with a frequency of 17.

$$\begin{aligned}M_o &= l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c \\ &= 10 + \frac{17 - 9}{2(17) - 9 - 15} \times 10 \\ &= 18 \text{ km}\end{aligned}$$

d. Median:

$$\begin{aligned}\text{M}_e &= l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c \\ &= 10 + \frac{25 - 9}{17} \times 10 \\ &= 19.4 \text{ km}\end{aligned}$$

e. Find the cumulative frequencies first and then plot against the distance to give the ogive below.

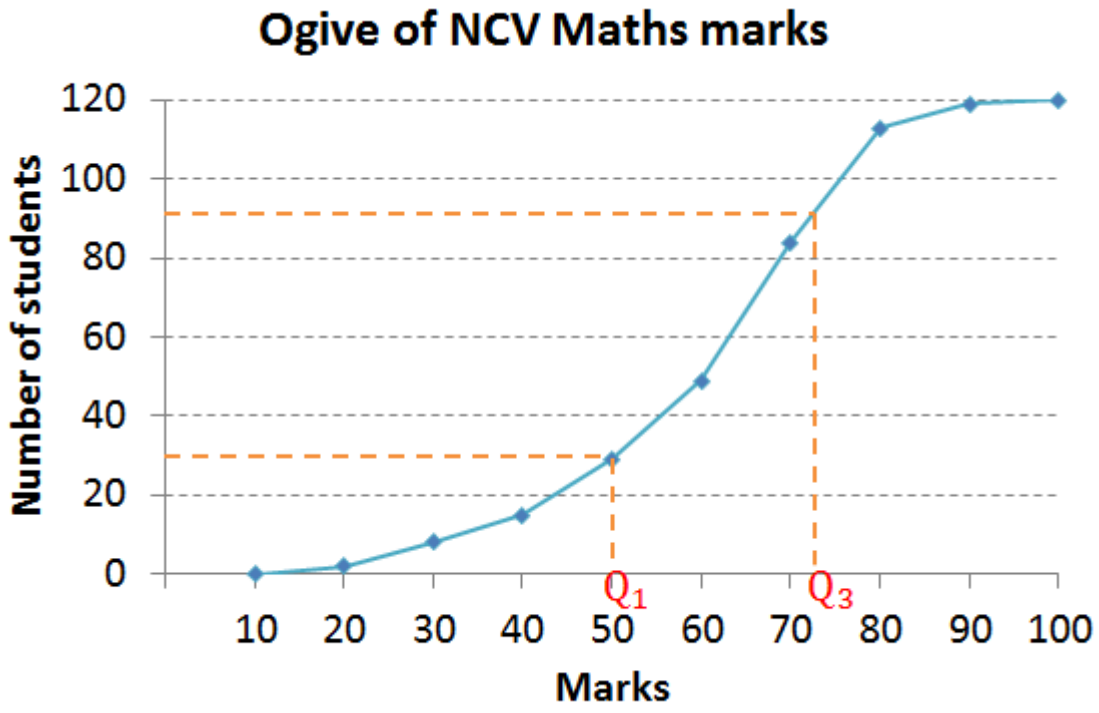


Back to Exercise 4.1

Exercise 4.2

1. There are 120 observations.

The ogive below is used to answer questions 2 to 5.



2. $0.75 \times 120 = 90$ so we must find the third quartile from position 90 on the vertical axis of ogive as

shown in the graph. We see that 75% of learners got less than 72%.

3. $0.25 \times 120 = 30$ so we must look for 30 on the y-axis and draw a horizontal line until we reach the graph and then read off the marks from the x-axis. We see that 25% of learners scored less than 50%.
4. Since 25% of learners scored less than 50% this means 75% of learners scored more than 50%.
5. $$\begin{aligned} \text{IQR} &= 72\% - 50\% \\ &= 22\% \end{aligned}$$

Back to Exercise 4.2

Unit 4: Assessment

1.

a.

Electricity cost (Rand amount)	Class midpoint	Number of families	Frequency $\times$ class midpoint
$450 < \text{R} \leq 600$	525	4	$2\ 100$
$600 < \text{R} \leq 750$	675	12	$8\ 100$
$750 < \text{R} \leq 900$	825	16	$13\ 200$
$900 < \text{R} \leq 1050$	975	10	$9\ 750$
$1050 < \text{R} \leq 1150$	$1\ 100$	8	$8\ 800$
	Total	50	$41\ 950$

b. Average amount spent:

$$\begin{aligned} \bar{x} &= \frac{\sum f_i x_i}{n} \\ &= \frac{41\ 950}{50} \\ &= \text{R}\ 839 \end{aligned}$$

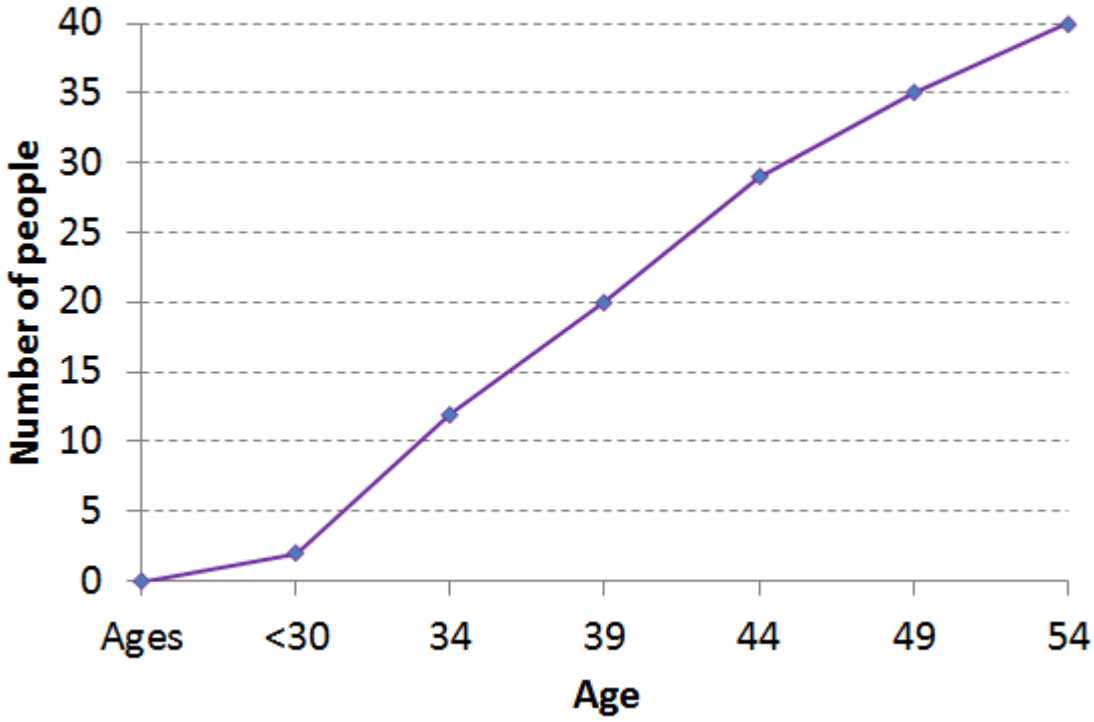
c. The median will be found in the interval $750 < \text{R} \leq 900$.

$$\begin{aligned} \text{M}_e &= l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c \\ &= 750 + \frac{25 - 16}{16} \times 150 \\ &= \text{R}834.38 \end{aligned}$$

d. mean $>$ median by a small amount so the data will be slightly skewed to the right.

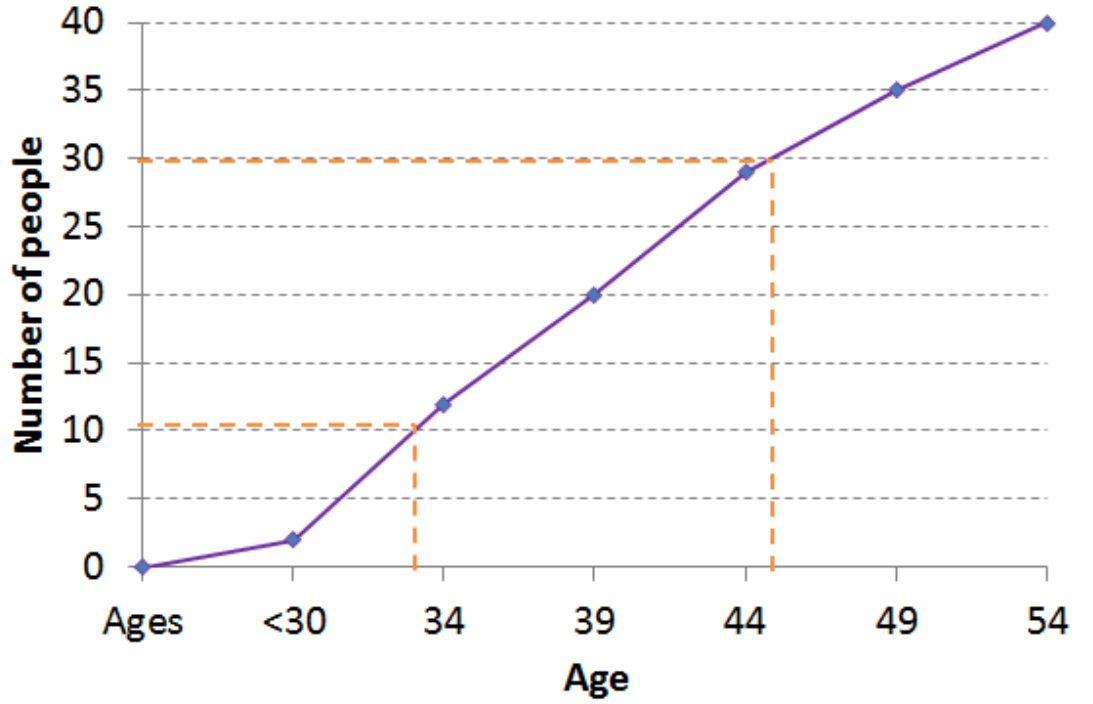
2.

a.



b.

Use the graph below to answer this part of the question by reading off from the graph.



$Q_1 \approx 33$

$Q_3 = 45$

$\begin{aligned} \text{IQR} &= 45 - 33 \\ &= 12 \end{aligned}$

Back to Unit 4: Assessment

Media Attributions

- Example 4.2 Ogive of ages © DHET is licensed under a CC BY (Attribution) license
- Example 4.2 Answer © DHET is licensed under a CC BY (Attribution) license
- Exercise 4.2 Q1 © DHET is licensed under a CC BY (Attribution) license
- Exercise 4.1 Q2e Ans © DHET is licensed under a CC BY (Attribution) license
- Exercise 4.2 Q1 Answers b-e © DHET is licensed under a CC BY (Attribution) license
- Assessment Q2 © DHET is licensed under a CC BY (Attribution) license
- Assess Q2 b Ans © DHET is licensed under a CC BY (Attribution) license

SUBJECT OUTCOME XIII

FINANCIAL MATHEMATICS: PLAN AND DESCRIBE HOW TO MANAGE FINANCES OF SOCIAL CLUBS

Subject outcome

Subject outcome 5.1: Plan and describe how to manage finances of social clubs.

Learning outcomes

- Describe financial concepts related to social clubs, methods of financing and financial control.
    - Range: normal operating expenses telephone, stationery, water & electricity, consumables plus club specific income and expenses: membership fees, donations, sponsorships, affiliation fees, license fees, honorarium, function fees, cutlery and crockery, catering and other expenses to host specific events e.g. prize giving functions, year-end functions, any other club events.
- Use given information to draw up a yearly budget for a social club.
  Note: Detailed examples and templates are available on the internet.
- Use given information to record actual income and expenditure for a year.
- Compare actual income and expenses to the projected budget figures.
- Identify variances and provide possible corrective actions to be taken to control future finances of the club.

Unit 1 outcomes

By the end of this unit you will be able to:

- Calculate operating expenses.
- Draw up a yearly budget.
- Use given information to record actual income and expenditure for a year.
- Compare actual income and expenses to the projected budget figures.
- Identify variances and provide possible corrective actions to be taken to control future finances of the club.

Unit 1: Financial control for social clubs

NATASHIA BEARAM-EDMUNDS

Unit 1 outcomes

By the end of this unit you will be able to:

- Calculate operating expenses.
- Draw up a yearly budget.
- Use given information to record actual income and expenditure for a year.
- Compare actual income and expenses to the projected budget figures.
- Identify variances and provide possible corrective actions to be taken to control future finances of the club.

What you should know

There is no prior learning required for this unit.

Introduction

In this unit we are going to learn about the finances involved in social clubs. Social clubs are membership organisations that are maintained mainly by fees paid by its members, from donations, from fundraising or grants. As these clubs are generally formed by like-minded people with a common interest or activity, they are allowed to limit their membership. There are many different types of social clubs in existence. Burial societies, book clubs, sports clubs, country clubs, Christian fellowships and gaming clubs are all examples of social clubs. Stokvels are social groups in which individuals regularly contribute a fixed amount of money into a common pool, taking turns on a rotational basis to use a lump sum for individual purpose.

Here are some of the common terms that you will need to be familiar with.

Income is the money received in exchange for goods or services. In social clubs income is received from membership fees and donations, for example.

An **expense** is a cost that is incurred.

A **budget** is a financial plan that shows expected income and expenses over a certain period.

Profit is money you are left with after deducting all expenses.

$$\text{profit} = \text{income} - \text{expenses}$$

If expenses are greater than income, then a financial loss is incurred.

Financial management

The size of the club will determine the financial management processes that are needed. A management

committee elected by club members will ensure that the club has good financial management processes in place to achieve its objectives.

Generally a treasurer, who is appointed by the club's members or management committee, will be responsible for the financial management of the club.

A simple computer accounting package may be used to keep track of the club's finances, however, if the club is small enough a manual system is sufficient.

The treasurer must keep track of:

- expenses
- income
- cash flow
- profit or loss
- necessary financial adjustments
- banking, bookkeeping and record keeping
- fundraising and ensuring appropriate use of funds
- tax considerations if applicable
- annual budget.

The club will have its own bank account where membership fees, donations, etc. can be deposited.

A **cashbook** is a journal in which all of the club's receipts and payments are recorded. Cash includes money, credit card slips, electronic payments and money orders. The details of all transactions should be recorded in the cashbook.

Budgets

Annual budgets are an important part of planning and organising the financial resources of a social club. Usually the treasurer will set up the annual budget.

A budget shows how much income the club expects to receive during the next financial year, how much the club expects to spend in the next financial year, and the predicted surplus (profit) or deficit (loss) for the budget period.

Once the budget is approved it is added to the accounting system so that it can be compared to the club's actual income and expenses for the year.

Figure 1 gives an example of a club's annual budget.

2020/2021 DNE club budget		
	Budgeted amount	
Income		
Membership fees	R 50 000	
Donations	R 12 000	
Fund-raising	R 25 000	
Total	**R 87 000**	
Expenses		
Rent	R 34 000	
Utilities	R 6 000	
Wages	R 24 500	
Events	R 18 000	
Total	**R 82 500**	
Projected profit/loss	R 4 500	

Figure 1: Sample of an annual budget

Looking at Figure 1, we can calculate that the projected profit for the year for the DNE club is: $\text{R}87\ 000 - \text{R}82\ 500 = \text{R}4\ 500$.

Recording income and expenditure

The treasurer must regularly compile reports on the financial situation of the club so that actual performance can be compared against the budget.

The profit and loss statement, also referred to as the income and expenditure statement, is a summary of a club's income and expenses over a specific period of time and shows how much money a club has made or lost as a result of its operations. Losses are sometimes shown in red on the statement.

The income and expenditure statement is prepared at regular intervals (usually monthly and at the financial year end) to show the results of operations for a given period. This statement is important for clubs, as it shows how received funds have been allocated against operating expenses during the reporting period.

Income

Clubs receive most of their income from membership fees, grants and donations. These are often referred to as 'contributions'. Some clubs may also receive income from trading activities, such as the sale of branded goods or goods produced from workshop activities.

Each source of income should be disclosed in the financial statement. Where income is received with specific conditions about how it is to be used, these conditions should be reported through the financial statements.

Expenses

For a complete financial picture of the club, all income must be matched against the expenses incurred during the reporting period. Expenses include rent, salaries, utilities (such as water and electricity), gifts, social functions, catering, and other operating expenses.

Variance

Variance is a method used to compare actual financial performance to budgeted expectations. Variance can be shown as percentages or monetary differences.

If the variance amount is about the same as what is budgeted for, then no action needs to be taken.

A budget variance is positive, or favourable, when the income results are higher than the budget expectations. But if actual expenses are greater than expected in the budget, the result is a negative variance.

The formula to calculate variance is:

$$\text{variance amount} = \text{actual number} - \text{budgeted number}$$

- If **income variance** is positive, with actual income greater than budgeted (or expected) income, this is favourable.
- If **expenditure variance** is positive, with actual expenditure greater than budgeted (or expected) expenditure, this is NOT favourable: spending was more than it was expected to be. So, when you see a negative in the **variance expense** column it is a good thing because it means expenses were less than expected.



Example 1.1

The treasurer of Mpower Club compared the actual income and expenses figures against the projected budget figures at the end of the financial year.

ITEM	BUDGETED (ZAR)	ACTUAL (ZAR)	VARIANCE (ZAR)
INCOME			
Membership fees	35 000	42 000	**A**
Income from sales	5 000	**B**	−500
Donations	12 000	6 000	**C**
Total	52 000	52 500	500
EXPENSES			
Club rental	15 000	15 000	0
Wages	**D**	7 000	2 000
Social events	32 000	38 000	6 000
Utilities	1 500	1 000	**E**
Total	57 500	61 000	3 500
Surplus/deficit	−5 500	**F**	

1. Complete the table by filling in the missing amounts for letters A to F.
2. Did Mpower perform better financially than expected?
3. List one possible way to decrease the club's deficit.

Solution

1. $$\begin{aligned}\text{A} &= 42\ 000 - 35\ 000\\ &= 7\ 000\\ \text{B} &= 5\ 000 + (-500)\\ &= 4\ 500\\ \text{C} &= 6\ 000 - 12\ 000\\ &= -6\ 000\\ \text{D} &= 7\ 000 - 2\ 000\\ &= 5\ 000\\ \text{E} &= 1\ 000 - 1\ 500\\ &= -500\\ \text{F} &= 42\ 000 + 4\ 500 + 6\ 000 - (15\ 000 + 7\ 000 + 38\ 000 + 1\ 000)\\ &= -8\ 500\end{aligned}$$
2. Mpower performed worse than expected with an additional $\text{R}3\ 000$ loss over what was budgeted for.
3. The club can reduce the costs by hosting fewer social functions.



Example 1.2

Ali manages the finances of the So we Sew Club. She is compiling the income and expenditure sheet for the club. Below are the transactions for the month. Use the following information to draw up the profit and loss statement.

Opening balance	$\text{R}0$
Rent	$\text{R}2\ 000$
Membership fees	$\text{R}100$ per member for 30 members
Sponsorships	$\text{R}3\ 000$
Purchase of knitting needles	$\text{R}600$
Transport costs	$\text{R}1\ 000$
Tea and coffee	$\text{R}500$

PROFIT & LOSS STATEMENT SO WE SEW	
INCOME	AMOUNT
Opening balance	R0
TOTAL INCOME	
EXPENDITURE	AMOUNT
TOTAL EXPENDITURE	
SURPLUS/DEFICIT	

Solution

We identify the income items: membership fees and sponsorships.

We identify the expense items: rent, knitting needles, transport costs, and tea and coffee.

Once you have identified which items fall under income and which are expenses you can complete the profit and loss statement.

PROFIT & LOSS STATEMENT SO WE SEW	
INCOME	AMOUNT
Opening balance	R0
Membership fees	$\text{R}100 \times 30 = \text{R}3\ 000$
Sponsorships	R3 000
TOTAL INCOME	R6 000
EXPENDITURE	AMOUNT
Rent	R2 000
Purchase of knitting needles	R600
Transport costs	R1 000
Tea and coffee	R500
TOTAL EXPENDITURE	R4 100
SURPLUS/DEFICIT	R1 900

Exercise 1.1

1. Siya is the financial secretary for the local church club. Use the information presented to complete the income and expenditure statement for the club.

Balance brought forward	R1 500
Sponsorship from local businesses	R1 500
Contribution from church board	R5 000
Fundraising	R3 000
Gifts to children's home	R900
Transport costs	R1 500
Purchase of bibles	R1 500

PROFIT & LOSS STATEMENT	
INCOME	AMOUNT
Balance brought forward	
TOTAL INCOME	
EXPENDITURE	AMOUNT
TOTAL EXPENDITURE	
SURPLUS/DEFICIT	

2. Leading Ladies is a social club that cares for abandoned children. The club was expected to receive the following amounts of money for the year ended 31/12/2019:

Grant from Lottery	R300 000
Donations	R25 000
Fees from members	R18 000
Total	R343 000

The club was expected to have the following expenditure for the same year:

Rent	R36 000
Stationery	R3 000
Baby formula	R89 000
Nappies	R76 000
Water and electricity	R12 000
Wages	R80 000
Baby clothes	R32 000
Medical expenses	R13 000
Total	R341 000

Listed below is the actual income and expenditure of the club for the year ended 31/12/2019:

Actual income:

Grant from Lottery	R250 000
Donations	R55 000
Fees from members	R20 000
Total	R325 000

Actual expenditure:

Rent	R36 000
Stationery	R2 800
Baby formula	R91 000
Nappies	R75 000
Water and electricity	R11 200
Wages	R80 000
Baby clothes	R29 000
Medical expenses	R10 000
Total	R335 000

1. Did the club expect to have a surplus or deficit at the end of the year? What amount were they expecting to have as a surplus or deficit?
2. Study the information provided and complete the table below. Write only the answers for each letter A to H.

ITEM	BUDGETED (ZAR)	ACTUAL (ZAR)	VARIANCE (ZAR)
INCOME			
Grant from lottery	300 000	250 000	**A**
Fees from members	18 000	**B**	+2 000
Donations	25 000	55 000	**C**
Total	343 000	**D**	−18 000
EXPENSES			
Rent	36 000	36 000	0
Stationery	3 000	2 800	+200
Baby formula	89 000	91 000	**E**
Diapers	76 000	**F**	+1 000
Water and electricity	12 000	11 200	+800
Wages	80 000	80 000	0
Baby clothes	32 000	29 000	+3 000
Medical expenses	13 000	10 000	+3 000
Total	341 000	335 000	+6 000
Surplus/deficit	**G**	**H**	

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How social club's finances are managed.
- How to prepare a profit and loss statement.
- How to complete and compare the annual budget.
- How to calculate the variance between the actual amounts and the budgeted amounts.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. The treasurer of Super Seven Club compared the actual income and expenses figures against the projected budget figures at the end of the financial year. Complete the table by filling in the missing amounts for letters A to J.

ITEM	BUDGETED (ZAR)	ACTUAL (ZAR)	VARIANCE (ZAR)
INCOME			
Membership fees	17 000	12 000	**A**
Income from sales	4 000	**B**	+1 000
Sponsorship	1 000	1 000	**C**
Total	**D**	18 000	−4 000
EXPENSES			
Club rental	0	0	0
Wages	**E**	2 000	0
Functions	4 000	6 000	**F**
Gift to orphanage	0	4 000	4 000
Total	**G**	12 000	**H**
Surplus/deficit	**I**	**J**	

2. Rachel is the financial secretary of the Women For Change club. Use the information presented to complete the income and expenditure statement for the club.

Opening balance	R3 500
Membership fees	R10 500
Income from sales	R8 000
Fundraising	R5 000
Donation to homeless shelter	R6 500
Transport costs	R1 000
Rent	R4 500

PROFIT & LOSS STATEMENT	
INCOME	AMOUNT
Opening balance	
TOTAL INCOME	
EXPENDITURE	AMOUNT
TOTAL EXPENDITURE	
SURPLUS/DEFICIT	

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

PROFIT & LOSS STATEMENT	
INCOME	AMOUNT
Balance brought forward	R1 500
Sponsorship from local businesses	R1 500
Contribution from church board	R5 000
Fundraising	R3 000
TOTAL INCOME	R11 000
EXPENDITURE	AMOUNT
Gifts to children's home	R900
Transport costs	R1 500
Purchase of bibles	R1 500
TOTAL EXPENDITURE	R3 900
SURPLUS	R7 100

2.

a. The club expects to have a surplus of R2 000.

b.

ITEM	BUDGETED (ZAR)	ACTUAL (ZAR)	VARIANCE (ZAR)
INCOME			
Grant from lottery	300 000	250 000	**A** −50 000
Fees from members	18 000	**B** 20 000	+2 000
Donations	25 000	55 000	**C** +30 000
Total	343 000	**D** 325 000	−18 000
EXPENSES			
Rent	36 000	36 000	0
Stationery	3 000	2 800	+200
Baby formula	89 000	91 000	**E** +2 000
Diapers	76 000	**F** 75 000	+1 000
Water and electricity	12 000	11 200	+800
Wages	80 000	80 000	0
Baby clothes	32 000	29 000	+3 000
Medical expenses	13 000	10 000	+3 000
Total	341 000	335 000	+6 000
Surplus/deficit	**G** +2 000	**H** −10 000	

Back to Exercise 1.1

Unit 1: Assessment

1.

ITEM	BUDGETED (ZAR)	ACTUAL (ZAR)	VARIANCE (ZAR)
INCOME			
Membership fees	17 000	12 000	**A** −5 000
Income from sales	4 000	**B** 5 000	+1 000
Sponsorship	1 000	1 000	**C** 0
Total	**D** 22 000	18 000	−4 000
EXPENSES			
Club rental	0	0	0
Wages	**E** 2 000	2 000	0
Functions	4 000	6 000	**F** +2 000
Gift to orphanage	0	4 000	4 000
Total	**G** 6 000	12 000	**H** 6 000
Surplus/deficit	**I** +16 000	**J** +6 000	

2.

PROFIT & LOSS STATEMENT	
INCOME	AMOUNT
Opening balance	R3 500
Membership fees	R10 500
Income from sales	R8 000
Fundraising	R5 000
TOTAL INCOME	R27 000
EXPENDITURE	AMOUNT
Donation to homeless shelter	R6 500
Transport costs	R1 000
Rent	R4 500
TOTAL EXPENDITURE	R12 000
SURPLUS	R15 000

Back to Unit 1: Assessment

Media Attributions

- Figure 1 Budget sample © DHET is licensed under a CC BY (Attribution) license

SUBJECT OUTCOME XIV

FINANCIAL MATHEMATICS: USE SIMPLE AND COMPOUND INTEREST TO EXPLAIN AND DEFINE A VARIETY OF SITUATIONS

Subject outcome

Subject outcome 5.2: Use simple and compound interest to explain and define a variety of situations.

Learning outcomes

- Construct and make use of timelines to solve problems relating to finance.
- Use the simple growth formula $A = P(1 + i.n)$ to solve real life problems.
- Use the compound growth formulae $A = P(1 + i)^n$ or $A_t = A_0\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$ to solve problems subject to the following compounding:
  - annually
  - semi-annually
  - quarterly
  - monthly
  - daily.

Range: unknown values to calculate will only include A, P and i.

Unit 1 outcomes

By the end of this unit you will be able to:

- Use the simple growth formula to solve problems related to finance.
- Solve for A, P, i and n using the simple interest formula.

Unit 2 outcomes

By the end of this unit you will be able to:

- Use the compound growth formula (or compound *interest* formula) to solve problems related to finance.
- Solve for A, P and i using the compound interest formula.



Unit 3 outcomes

By the end of this unit you will be able to:

- Interpret financial questions using timelines.

Unit 1: Calculations involving simple interest

GILL SCOTT

Unit outcomes

By the end of this unit you will be able to:

- Use the simple growth formula to solve problems related to finance.
- Solve for A, P, i and n using the simple interest formula.

What you should know

Before you start this unit, make sure that you can:

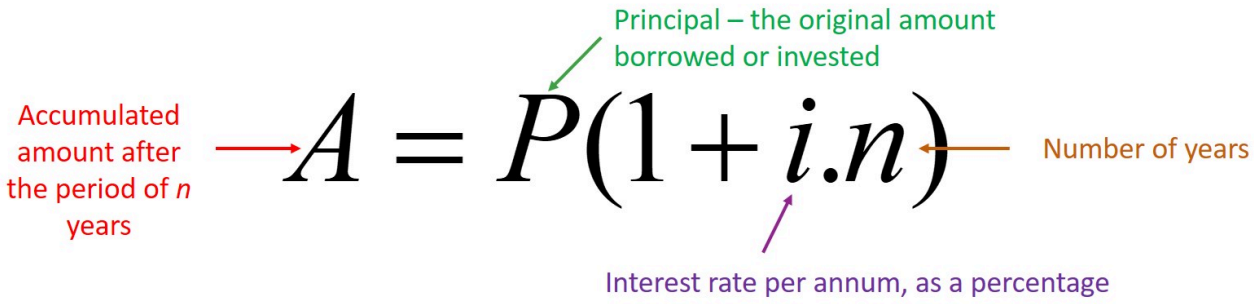
- Use a scientific calculator. Look back to level 2 subject outcome 1.1 to revise this.
- Use basic laws of arithmetic.
- Solve simple equations. Level 2 subject outcome 2.2 unit 1 will provide help with these.
- Complete basic simple interest calculations. Refer to level 2 subject outcome 5.2 unit 1 for help with this.

Introduction

Simple and compound growth formulae are used most commonly when talking about financial loans and investments. Growth is calculated as 'interest' on an amount that is invested or borrowed for a period of time. In this unit we will look at simple interest and how it is calculated.

Calculating simple interest

The elements of the simple interest/growth formula are as follows:



If you multiply out the brackets in this formula, you see that the accumulated amount is made up of the

sum of the **principal** – the original amount that is borrowed or invested – and the increase (by the given percentage) of the principal over the specified number of years:

$A = P + P.i.n$

Let's look at a few examples of how to use this simple interest formula.



Example 1.1

A TV costs $\text{R}4\ 500$ on a hire purchase loan agreement. The interest charged is 12% simple interest each year (per annum, or 'p.a.') based on the full purchase price of the TV. Portia repays the loan over three years.

1. Calculate the full amount of money paid for the TV.
2. Calculate how much must be paid each month if equal payments are made every month.

Solution

It is a good idea to list all the information given, and to identify what variable must be calculated.

$P = 4\ 500$
$i = 12\%$
$n = 3$
$A = ?$

The interest is always a percentage. You can express this as a fraction over 100, or substitute the decimal value right away: $12\% = \frac{12}{100} = 0.12$

1. $$\begin{aligned} A &= P(1 + i.n) \\ &= 4\ 500\left[1 + \left(\frac{12}{100}\right)(3)\right] \\ &= 4\ 500\,[1 + 0.36] \\ &= 4\ 500(1.36) \\ &= 6\ 120 \end{aligned}$$
   A total of $\text{R}6\ 120$ is paid over the period of three years or 36 months.
2. Each monthly payment should be $\frac{6\ 120}{36} = \text{R}170$



Example 1.2

Pumla borrows $\text{R}150\ 000$ to start a business and repays the loan in one lump sum after five years. The total amount that she is required to pay is $\text{R}243\ 750$. If the interest charged on the loan was simple interest, calculate the annual interest rate.

Solution

List all the information given and identify what variable must be calculated:

$P = 150\ 000$
$A = 243\ 750$
$n = 5$
$i = ?$

Therefore:

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{n}\left(\frac{A}{P} - 1\right) = i$$
$$i = \frac{1}{5}\left(\frac{243\ 750}{150\ 000} - 1\right)$$
$$= \frac{1}{5}(1.625 - 1)$$
$$= \frac{1}{5}(0.625)$$
$$= 0.125 \text{ or } 12.5$$



Example 1.3

A cell phone is advertised for R280 per month on hire purchase over a period of three years. If the simple interest rate charged is 33.9%, calculate the original cash price of the phone. What would the buyer save by buying the phone for cash rather than on hire purchase?

Solution

List all the information given and identify what variable must be calculated:

$n = 3 \text{ years}$
$i = 33.9\%$
$A = 280 \times (3 \times 12) = 10\ 080$
$P = ?$

Therefore:

$$A = P(1 + i.n)$$
$$\frac{A}{1 + i.n} = P$$
$$P = \frac{10\ 080}{1 + \left(\frac{33.9}{100}\right)3}$$
$$= \frac{10\ 080}{1 + (0.339 \times 3)}$$
$$= \frac{10\ 080}{1 + 1.017}$$
$$= \frac{10\ 080}{2.017}$$
$$= 4\ 997.52$$

The cash price for the phone is $\text{R}4\ 997.52$.

The buyer would save $10\ 080 - 4\ 997.52 = \text{R}5\ 082.48$ by paying cash rather than buying on hire purchase.



Example 1.4

How long will it take for an investment of $\text{R}7\ 500$ to amount to $\text{R}10\ 125$ at 3.5% simple interest per annum?

Solution

List all the information given and identify what variable must be calculated:

$P = 7\ 500$
$A = 10\ 125$
$i = 3.5\% = \frac{3.5}{100} = 0.035$
$n = ?$

Therefore:

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{i}\left(\frac{A}{P} - 1\right) = n$$
$$n = \frac{1}{0.035}\left(\frac{10\ 125}{7\ 500} - 1\right)$$
$$= \frac{1}{0.035}(1.35 - 1)$$
$$= \frac{1}{0.035}(0.35)$$
$$= 10$$

It will take 10 years.

Exercise 1.1

Note: Save your calculations and answers from this exercise to compare with your work in unit 2 in this subject outcome.

1. $\text{R}3\ 750$ is invested in a savings account at 3.2% simple interest per annum. What is the total value of the investment after a period of four years?
2. A loan of $\text{R}30\ 000$ is offered, to be paid off over a period of two years. By the time the debt is paid off, Karabo has paid a total of $\text{R}9\ 000$ in interest. What is the rate of simple interest that was charged on the loan?
3. A laptop is advertised for $\text{R}299$ per month on hire purchase over a period of three years. If the simple interest rate charged is 35%, calculate the original cash price of the laptop. How much would the buyer save by buying the laptop for cash rather than on hire purchase?
4. Calculate how long it would take to generate an amount of $\text{R}1\ 050$ in interest from an investment of $\text{R}5\ 000$, at 3.5%simple interest per annum.

The full solutions are at the end of the unit.

Note

When you get to unit 2 of this subject outcome, you might wonder if there are any real-life applications of simple interest. The SA Shares website explains that simple interest applies to loans for the purchase of vehicles:

"Vehicle loans are **amortised** (gradually written off over a period) monthly, meaning that part of the monthly instalment is allocated to pay the outstanding monthly balance of the loan, and the **remainder covers the interest payment**. As the outstanding balance declines, the monthly interest payable reduces. There is no compounding effect on the interest itself, as it is always considered as paid in full for the specific month."

In the level 4 topic on Financial Mathematics, you will work with examples of this type, where simple and compound interest relate to *reducing* balances, rather than growth and increasing balances.

Summary

In this unit you have learnt the following:

- How to use the simple growth formula to solve problems related to finance.
- How to solve for A, P, i and n using the simple interest formula.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. A laptop is advertised for $\text{R}469$ per month over a period of three years on hire purchase. If the rate of simple interest charged is 22.95%, calculate the original cash price of the laptop. Calculate the amount saved by buying for cash rather than on hire purchase.
2. After a period of 20 years, an investment of $\text{R}12\ 750$ has a total value of $\text{R}23\ 205$. Calculate the rate of simple interest that was paid on this investment.
3. A new fridge is advertised for $\text{R}262$ per month over a period of two years. If the simple interest charged is 45.3% per annum, calculate the cash price of the fridge.
4. Calculate the time it will take $\text{R}8\ 000$ to amount to $\text{R}8\ 960$ at 4% simple interest per annum.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $P = \text{R}3\ 750$
   $i = 3.2\%$
   $n = 4$
   $A = ?$

   $$\begin{aligned} A &= P(1 + i.n) \\ &= 3750\left[1 + \frac{3.2}{100}(4)\right] \\ &= 3750[1 + (0.032)(4)] \\ &= 3750[1 + 0.128] \\ &= 3750[1.128] \\ &= 4230 \end{aligned}$$

   The total value of the investment after the period of four years is $\text{R}4\ 230$.
2. $P = 30\ 000$
   $n = 2$
   $A = 30\ 000 + 9\ 000 = 39\ 000$
   $i = ?$

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{n}\left(\frac{A}{P} - 1\right) = i$$
$$\begin{aligned} i &= \frac{1}{2}\left(\frac{39000}{30000} - 1\right) \\ &= \frac{1}{2}(1.3 - 1) \\ &= \frac{1}{2}(0.3) \\ &= 0.15 \text{ or } 15\% \end{aligned}$$

The rate of simple interest charged was 15%.

3.

$n = 3 \text{ years} = 36 \text{ months}$
$i = 35\%$
$A = 299 \times 36 = 10764$

$$A = P(1 + i.n)$$
$$\frac{A}{(1 + i.n)} = P$$
$$\begin{aligned} P &= \frac{10\ 764}{1 + \left(\frac{35}{100}\right)(3)} \\ &= \frac{10\ 764}{1 + (0.35)(3)} \\ &= \frac{10\ 764}{1 + 1.05} \\ &= \frac{10\ 764}{2.05} \\ &= 5\ 250.732 \end{aligned}$$

The original cash price of the laptop is $\text{R}5\ 250.73$.
Buying for cash rather than hire purchase would save $10\ 764 - 5\ 250.73 = \text{R}5\ 513.27$.

4.

$P = 5\ 000$
$A = 1\ 050 + 5\ 000 = 6\ 050$
$I = 3.5\% = \frac{3.5}{100} = 0.035$

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{i}\left(\frac{A}{P} - 1\right) = n$$
$$n = \frac{1}{0.035}\left(\frac{6\ 050}{5\ 000} - 1\right)$$
$$= \frac{1}{0.035}(1.21 - 1)$$
$$7 = \frac{1}{0.035}(0.21)$$
$$= 6$$

It would take six years.

Back to Exercise 1.1

Unit 1: Assessment

1. $n = 3 \text{ years} = 36 \text{ months}$
   $A = 469 \times 36 = 16\ 884$
   $i = 22.95\%$
   $P = ?$

$$A = P(1 + i.n)$$
$$\frac{A}{1 + i.n} = P$$
$$P = \frac{16\ 884}{1 + \left(\frac{22.95}{100}\right)(3)}$$
$$= \frac{16\ 884}{1 + (0.2295)(3)}$$
$$= \frac{16\ 884}{1 + 0.6885}$$
$$= \frac{16\ 884}{1.6885}$$
$$= 9\ 999.408$$

   The cash price of the laptop is $\text{R}9\ 999.41$.
   The amount saved by buying for cash rather than on hire purchase would be
   $16884 - 9999.41 = R6884.59$

2. $A = 23\ 205$
   $P = 12\ 750$
   $n = 20$
   $i = ?$

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{n}\left(\frac{A}{P} - 1\right) = i$$
$$\begin{aligned} i &= \frac{1}{20}\left(\frac{23\ 205}{12\ 750} - 1\right) \\ &= \frac{1}{20}(1.82 - 1) \\ &= \frac{1}{20}(0.82) \\ &= 0.041 \text{ or } 4.1\ \% \text{ p.a.} \end{aligned}$$

The rate of simple interest paid was 4.1% per annum.

3.

$$i = 45.3\%$$
$$n = 2 \text{ years} = 24 \text{ months}$$
$$A = 262 \times 24 = 6\ 288$$

$$A = P(1 + i.n)$$
$$\frac{A}{(1 + i.n)} = P$$
$$\begin{aligned} P &= \frac{6\ 288}{1 + \left(\frac{45.3}{100}\right)(2)} \\ &= \frac{6\ 288}{1 + (0.453)(2)} \\ &= \frac{6\ 288}{1 + 0.906} \\ &= \frac{6\ 288}{1.906} \\ &= 3\ 299.056 \end{aligned}$$

The cash price of the fridge is $\text{R}3\ 299.06$

4.

$$P = 8\ 000$$
$$A = 8\ 960$$
$$i = 4\% = \frac{4}{100} = 0.04$$

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{i}\left(\frac{A}{P} - 1\right) = n$$
$$\begin{aligned} n &= \frac{1}{0.04}\left(\frac{8\ 960}{8\ 000} - 1\right) \\ &= \frac{1}{0.04}(1.12 - 1) \\ &= \frac{1}{0.04}(0.12) \\ &= 3 \end{aligned}$$

It will take three years.

Back to Unit 1: Assessment

Media Attributions

- M3 SO5.2 Unit1 Image1 Simple interest formula © DHET is licensed under a CC BY (Attribution) license

Unit 2: Calculations with compound interest

Unit outcomes

By the end of this unit you will be able to:

- Use the compound growth formula (or compound *interest* formula) to solve problems related to finance.
- Solve for A, P and i using the compound interest formula.

What you should know

Before you start this unit, make sure that you can:

- Use a scientific calculator. Look back to level 2 subject outcome 1.1 to revise this.
- Use basic laws of arithmetic.
- Solve simple equations. Level 2 subject outcome 2.2 unit 1 will provide help with these.
- Complete basic simple interest calculations. Refer to level 2 subject outcome 5.2 unit 1 for help with this.

Introduction

Simple and compound growth formulae are used most commonly when talking about financial loans and investments. Growth is calculated as 'interest' on an amount that is invested or borrowed for a period of time, and added to the investment or loan, so that it grows bigger. In this unit we will look at compound interest and how it is calculated.

Calculating compound interest

The elements of the compound growth formula are as follows:

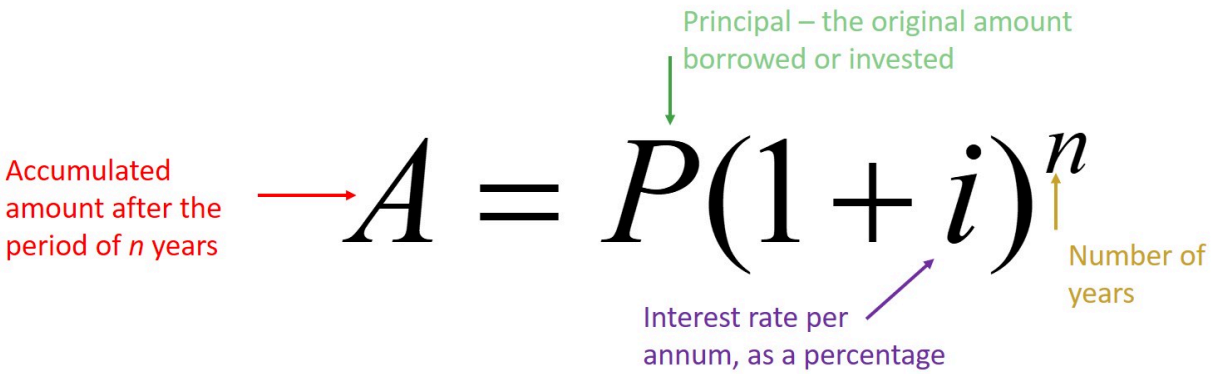


Image 1: Compound interest formula

See how this formula differs from that for simple interest, where n, the time period, is not in the exponent. Simple interest formula: $A = P(1 + i.n)$.

With compound growth, interest is calculated on the sum of the principal and the interest accumulated in the previous time periods. The interest is repeatedly calculated on an increasing amount. Unless stated otherwise, interest is compounded annually (i.e. added each year to the existing sum of principal and accumulated interest).

Let's look at a few of the same examples from unit 1, but this time, learning how to use this compound interest formula.

Example 2.1

A TV costs $\text{R}4\ 500$ on a hire purchase loan agreement. The interest charged is 12% compound interest each year (or per annum, 'p.a.') based on the full purchase price of the TV. Portia repays the loan over three years.

1. Calculate the full amount of money that is paid for the TV.
2. Calculate how much must be paid each month if equal payments are made every month.

Solution

Remember to list all the information given so you can identify what variable must be calculated.

$P = 4\ 500$
$i = 12\%$
$n = 3$
$A = ?$

1.

$$
\begin{aligned}
A &= P(1+i)^n \\
&= 4\ 500\left(1+\frac{12}{100}\right)^3 \\
&= 4\ 500(1+0.12)^3 \\
&= 4\ 500(1.12)^3 \\
&= 4\ 500(1.405) && \text{This is the answer of } 1.12^3\text{, correct to 3 decimal places} \\
&= \text{R}6\ 322.18 && \text{The more accurate form of } 1.12^3 \text{ is used for the calculation, with rounding off done only in the final answer}
\end{aligned}
$$

A total of $\text{R}6\ 322.18$ is paid over the period of three years or 36 months.

Note: If you compare the amount calculated as compound interest here to that for simple interest, as calculated in example 1.1 in unit 1 ($\text{R}6\ 120$), you will see that the compound interest total is greater.

2. Each monthly payment should be $\frac{6\ 322.18}{36} = \text{R}175.62$

Example 2.2

Pumla borrows $\text{R}175\ 000$to start a business and repays the loan in one lump sum after six years. The total amount that she is required to pay is $\text{R}354\ 775.14$. If the interest on the loan was compound interest, calculate the annual interest rate.

Solution

$P = 175\ 000$
$A = 354\ 775.14$
$n = 6$
$i = ?$

$$
\begin{aligned}
A &= P(1+i)^6 \\
\frac{A}{P} &= (1+i)^6 \\
\sqrt[6]{\frac{A}{P}} &= 1+i \\
\sqrt[6]{\frac{A}{P}} - 1 &= i \\
i &= \sqrt[6]{\frac{354\ 775.14}{175\ 000}} - 1 \\
&= \sqrt[6]{2.0273} - 1 \\
&= 1.125 - 1 \\
&= 0.125 \text{ or } 12.5\ \%
\end{aligned}
$$

Note: To avoid round-off errors, it is a good idea to do the calculation as one complete process on your calculator, rather than inputting sub-answers step by step. In this case the calculator steps will be:

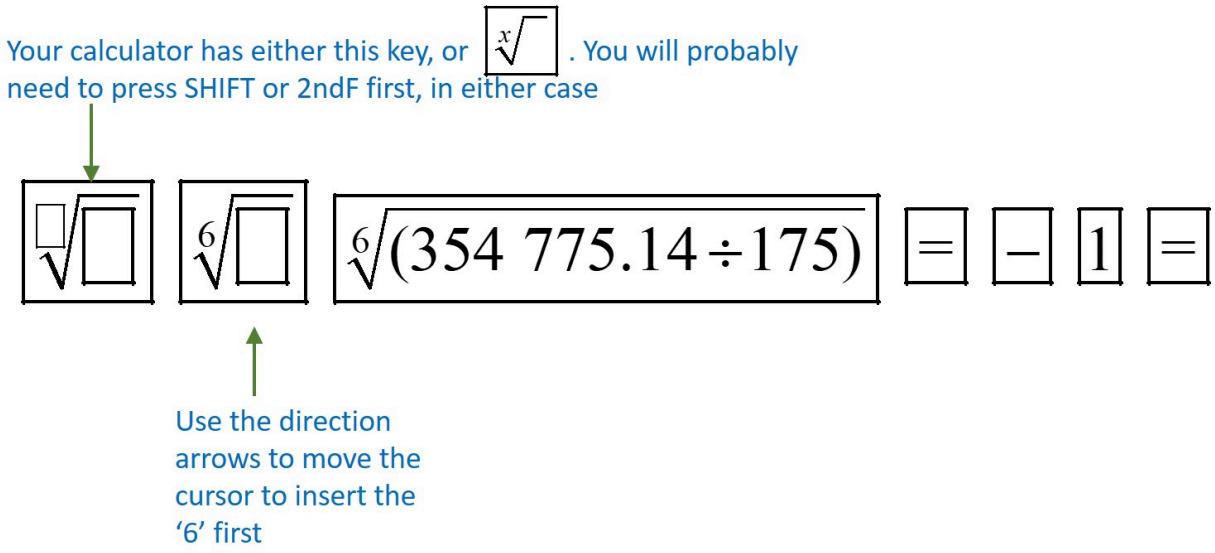


Image 2: Calculator refresher

Example 2.3

A cell phone is advertised for R280 per month on hire purchase over a period of three years. If the compound interest rate charged is 33.9%, calculate the original cash price of the phone. What would the buyer save by buying the phone for cash rather than on hire purchase?

Solution

$n = 3 \text{ years}$
$i = 33.9\%$
$A = 280 \times (3 \times 12) = 10\ 080$
$P = ?$

$$A = P(1+i)^3$$
$$\frac{A}{(1+i)^3} = P$$
$$P = \frac{10\ 080}{(1+0.339)^3}$$
$$= \frac{10\ 080}{(1.339)^3}$$
$$= \frac{10\ 080}{2.4007}$$
$$= 4\ 198.738$$

The cash price for the phone is R4 198.74.
The buyer would save $10\ 080 - 4\ 198.74 = R5\ 881.26$ by paying cash rather than buying on hire purchase.

Note: If you compare this to Example 1.3 in unit 1, you will see that the compound interest on a phone

costing $\text{R}4\ 198.74$ is more than the simple interest at the same rate and over the same period on a phone costing $\text{R}4\ 997.52$. Compound interest thus accumulates faster than simple interest.

Exercise 2.1

1. $\text{R}3\ 750$ is invested in a savings account at 3.2% compound interest per annum.
   a. What is the total value of the investment after a period of four years?
   b. Explain how this answer compares to question 1 of Exercise 1.1 in unit 1, in which the given values are the same, but simple interest is calculated.
2. A laptop is advertised for $\text{R}299$ per month on hire purchase over a period of three years.
   a. If the compound interest rate charged is 35%, calculate the original cash price of the laptop. How much would the buyer save by buying for cash rather than on hire purchase?
   b. Explain how this answer compares to question 3 of Exercise 1.1 in unit 1, in which the figures are the same, but simple interest is calculated.
3. Lusanda invests $\text{R}10\ 000$, and after eight years her investment triples. Calculate the interest rate per annum if the interest was:
   a. simple interest
   b. compounded annually?
4. Thabiso needs to borrow $\text{R}75\ 000$ to buy a machine for his business. The debt will be paid in six years' time when one of his investments matures. There are two options:
   Option 1: Borrow the money from a relative at $18\ \%$ p.a. simple interest.
   Option 2: Borrow from a bank at an interest rate of $15\ \%$ p.a. compound interest.

   Indicate which is the cheaper option and calculate the amount of money that will be saved by choosing the cheaper option.

The full solutions are at the end of the unit.

Note

Compound interest is accumulated on the original amount saved/borrowed and on the interest already accumulated. Think of it like earning interest on interest. For example, the graph below shows the growth of $\text{R}100$ saved in each of two different accounts. The green line represents growth of $10\ \%$ simple interest over 40 years, while the purple line represents growth of $10\ \%$ compound interest over the same period.

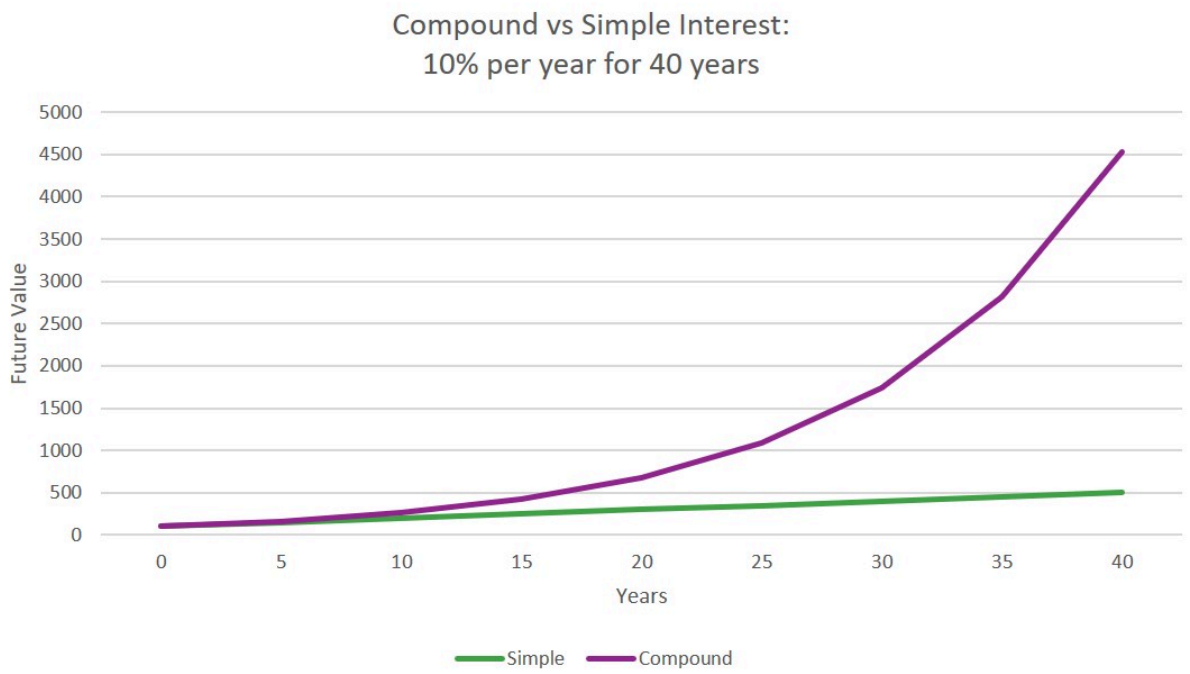


Image 3: Graph

As you can see, compound interest grows your money much faster than simple interest, over the longer term.

Compounding interest more frequently than annually

The interest calculated on a principal amount can be added (or compounded) more often than annually, and this can be any regular period of time during the year. The most common are:

- semi-annually (i.e. every six months)
- quarterly (i.e. every three months)
- monthly
- daily.

The following slight changes to the compound interest formula allow these variations in the frequency of compounding to be taken into account:

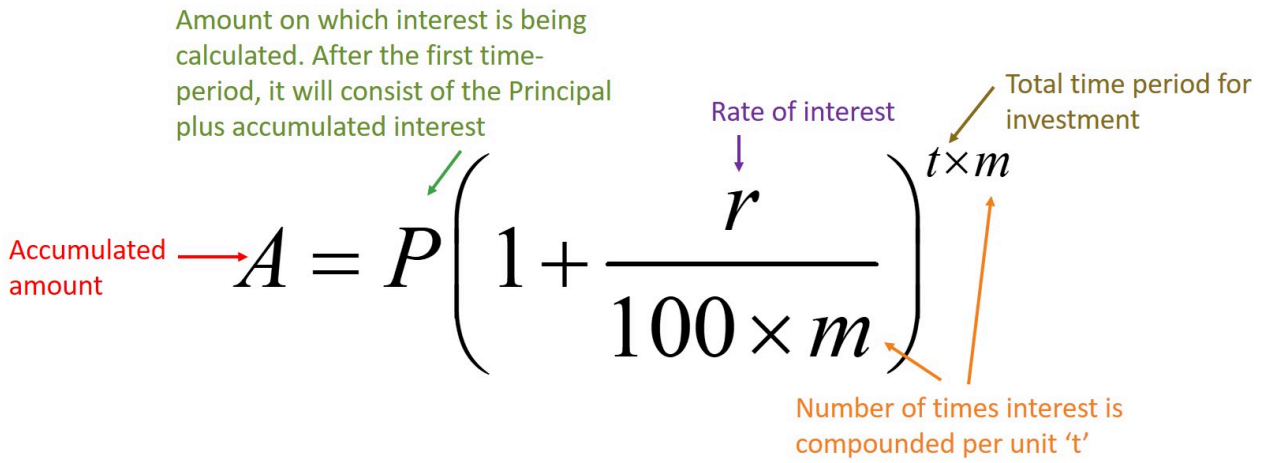


Image 4: Formula for compounding interest more often

- Dividing the interest rate by m (the number of times it is compounded per unit time) results in the total amount of interest added per unit time being split into m equal portions.
- Multiplying the index t by m results in the total number of additions of interest being m every time period for t time periods, i.e. $t \times m$.
- In the formula $A = P(1+i)^n$, i represents the interest rate expressed as a percentage. If we were doing a calculation based on $12\ \%$ interest, we would substitute $i = \frac{12}{100} = 0.12$. However, in the formula above, the conversion to a percentage (dividing by 100) is included in the formula. Doing a calculation based on $12\ \%$ interest using this formula, we would substitute $r = 12$, seeing that the division by 100 is built into the formula.

Take note!

If the compounding period is not specifically mentioned, it should be assumed that compounding of interest takes place annually.

Example 2.4

$\text{R}25\ 000$ is deposited into a savings account. Calculate the value of the investment at the end of eight years, if the interest rate is:

1. $12\ \%$ p.a. compounded annually
2. $12\ \%$ p.a. compounded semi-annually
3. $12\ \%$ p.a. compounded quarterly
4. $12\ \%$ p.a. compounded monthly
5. $12\ \%$ p.a. compounded daily.

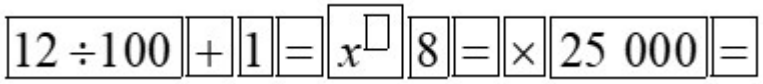
Solution

$P = 25\ 000;\ t = 8;\ r = 12$

1. Annual compounding: $m = 1$

$$\begin{aligned} A &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 25\ 000\left(1 + \frac{12}{100 \times 1}\right)^{8 \times 1} \\ &= 25\ 000(1 + 0.12)^{8} \\ &= 25\ 000(1.12)^{8} \\ &= 25\ 000(2.47596) \\ &= \text{R}61\ 899.08 \end{aligned}$$

Note: Remember that it is important to avoid round-off errors, so you need to think about how you will use your calculator so that you round off only at the final answer. BODMAS is a good guide to operations that should take priority, starting with Brackets, and Of (here – powers Of), as follows:



It is a good exercise to do the simpler calculations mentally, although you can always check your work with the calculator.

2. Semi-annual compounding: $m = 2$

$$\begin{aligned} A &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 25\ 000\left(1 + \frac{12}{100 \times 2}\right)^{8 \times 2} \\ &= 25\ 000(1 + 0.06)^{16} \\ &= 25\ 000(1.06)^{16} \\ &= 25\ 000(2.54035) \\ &= \text{R}63\ 508.79 \end{aligned}$$

3. Quarterly compounding: $m = 4$

$$\begin{aligned} A &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 25\ 000\left(1 + \frac{12}{100 \times 4}\right)^{8 \times 4} \\ &= 25\ 000(1 + 0.03)^{32} \\ &= 25\ 000(1.03)^{32} \\ &= 25\ 000(2.575083...) \\ &= \text{R}64\ 377.07 \end{aligned}$$

4. Monthly compounding: $m = 12$

$$\begin{aligned} A &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 25\ 000\left(1 + \frac{12}{100 \times 12}\right)^{8 \times 12} \\ &= 25\ 000(1 + 0.01)^{96} \\ &= 25\ 000(1.01)^{96} \\ &= 25\ 000(2.599272..) \\ &= \text{R}64\ 981.82 \end{aligned}$$

5. Daily compounding: $m = 365$

$$\begin{aligned} A &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 25\ 000\left(1 + \frac{12}{100 \times 365}\right)^{8 \times 365} \\ &= 25\ 000(1 + 0.000328767...)^{2\ 920} \\ &= 25\ 000(1.000328767...)^{2\ 920} \\ &= 25\ 000(2.611284..) \\ &= \mathrm{R}65\ 282.11 \end{aligned}$$

Note

The more often interest is compounded on an investment, the more interest is earned. The more interest is compounded on a loan, the more interest is paid, and the more expensive the loan is.



Example 2.5

$\mathrm{R}120\ 000$ was deposited into a fixed deposit savings account for five years. The accumulated amount in the account at the end of the five-year period is $\mathrm{R}195\ 000$. Calculate correct to one decimal place the interest rate per annum paid by the bank if:

1. the interest was compounded semi-annually
2. the interest was compounded daily.

Solution

1. $A = 195\ 000;\ \mathrm{P} = 120\ 000;\ t = 5;\ m = 2$

$$\begin{aligned} A &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ \frac{A}{P} &= \left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ \sqrt[t \times m]{\frac{A}{P}} &= 1 + \frac{r}{100 \times m} \\ \frac{r}{100 \times m} &= \sqrt[t \times m]{\frac{A}{P}} - 1 \\ r &= 100m\left(\sqrt[t \times m]{\frac{A}{P}} - 1\right) \\ r &= 200\left(\sqrt[10]{\frac{195\ 000}{120\ 000}} - 1\right) \\ &= 200\left(\sqrt[10]{1.625} - 1\right) \\ &= 200\,(0.0497...) \\ &= 9.5\ \% \end{aligned}$$

2. $A = 195000;\ \mathrm{P} = 120000;\ t = 5;\ m = 365$

$$A = P\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$

$$\frac{A}{P} = \left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$

$$\sqrt[t \times m]{\frac{A}{P}} = 1 + \frac{r}{100 \times m}$$

$$\frac{r}{100 \times m} = \sqrt[t \times m]{\frac{A}{P}} - 1$$

$$r = 36\ 500\left(\sqrt[t \times m]{\frac{A}{P}} - 1\right)$$

$$r = 36\ 500\left(\sqrt[5 \times 365]{\frac{195\ 000}{120\ 000}} - 1\right)$$

$$= 36\ 500\left(\sqrt[825]{1.625}^{1} - 1\right)$$

$$= 36\ 500\,(0.000266...)$$

$$= 9.7\ \%$$

Example 2.6

Find the amount of money that must be invested for five years at $3\ \%$ per annum, compounded monthly, to accumulate to $\mathrm{R}35\ 000$ by the end of the period.

Solution

$A = 35\ 000;\ t = 5;\ r = 3\%; \quad m = 12$

$$A = P\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$

$$P = \frac{A}{\left(1 + \frac{r}{100 \times m}\right)^{t \times m}}$$

$$= \frac{35\ 000}{\left(1 + \frac{3}{100 \times 12}\right)^{5 \times 12}}$$

$$= \frac{35\ 000}{\left(1 + \frac{3}{1200}\right)^{60}}$$

$$= \frac{35\ 000}{(1 + 0.0025)^{60}}$$

$$= \frac{35\ 000}{(1.0025)^{60}}$$

$$= \frac{35\ 000}{1.6162}$$

$$= \mathrm{R}30\ 130.42$$

The power of compound interest lies in the length of time that an investment is allowed to grow. Our patience is the deciding factor.

Example 2.7

Suppose Kekeletso and Tshiamo start working at age 24. Kekeletso works until she is 65 years old, and Tshiamo until he is 55 years old. At the end of their first year of work, they each invest $\text{R}1\ 500$ in a savings plan that offers $11.11\ \%$ interest compounded monthly. What difference will the extra 10 years make to the value of their investments?

Solution

The initial investment amounts and interest rates are the same for both investments. Only the time periods differ.

$A = 1\ 500;\ r = 11.11\ \%;\ m = 12$
For Kekeletso, $t = 65 - 25 = 40$ years

$$\begin{aligned} A_K &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 1\ 500\left(1 + \frac{11.11}{100 \times 12}\right)^{40 \times 12} \\ &= 1\ 500\left(1 + \frac{11.11}{1\ 200}\right)^{480} \\ &= 1\ 500(1 + 0.009258)^{480} \\ &= 1\ 500(1.009258)^{480} \\ &= 1\ 500(83.392) \\ &= \text{R}125\ 088.21 \end{aligned}$$

So, after a period of 40 years, Kekeletso's investment of $\text{R}1\ 500$ would have grown to the amount of $\text{R}125\ 088.21$.

On the other hand, Tshiamo's investment had $t = 55 - 25 = 30$ years to grow.

$$\begin{aligned} A_T &= P\left(1 + \frac{r}{100 \times m}\right)^{t \times m} \\ &= 1\ 500\left(1 + \frac{11.11}{100 \times 12}\right)^{30 \times 12} \\ &= 1\ 500\left(1 + \frac{11.11}{1\ 200}\right)^{360} \\ &= 1\ 500(1 + 0.009258)^{360} \\ &= 1\ 500(1.009258)^{360} \\ &= 1\ 500(27.59585) \\ &= \text{R}41\ 393.78 \end{aligned}$$

Although Tshiamo's $\text{R}1\ 500$ investment had grown impressively to $\text{R}41\ 393.78$, Kekeletso's investment was worth $\text{R}125\ 088.21 - \text{R}41\ 393.78 = \text{R}83\ 694.43$ more, as a result of the additional 10 years. In total Kekeletso's investment was worth just over three times as much as Tshiamo's!

Note

The most important thing to remember is to start saving in the first place. Your money may grow slowly at first, but over the years it will grow faster and faster with compound interest.

Exercise 2.2

1. $\mathrm{R}120\ 000$ is needed to start a business. A loan can be obtained from the small business development fund or from a bank. The loan must be repaid in one amount at the end of three years. The small business development fund charges $9\ \%$ p.a. simple interest. The bank charges $7.2\ \%$ p.a. compounded monthly. Which loan is cheaper, and how much cheaper is it?
2. $\mathrm{R}750\ 000$ is deposited into a fixed deposit savings account for six years, and amounts to $\mathrm{R}1\ 094\ 505.70$ by the end of this period. Calculate the interest rate per annum paid by the bank if:
   a. the interest is compounded annually
   b. the interest is compounded daily.
3. Calculate the amount of money that must be invested at $11\ \%$ p.a. compounded monthly, if the accumulated investment is $\mathrm{R}135\ 000$ after a period of 15 years.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to use the compound growth formula (or compound interest formula) to solve problems related to finance.
- How to solve for A, P and i using the compound interest formula.
- How to compare simple interest growth to compound interest growth.
- How to solve compound growth problems with interest compounded at various different regular intervals (semi-annually, quarterly, monthly and daily).

Unit 2: Assessment

Suggested time to complete: 40 minutes

1. A shop offers a $\mathrm{R}5\ 299.00$ laptop that can be paid off in three ways:
   a. over 12 months at $\mathrm{R}627.67$ per month
   b. over 24 months at $\mathrm{R}382.88$ per month
   c. over 36 months at $\mathrm{R}303.70$ per month.

   Calculate the total amount paid for each option, and the compound interest rate charged for each.

2. $\mathrm{R}2\ 500$ is deposited in each of the following:
   Option A: a savings account that pays $3.25\ \%$ compound interest, and permits immediate access to its funds
   Option B: a savings account paying $4.35\ \%$ compound interest, where withdrawals require 32 days' advance notice, or a penalty to be paid.
   If no withdrawals are made from either account, what will their balances be after three years?
3. What amount of money should be invested at $4\ \%$ p.a. compound interest to amount to $\mathrm{R}350\ 000$ in 20

years' time?

4. $\text{R}5\ 700$ was invested in a savings account at $8.75\ \%$ p.a. Calculate how much more money is in the account at the end of 18 months if interest is compounded daily than if interest is compounded semi-annually.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

a. $P = \text{R}3\ 750$
$i = 3.2\ \%$
$n = 4$
$A = ?$

$$\begin{aligned} A &= P(1+i)^n \\ &= 3\ 750\left(1+\frac{3.2}{100}\right)^4 \\ &= 3\ 750(1+0.032)^4 \\ &= 3\ 750(1.032)^4 \\ &= 3\ 750(1.134) \\ &= 4\ 253.54 \end{aligned}$$

The total value of the investment after the period of four years is $\text{R}4\ 230$.

b. This is $\text{R}23.54$ more than if simple interest at the same rate had been calculated over the same period.

2.

a. $n = 3 \text{ years} = 36 \text{ months}$
$i = 35\ \%$
$A = 299 \times 36 = 10\ 764$

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{(1+i)^n} &= P \\ P &= \frac{10\ 764}{\left(1+\frac{35}{100}\right)^3} \\ &= \frac{10\ 764}{(1+0.35)^3} \\ &= \frac{10\ 764}{1.35^3} \& = \frac{10\ 764}{2.460} \\ &= 4\ 374.943 \end{aligned}$$

The original cash price of the laptop is $\text{R}4\ 374.94$.
Buying for cash rather than on hire purchase would save $\text{R}10\ 764 - \text{R}4\ 374.94 = \text{R}5\ 504.27$

b. Comparing this to question 3 of Exercise 1.1 in unit 1, you can see that at the same rate and over the same period, the compound interest is greater.

3.

a. $P = 10\ 000$
$A = 3 \times 10\ 000 = 30\ 000$
$n = 8$

$$A = P(1 + i.n)$$
$$\frac{A}{P} = 1 + i.n$$
$$\frac{A}{P} - 1 = i.n$$
$$\frac{1}{n}\left(\frac{A}{P} - 1\right) = i$$
$$i = \frac{1}{8}\left(\frac{30\ 000}{10\ 000} - 1\right)$$
$$= \frac{1}{8}(3 - 1)$$
$$= \frac{1}{8}(2)$$
$$= 0.25\ \%\ \text{simple interest}$$

b.
$$A = P(1 + i)^n$$
$$\frac{A}{P} = (1 + i)^n$$
$$\sqrt[8]{\frac{A}{P}} = 1 + i$$
$$i = \sqrt[8]{\frac{A}{P}} - 1$$
$$= \sqrt[8]{\frac{30\ 000}{10\ 000}} - 1$$
$$= \sqrt[8]{3} - 1$$
$$= 1.147 - 1$$
$$= 0.147\ \%$$

Note: A lower rate of compound interest achieves the same amount of growth as a higher rate of simple interest.

4. $P = 75\ 000$
$n = 6$ years
$A =$?
Option A: $i = 18\ \%$ p.a. simple interest

$$A = P(1 + i.n)$$
$$= 75\ 000[1 + (0.18)(6)]$$
$$= 75\ 000[1 + 1.08]$$
$$= 75\ 000(2.08)$$
$$= \text{R}156\ 000$$

Option B: $i = 15\ \%$ compound interest

$$A = P(1 + i)^n$$
$$= 75\ 000(1 + 0.15)^6$$
$$= 75\ 000(1.15)^6$$
$$= 75\ 000(2.3131)$$
$$= \text{R}173\ 479.56$$

Borrowing at $18\ \%$ simple interest is cheaper than $15\ \%$ compound interest. By doing this, there will be a saving of $\text{R}173\ 479.56 - \text{R}156\ 000 = \text{R}17\ 479.56$.

Back to Exercise 2.1

Exercise 2.2

1. $P = \text{R}120\ 000$
   $n = 3$ years
   Small business development fund: $i = 9\ \% = 0.09$ p.a. simple interest
   $$\begin{aligned} A &= 120\ 000(1 + 0.09 \times 3) \\ &= 120\ 000(1.27) \\ &= \text{R}152\ 000 \end{aligned}$$

   Bank: $i = 7.2\ \%$ p.a. compounded monthly
   $$\begin{aligned} A &= P\left(1 + \frac{7.2}{100 \times 12}\right)^{3 \times 12} \\ &= 120\ 000\left(1 + \frac{7.2}{1200}\right)^{36} \\ &= 120\ 000(1.006)^{36} \\ &= 120\ 000(1.240) \\ &= \text{R}148\ 836.19 \end{aligned}$$

   The bank loan at $7.2\ \%$ p.a. compounded monthly is cheaper:
   $\text{R}152\ 000 - \text{R}148\ 836.19 = \text{R}3\ 163.81$

2.
   a. If the interest is compounded annually:
   $$\begin{aligned} A &= P(1+i)^n \\ \sqrt[n]{\frac{A}{P}} &= 1 + i \\ i &= \sqrt[n]{\frac{A}{P}} - 1 \\ &= \sqrt[6]{\frac{1\ 094\ 505.70}{750\ 000}} - 1 \\ &= 1.065 - 1 \\ &= 6.5\ \% \end{aligned}$$

   b. If the interest is compounded daily:
   $$\begin{aligned} A &= P(1 + \frac{r}{100 \times m})^{t \times m} \\ \frac{A}{P} &= (1 + \frac{r}{100 \times 365})^{6 \times 365} \\ \sqrt[2190]{\frac{1094505.70}{750000}} &= 1 + \frac{r}{36\ 500} \\ \frac{r}{36500} &= \sqrt[2190]{\frac{1094505.70}{750000}} - 1 \\ r &= 36500\left(\sqrt[2190]{\frac{1094505.70}{750000}} - 1\right) \\ r &= 36500(1.0001726 - 1) \\ r &= 6.3\ \% \end{aligned}$$

3. $A = \text{R}135\ 000$ $i = 11\ \%$ p.a. compounded monthly $t = 15$ years

$$A = P\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$

$$\frac{A}{\left(1 + \frac{r}{100 \times m}\right)^{t \times m}} = P$$

$$\begin{aligned} P &= \frac{135\ 000}{\left(1 + \frac{11}{100 \times 12}\right)^{15 \times 12}} \\ &= \frac{135\ 000}{(1.00917)^{180}} \\ &= \frac{135\ 000}{5.168} \\ &= \text{R}26\ 122.35 \end{aligned}$$

$\text{R}26\ 122.35$ would need to be invested.

Back to Exercise 2.2

Unit 2: Assessment

1. $P = 5\ 299$
   a. $A = 12 \times 627.67 = 7\ 532.04$; this is the cost of the laptop in this option

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \\ (1+i)^1 &= \frac{A}{P} \\ 1 + i &= \frac{7\ 532.04}{5\ 299} \\ i &= 1.4214 - 1 \\ i &= 0.4214\%\ \text{p.a.} \end{aligned}$$

   Note: With the period being one year, the simple interest formula and compound interest formula produce the same answer.

   b. $A = 24 \times \text{R}382.88 = \text{R}9\ 189.12$; this is the cost of the laptop in this option

$$\begin{aligned} (1+i)^2 &= \frac{A}{P} \\ 1 + i &= \sqrt{\frac{A}{P}} \\ i &= \sqrt{\frac{9189.12}{5299}} - 1 \\ &= \sqrt{1.7341} - 1 \\ &= 1.3169 - 1 \\ &= 0.3169\%\ \text{p.a.} \end{aligned}$$

   c. $A = 36 \times \text{R}303.70 = \text{R}10\ 933.20$; this is the cost of the laptop in this option

$$\begin{aligned} (1+i)^3 &= \frac{A}{P} \\ 1 + i &= \sqrt[3]{\frac{10\ 933.2}{5\ 299}} \\ i &= \sqrt[3]{2.06326} - 1 \\ &= 1.2731 - 1 \\ &= 0.2731\%\ \text{p.a.} \end{aligned}$$

2. $P = 2\ 500$
   Option A: $i = 3.25\ \%$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2\ 500(1+0.0325)^3 \\ &= 2\ 500(1.0325)^3 \\ &= 2\ 500(1.1007) \\ &= \text{R}2\ 751.76 \end{aligned}$$

Option B: $i = 4.35\ \%$

$$\begin{aligned} A &= P(1+i)^n \\ &= 2\ 500(1+0.0435)^3 \\ &= 2\ 500(1.0435)^3 \\ &= 2\ 500(1.1353) \\ &= \text{R}2\ 840.65 \end{aligned}$$

3. $A = 350\ 000;\ n = 20;\ i = 4\ \%\ p.a.$

$$A = P(1+i)^n$$

$$\frac{A}{(1+i)^n} = P$$

$$\begin{aligned} P &= \frac{350\ 000}{(1+0.04)^{20}} \\ &= \frac{350\ 000}{(1.04)^{20}} \\ &= \frac{350\ 000}{2.1912} \\ &= \text{R}159\ 735.43 \end{aligned}$$

Back to Unit 2: Assessment

Media Attributions

- M3 SO5.2 Unit2 Image1 Compound interest formula © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit2 Image2 Calculator refresher © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit 2 Image3 Compound vs simple interest © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit2 Image4 Compounding interest more often © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 UNit 2 Note on calculator use

Unit 3: Using timelines for calculations of simple and compound interest

Unit outcomes

By the end of this unit you will be able to:

- Interpret financial questions using timelines.

What you should know

Before you start this unit, make sure that you can:

- Use a scientific calculator. Look back to level 2 subject outcome 1.1 to revise this.
- Make calculations involving simple interest.
- Make calculations using compound interest.

For help with calculations of simple and compound interest, and the distinctions between these, you could review:

- Level 2 subject outcome 5.2
- Level 3 subject outcome 5.2 units 1 and 2.

Introduction

In units 1 and 2 of this subject outcome, you have seen how to calculate with the simple and compound interest formulae. You have also done calculations to investigate variations on the compound growth formula that involve different rates of compounding.

Very often, changes are made to investments and loans during the course of the period of the investment or loan, and these have the effect of changing the details of the formulae that should be used for parts of that investment period.

For example, an investor may need to withdraw a portion of their investment for an emergency expense, or may have an unexpected windfall, which may allow them to invest an additional amount for part of the period.

A **timeline** is useful for representing this information when there are changes to the variables (A, P and i) during the period of the investment or the loan.

Using time lines to illustrate changes made to investments or loans over their period

You will need to use the statement of each given problem to draw a line that represents the entire duration of the investment from the first deposit/loan to the end, whether it ends with a withdrawal or a final repay-

ment. The line is marked off, usually in periods of years, although this depends on the calculations to be made. Interest rates, and amounts of deposits or loans, and changes in these, are indicated against the line. Let's take a look at an example.

Example 3.1

R2 500 is invested for two years in a savings account at 3.5 % compound interest per annum. After that period, a further R1 500 is deposited into the account. The interest rate is adjusted to 4.2 % compound interest per annum at the end of the third year. Calculate the total amount in the account after a total period of five years, if no further adjustments are made.

Solution

The timeline drawn below shows the initial deposit of R2 500 at T_0 (the start of the investment period), with the entire investment period from T_0 to T_5 – the five-year period. After two years, represented by T_2, the deposit of R1 500 is shown: all deposits are aligned in the timeline diagram.

Interest rates are shown with arrows extending over the period for which they are relevant: so the rate of 3.5 % compound interest per annum is paid from the start to the end of the third year – from T_0 to T_3, when it changes to 4.2 % compound interest per annum for the remainder of the investment period.

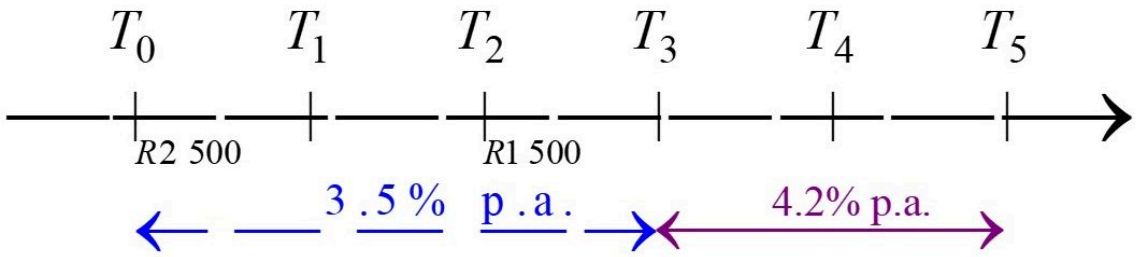


Image 1: Timeline example 3.1

In the calculations, investments are treated separately.

Period 1:
In the period T_0 to T_2, the investment consists of R2 500 invested at 3.5 % p.a. compound interest per annum.

$$\begin{aligned} A_1 &= P_1(1+i)^n \\ &= 2\ 500(1+0.035)^2 \\ &= 2\ 500\,(1.071225) \\ &= 2\ 678.0625 \end{aligned}$$

The investment had a value of R2 678.06 (correct to two decimal places) at the end of the second year. To avoid introducing round-off errors, we prefer to use $A_1 = 2\ 500(1+0.035)^2$ in subsequent calculations.

Period 2:
In the period T_2 to T_3, there was a further deposit of R1 500. While the interest rate remained the same for one more year, substituting the expression for A_1 shows that the principal after T_2, at the start of the third year was

$$\begin{aligned} P_2 &= 1\ 500 + A_1 \\ &= 1\ 500 + 2\ 500(1+0.035)^2 \end{aligned}$$

Using this in the compound interest formula for this second period, we arrive at an expression of the

value of the investment by the end of the third year:

$$\begin{aligned} A_2 &= P_2(1+i)^n \\ &= \left[1\ 500 + 2\ 500(1+0.035)^2\right](1+0.035)^1 \\ &= [4\ 178.0625]\,(1.035) \\ &= 4\ 324.294688 \end{aligned}$$

The investment had a value of $\text{R}4\ 324.30$ (correct to two decimal places) at the end of the third year. So $P_3 = A_2$.

Once again, to avoid introducing round-off errors, we prefer to use $A_2 = \left[1\ 500 + 2\ 500(1+0.035)^2\right](1+0.035)^1$ in subsequent calculations.

Period 3:

In the remaining two years from T_3 to T_5, the interest rate was increased to $4.2\ \%$ compound interest per annum. The value of the investment after the period of five years will be:

$$\begin{aligned} A_3 &= P_3(1+i)^n \\ &= \left[\left[1\ 500 + 2\ 500(1.035)^2\right](1.035)^1\right](1+0.035)^2 \\ &= \left[[4\ 178.0625]\,(1.035)^1\right](1.042)^2 \\ &= [4\ 324.294688]\,(1.042)^2 \\ &= 4\ 695.163497 \\ &= \text{R}4\ 695.16 \end{aligned}$$

The value of the investment at the end of the five year period is $\text{R}4\ 695.16$.

You will see from the example above that it is only necessary to work out the answer to the calculation in the final step.



Take note!

It's important to know how to use your calculator memory in complicated calculations.

(*These instructions are for a CASIO fx-991ES PLUS calculator. They might also work for your later model CASIO, or SHARP calculators. If not, you need to refer to an operating manual for your calculator.*)

Although each subsection of the calculation can be worked out completely, it is inconvenient, and sometimes impossible, to write down long, trailing decimal fractions. Rounding answers off for each part of the calculation might lead to round-off errors in your final answer. It is best to build up the calculation using the timeline, until you have all its parts, and then to work out the answer, rounding off only the final number.

Your calculator's memory function can be very useful in long calculations that use a repeated factor. For example, in the above calculation of A_3 it is useful to store the $1 + i = 1.035$ in the memory. Key the following into your calculator to do so (making sure that the value in the calculator memory is 0 before you start):



This stores 1.035 in the memory, and clears the screen for your calculation.

To recall what is stored in the memory, press the following keys:

RCL M+

This brings the value in the memory to the screen, ready to be used immediately in the calculation. So, once you have worked out what to calculate for A_3, you could proceed as follows:

$$A_3 = P_3(1+i)^n$$

$$= \left[\left[1\,500 + 2\,500(1.035)^2\right](1.035)^1\right](1+0.035)^2$$

Start at the first step of the calculation of A_1 and make sure that the repeated factor is in the memory. Then do the calculation using the number in the memory

Image 2: NOTE on calculator memory

1 + 0.35 = M+	This puts 1.035 into Memory, and leaves 1.035 in the screen to work with
x^2 = × 2 500 =	This squares 1.035 and multiplies that answer by 2 500
	This is the value of A_1
+ 1 500 =	This is the answer oif the first square brackets
× RCL M+ =	This is the value of A_2
× RCL M+ x^2 =	This is the value of A_3

After you have written down your answer, you need to clear your calculator screen AND memory for the next calculation. This is how to clear the calculator memory:

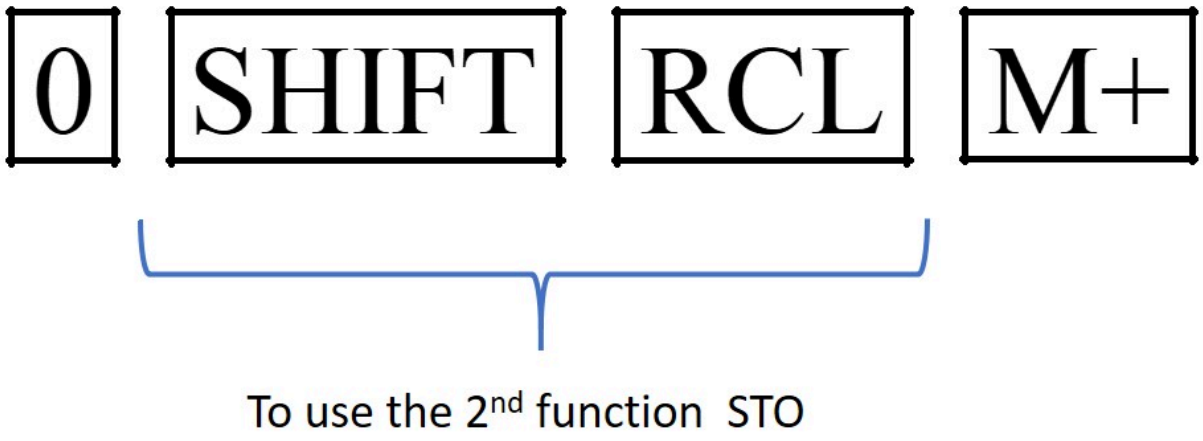


Image 3: NOTE 2 on calculator memory (clearing memory)

Example 3.2

$\mathrm{R2\ 800}$ is invested in a savings account and 12 months later, a further $\mathrm{R1\ 400}$ is deposited into the account. Calculate the total amount saved by the end of the second year if the interest rate is $4.85\ \%$ compounded quarterly for the first year, and is then adjusted to $5.90\ \%$ compounded monthly for the second year.

Solution

The timeline below shows the deposits and interest rates over the entire T_0 to T_2 two-year period.

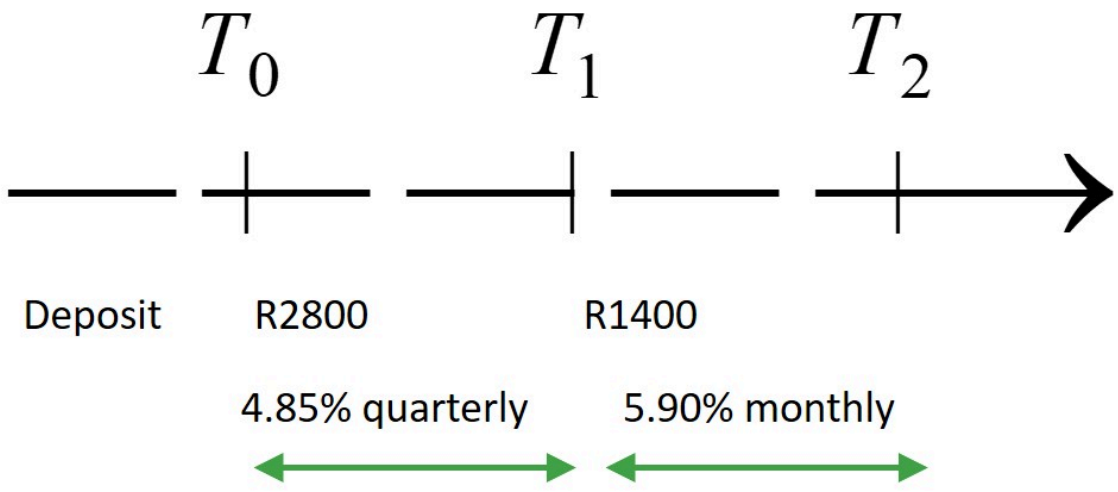


Image 4: Solution for Example 3.2

Period 1:
Consider the first deposit:

$$A_1 = P\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$
$$= 2\ 800\left(1 + \frac{4.85}{100 \times 4}\right)^{1 \times 4}$$

This is the expression for the total value of the first investment after one year.

Period 2:
The second deposit is added to this amount at T_1, and the interest rate is changed.

$$P_2 = 1\ 400 + A_1$$
$$A_2 = P_2\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$
$$= \left[1\ 400 + 2\ 800\left(1 + \frac{4.85}{400}\right)^{4}\right]\left(1 + \frac{5.90}{100 \times 12}\right)^{1 \times 12}$$
$$= \left[1\ 400 + 2\ 800(1.012125)^{4}\right]\left(1 + \frac{5.90}{100 \times 12}\right)^{1 \times 12}$$
$$= 4\ 338.29(1 + 0.00491667)^{12}$$
$$= 4\ 338.29(1.00491667)^{12}$$
$$= \text{R}4\ 601.29$$

Exercise 3.1

1. $\text{R}2\ 500$ is deposited in a savings account for 12 years. For the first seven years, the interest rate is $8.2\ \%$ p.a. compounded quarterly, and then it is increased to $10.5\ \%$ p.a. compounded monthly.
   a. Use a timeline to show the status of the investment.
   b. Calculate the amount of money accumulated by the end of 12 years.
2. $\text{R}12\ 000$ is deposited into a savings account, and three years later, a further deposit of $\text{R}4\ 000$ is added to the savings. Calculate the amount of money in the savings account at the end of eight years if the interest rate is $7.5\ \%$ compounded monthly.
3. Every six months for three years $\text{R}500$ is deposited into a Fixed Deposit savings account, with the first deposit made immediately, and the sixth and final deposit made six months before the end of the third year. Calculate the value of the investment at the end of the third year, assuming a $7.37\ \%$ interest rate, compounded semi-annually.
4. Nonhlanhla borrowed $\text{R}1\ 400$, and repaid $\text{R}365$ at the end of the first year, and made two further instalments of $\text{R}365$ at the end of the second and third years. If a simple interest rate of $11\ \%$ is charged on the loan, use a timeline to show how you calculate what amount must be repaid to settle the debt at the end of the fourth year.
5. $\text{R}50\ 000$ is deposited into a savings account, and 18 months later $\text{R}12\ 000$ is withdrawn from the account. How much money will be in the account at the end of five years if the interest is calculated at 8% compounded quarterly?
6. $\text{R}75\ 000$ is invested in the bond market for nine years. The interest rate for the first three years is $8.5\ \%$ compounded annually. For the next four years the interest rate increased to $9.75\ \%$ compounded quarterly. During the final two years, the interest rate is $10.25\ \%$ compounded monthly.

Calculate the total value of the investment at the end of the nine years.

The full solutions are at the end of the unit.

Note

If you would like more worked examples, you can find some at this link.

Summary

In this unit you have learnt the following:

- How to interpret financial questions using timelines.
- How to use your calculator's memory to do calculations using repeated factors.

Unit 3: Assessment

Suggested time to complete: 20 minutes

1. $\mathrm{R}3\ 500$ is deposited into a savings account, and 18 months later $\mathrm{R}1\ 800$ is added to the amount. Calculate the total amount saved by the end of four years if the interest rate is $7.2\ \%$ compounded quarterly for the first year, and is then increased to $8.5\ \%$ compounded monthly for the remainder of the period. Use a timeline to show the changes to the investment over the time period.
2. $\mathrm{R}10\ 200$ is deposited into a savings account, and two years later $\mathrm{R}4\ 500$ is withdrawn from the account. How much money will be in the account at the end of five years if the interest rate is $12.2\ \%$ compounded quarterly? Use a timeline to show the changes to the investment over the time period.

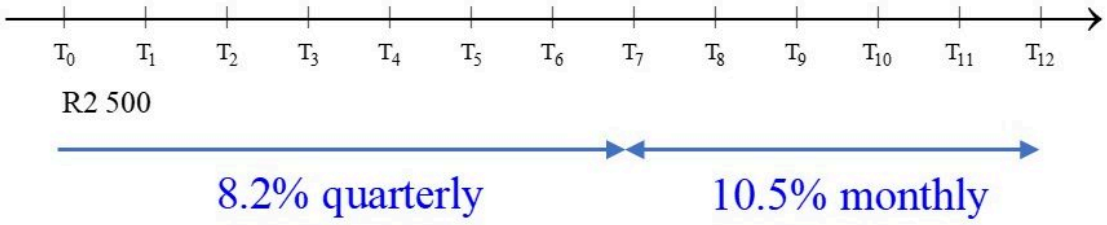
The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.

a. The timeline for the investment can be drawn as follows:



b. Consider the first deposit of $\text{R}2\ 500$.
At the end of seven years, the amount accumulated will be:

$$A_1 = P_1\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$
$$= 2\ 500\left(1 + \frac{8.2}{100 \times 4}\right)^{7 \times 4}$$

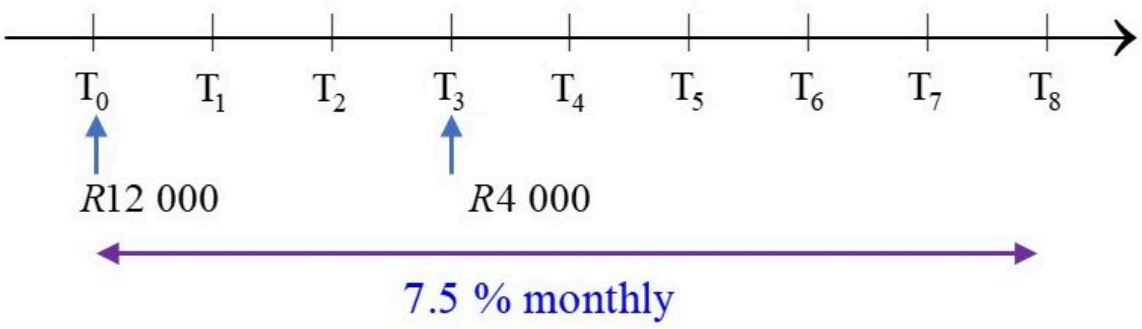
Consider the second period of the saving account.
After seven years, the interest rate changes, so for the next five years, the interest rate is $10.5\ \%$ p.a. compounded monthly.

At the start of this second period, the initial investment amount is the accumulated total from the first period. So, we could say that $P_2 = A_1$.
So, the total amount by the end of the 12 years will be:

$$A_{Total} = P_2\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$$
$$= \left[2\ 500\left(1 + \frac{8.2}{100 \times 4}\right)^{7 \times 4}\right]\left(1 + \frac{10.5}{100 \times 12}\right)^{5 \times 12}$$
$$= \left[2\ 500(1.0205)^{28}\right](1.00875)^{60}$$
$$= (4\ 412.699)(1.687)$$
$$= \text{R}7\ 442.47$$

Comment: Notice that after 12 years of schooling, a child is eligible for tertiary education. Young parents are likely to start thinking about ways to save to pay for their children's education when they start school. Although this investment would be a good contribution, it would not be enough to pay for tertiary education. Parents would do better to start saving when their children are younger (if possible) and to continue to save while they are at primary and secondary school.

2.



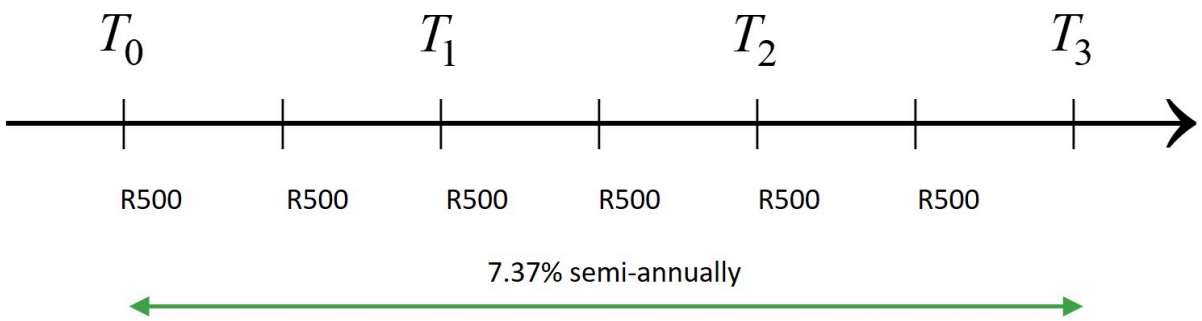
Consider the first period:

$$A_1 = P_1\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= 12\ 000\left(1+\frac{7.5}{100\times 12}\right)^{3\times 12}$$
$$= 12\ 000(1.00625)^{36}$$

Consider the second period:
The investment principal for this period is the amount accumulated by the end of the first period:
$P_2 = A_1$
$$A_2 = P_2\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= \left[12\ 000(1.00625)^{36}\right](1.00625)^{5\times 12}$$
$$= [15\ 017.35363](1.00625)^{60}$$
$$= 21\ 824.63605$$

At the end of eight years, there will be $\mathrm{R}21\ 824.64$ in the account.

3.



There are different ways of approaching this problem:
Each deposit can be treated separately, and the individual amounts added together for the total value for the entire period.
1st deposit:
$$A = P\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= 500\left(1+\frac{7.37}{100\times 2}\right)^{3\times 2}$$
$$= 500(1+0.03685)^{6}$$
$$= \mathrm{R}621.249 \text{ correct to 3 decimal places}$$

2nd deposit: Notice that the investment period for the second amount is 2.5 years.
$$A = P\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= 500\left(1+\frac{7.37}{100\times 2}\right)^{2.5\times 2}$$
$$= 500(1+0.03685)^{5}$$
$$= \mathrm{R}599.169 \text{ correct to 3 decimal places}$$

3rd deposit:
$$A = P\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= 500\left(1+\frac{7.37}{100\times 2}\right)^{2\times 2}$$
$$= 500(1+0.03685)^{4}$$
$$= \mathrm{R}577.875 \text{ correct to 3 decimal places}$$

4th deposit:

$$A = 500\left(1 + \frac{7.37}{100 \times 2}\right)^{1.5\times 2}$$
$$= 500(1 + 0.03685)^3$$
$$= \text{R}557.337 \text{ correct to 3 decimal places}$$

5th deposit:

$$A = 500\left(1 + \frac{7.37}{100 \times 2}\right)^{1\times 2}$$
$$= 500(1 + 0.03685)^2$$
$$= \text{R}537.529 \text{ correct to 3 decimal places}$$

6th deposit:

$$A = 500\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}$$
$$= 500(1 + 0.03685)^1$$
$$= \text{R}518.425$$

The value of investment after three years is the sum of the values for all the deposits, which is $\text{R}3\ 411.58$

Alternatively

1st deposit:

$$A_1 = P_1\left(1 + \frac{r}{100 \times m}\right)^{0.5\times m}$$
$$= 500\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}$$

2nd deposit:

At the start of the second period, the initial investment amount is the accumulated total from the first period with an additional $R500$. So, we could say that $P_2 = 500 + A_1$. This second investment is also for a six-month period. So:

$$A_2 = P_2\left(1 + \frac{r}{100 \times m}\right)^{0.5\times m}$$
$$= \left[500 + 500\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}$$

3rd deposit:

At the start of the third period, the initial investment amount is the accumulated total from the second period with an additional $\text{R}500$. So, we could say that $P_3 = 500 + A_2$. This third investment is also for a six-month period.

$$A_3 = P_3\left(1 + \frac{r}{100 \times m}\right)^{0.5\times m}$$
$$= \left[500 + \left[500 + 500\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}$$

And so on, as follows:

$$A_4 = P_4\left(1 + \frac{r}{100 \times m}\right)^{0.5\times m}$$
$$= \left[500 + \left[500 + \left[500 + 500\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}$$

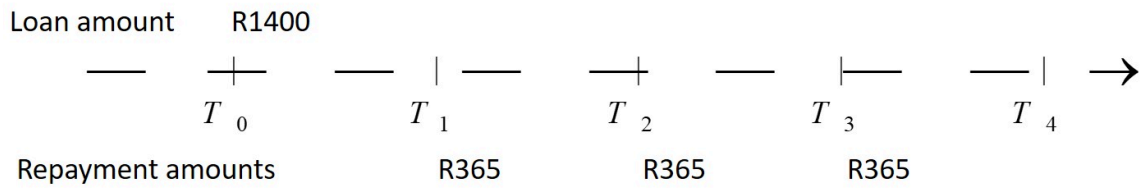
$$A_5 = \left[500 + \left[500 + \left[500 + \left[500 + 500\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}\right]\left(1 + \frac{7.37}{100 \times 2}\right)^{0.5\times 2}$$

The formula for A_6 is getting far too long, so instead we can calculate the exact decimal value of the factor that increases the principal for each period:

$$
\begin{aligned}
A_6 &= [500 + [500 + [500 + [500 + [500 + 500\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + [500 + [500 + [500 + 518.425]\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + [500 + [500 + 1\ 018.425\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + [500 + [500 + 1\ 055.953961]\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + [500 + 1\ 555.953961\,(1.03685)]\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + [500 + 1\ 613.290865]\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + 2\ 113.2908525\,(1.03685)]\,(1.03685)]\,(1.03685) \\
&= [500 + [500 + 2\ 191.165633]\,(1.03685)]\,(1.03685) \\
&= [500 + 2\ 691.165633\,(1.03685)]\,(1.03685) \\
&= [500 + 2\ 790.335087]\,(1.03685) \\
&= 3\ 290.335087\,(1.03685) \\
&= 3\ 411.583935 \\
&= \mathrm{R}3\ 411.58
\end{aligned}
$$

This is obviously a case for using the calculator memory, which will simplify the workings. However, the first approach to the problem is more straightforward, with fewer opportunities for mistakes or confusion.

4.



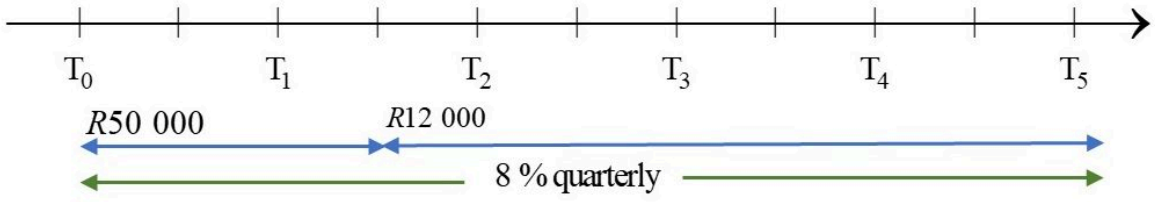
Initial loan = $\mathrm{R}1\ 400$
Total amount owing at the end of the four years:
$= P\,(1 + 0.11 \times 4)$
$= 1\ 400(1.44)$
$= \mathrm{R}2\ 016$

The first repayment reduces the total loan by the amount repaid as well as the interest on that repayment amount for the remaining period of three years:
Reduction effect of 1st repayment: $365(1 + 0.11 \times 3) = 365(1.33) = \mathrm{R}485.45$

Similarly, with subsequent repayments:
Reduction effect of 2nd repayment: $365\,(1 + 0.11 \times 2) = 365\,(1.22) = \mathrm{R}445.30$
Reduction effect of 3rd repayment: $365\,(1 + 0.11) = 365\,(1.11) = \mathrm{R}405.15$

The outstanding amount to be settled at the end of the 4th year:
$2\ 016 - 485.45 - 445.30 - 405.15 = \mathrm{R}680.10$

5.



Consider the first period:

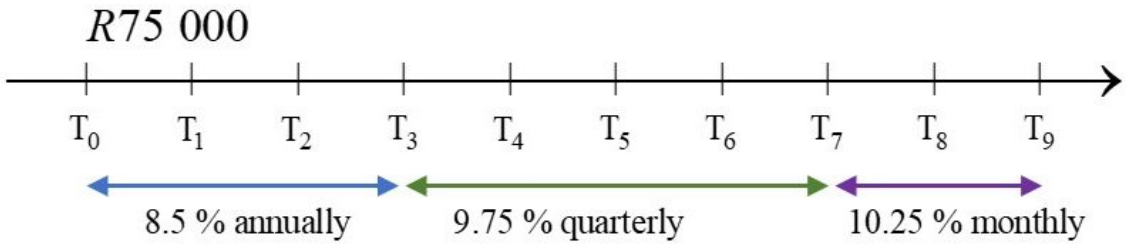
$$A_1 = P_1\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= 50\ 000\left(1+\frac{8}{100\times 4}\right)^{1.5\times 4}$$
$$= 50\ 000(1.02)^6$$

Consider the second period:
At the start of the period, the principal will be the amount accumulated by the end of the first period: $P_2 = A_1$.

$$A_2 = P_2\left(1+\frac{r}{100\times m}\right)^{t\times m}$$
$$= \left[50\ 000(1.02)^6 - 12000\right](1.02)^{3.5\times 4}$$
$$= [44\ 308.12098](1.02)^{14}$$
$$= 58\ 463.62$$

By the end of five years, there will be $\mathrm{R}58\ 463.62$ in the account.

6.



Consider the first period:

$$A_1 = P_1\left(1+\frac{8.5}{100\times 1}\right)^{3\times 1}$$
$$= 75\ 000(1.085)^3$$

Consider the second period:
The principal at the start of the second period is the accumulated amount of the first period: $P_2 = A_1$

$$A_2 = P_2\left(1+\frac{9.75}{100\times 4}\right)^{4\times 4}$$
$$= \left[75\ 000(1.085)^3\right](1.024375)^{16}$$

Consider the third period:
At the start of the third period, the principal will be the amount accumulated by the end of the second period: $P_3 = A_2$

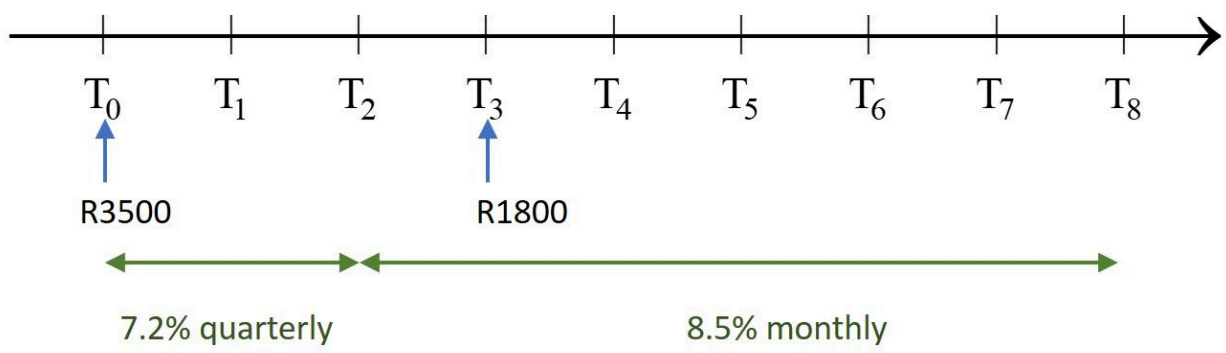
$$A_3 = P_3\left(1+\frac{10.25}{100\times 12}\right)^{2\times 12}$$
$$= \left[\left[75\ 000(1.085)^3\right](1.024375)^{16}\right](1.008541...)^{24}$$
$$= \left[[95\ 796.68...](1.024375)^{16}\right](1.008541...)^{24}$$
$$= [140\ 829.6217](1.008541...)^{24}$$
$$= 172\ 721.4594$$

At the end of the nine years, the total value of the investment will be $\mathrm{R}172\ 721.46$.

Back to Exercise 3.1

Unit 3: Assessment

1.



Consider the first period – the first year:

$$\begin{aligned} A_1 &= P_1\left(1+\frac{r}{100\times m}\right)^{t\times m} \\ &= 3\ 500\left(1+\frac{7.2}{100\times 4}\right)^{1\times 4} \\ &= 3\ 500(1.018)^4 \end{aligned}$$

Consider the second time period:
The principal at the start of the period is the accumulated amount at the end of the first period ($P_2 = A_1$), and the new interest rate applies.

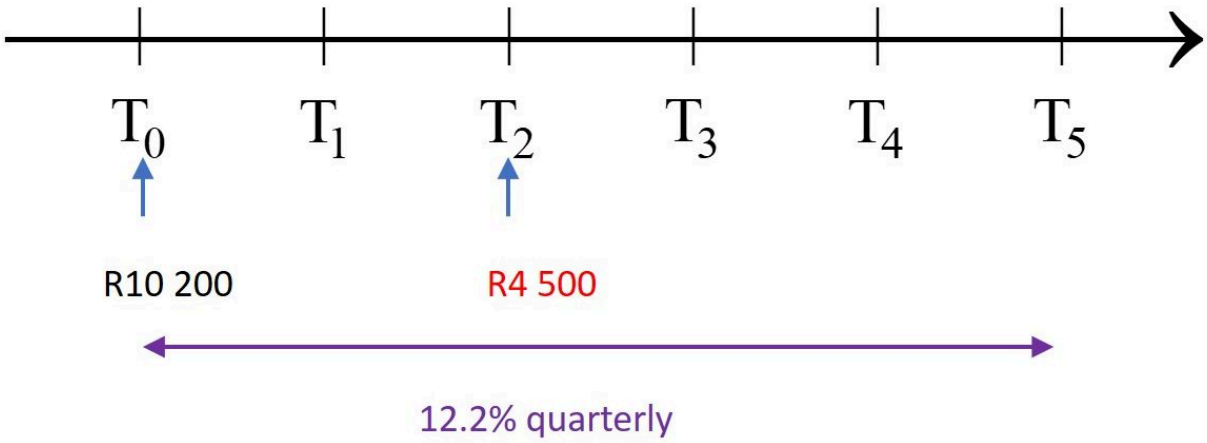
$$\begin{aligned} A_2 &= P_2\left(1+\frac{r}{100\times m}\right)^{t\times m} \\ &= \left[3\ 500(1.018)^4\right]\left(1+\frac{8.5}{100\times 12}\right)^{0.5\times 12} \\ &= \left[3\ 500(1.018)^4\right](1.00708833...)^6 \end{aligned}$$

Consider the third time period:
The principal at the start of the period is the accumulated amount at the end of the second period (A_2) with the additional amount deposited.

$$\begin{aligned} A_3 &= P_3\left(1+\frac{r}{100\times m}\right)^{t\times m} \\ &= \left[1\ 800+\left[3\ 500(1.018)^4\right]\left(1+\frac{8.5}{100\times 12}\right)^{0.5\times 12}\right]\left(1+\frac{8.5}{100\times 12}\right)^{2.5\times 12} \\ &= \left[1\ 800+[3\ 758.886...]\left(1+\frac{8.5}{100\times 12}\right)^{6}\right]\left(1+\frac{8.5}{100\times 12}\right)^{30} \\ &= [5\ 721.494485]\left(1+\frac{8.5}{100\times 12}\right)^{30} \\ &= 7\ 070.8519.. \end{aligned}$$

The total amount saved after four years will be $\mathrm{R}7\ 070.85$.

2.



Consider the first period:

$$\begin{aligned} A_1 &= P_1\left(1+\frac{r}{100\times m}\right)^{t\times m} \\ &= 10\ 200\left(1+\frac{12.2}{100\times 4}\right)^{2\times 4} \\ &= 10\ 200(1.0305)^{8} \end{aligned}$$

Consider the second period:
The principal at the start of the second period is $\mathrm{R}4\ 500$ less than the amount accumulated by the end of the first period.

$$\begin{aligned} A_2 &= P_2\left(1+\frac{r}{100\times m}\right)^{t\times m} \\ &= \left[10\ 200(1.0305)^{8}-4\ 500\right](1.0305)^{3\times 4} \\ &= [12\ 971.31902-4\ 500](1.0305)^{12} \\ &= 12\ 148.62119 \end{aligned}$$

At the end of five years there will be $\mathrm{R}12\ 148.62$ in the account.

Back to Unit 3: Assessment

Media Attributions

- M3 SO5.2 Unit3 Image1 Example 3.1 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 NOTE-calculator memory
- M3 SO5.2 Unit 3 Take Note RCL M+
- M3 SO5.2 Unit3 Image2 NOTE on calculator memory © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 demonstrating calculator use
- M3 SO5.2 Unit3 Image3 NOTE2 on calculator memory © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image4 Example3.2 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image5 Solution Ex3.1 No1a © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image6 Solution Ex3.1 No2 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image7 Solution Ex3.1 No3 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image8 Solution Ex3.1 No4 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image9 Solution Ex3.1 No5 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image10 Solution Ex3.1 No6 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image11 Solution Assessment No1 © DHET is licensed under a CC BY (Attribution) license
- M3 SO5.2 Unit3 Image12 Solution Assessment No2 © DHET is licensed under a CC BY (Attribution)

license